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Distance Laplacian Eigenvalues and Chromatic Number in Graphs

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Abstract. In the present paper we are interested in the study of the distance Laplacian eigenvalues of a connected graph with fixed order n and chromatic number χ. We prove lower bounds on the distance Laplacian spectral radius in terms of n and χ. We also prove results related to the distribution of the distance Laplacian eigenvalues with respect to the values of the chromatic number χ. For some of the results, we characterize the extremal graphs, for others, we give examples of extremal graphs.

1. Introduction

In this paper, we consider only connected simple, undirected and finite graphs, *i.e*, undirected graphs on a finite number of vertices without multiple edges or loops and in which any two vertices are connected by a sequence of edges. A graph is (usually) denoted by $G = G(V, E)$, where V is its vertex set and E its edge set. The *order* of G is the number $n = |V|$ of its vertices and its *size* is the number $m = |E|$ of its edges.

The *adjacency matrix* of G is a 0–1 $n \times n$–matrix indexed by the vertices of G and defined by $a_{ij} = 1$ if and only if $ij \in E$. The *adjacency spectrum* of G is the spectrum of its adjacency matrix. For more details about the adjacency spectrum of a graph see the books [9, 14, 15, 18].

The matrix $L = Diag(Deg) - A$, where $Diag(Deg)$ is the diagonal matrix whose main entries are the degrees in G, is called the *Laplacian* of G. The Laplacian spectrum of G is the spectrum of L. More details about L and its spectrum can be found in the books [9, 18] and in the survey papers [39, 40].

The matrix $Q = Diag(Deg) + A$, where $Diag(Deg)$ is the diagonal matrix whose main entries are the degrees in G, is called the *signless Laplacian* of G. The signless Laplacian spectrum of G is the spectrum of Q. More details about Q and its spectrum can be found in [7, 16, 17, 19–21].

Given two vertices u and v in a graph G, $d(u, v) = d_G(u, v)$ denotes the *distance* (the length or number of edges of a shortest path) between u and v. The *transmission* $Tr(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, *i.e.*,

$$Tr(v) = \sum_{u \in V} d(u, v).$$

The *distance matrix* $\mathcal{D}$ of a graph G is the matrix indexed by the vertices of G with $\mathcal{D}_{i,j} = d(v_i, v_j)$ and where $d(v_i, v_j)$ denotes the distance between the vertices v_i and v_j. Let $(\partial_1, \partial_2, \ldots, \partial_n)$ denote the spectrum

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of $\mathcal{D}$. It is called the *distance spectrum* of the graph G. We assume that the distance eigenvalues are labeled such that $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. For more about the distance matrix and its spectrum see the survey [6] as well as the references therein.

Introduced in [3] the *distance Laplacian* $\mathcal{D}^L$ of a graph G is defined as $\mathcal{D}^L = Diag(Tr) - \mathcal{D}$, where $Diag(Tr)$ denotes the diagonal matrix of the vertex transmissions in G. Let $(\partial_1^L, \partial_2^L, \ldots, \partial_n^L)$ denote the spectrum of $\mathcal{D}^L$.It is called the *distance Laplacian spectrum* of the graph G. We assume that the distance Laplacian eigenvalues are labeled such that $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L$. For more results and details about distance Laplacian spectra of graphs see [5, 37, 41, 45]

Recall that the *chromatic number* $\chi = \chi(G)$ of a graph G is the smallest number of colors to be assigned to G's vertices such that no pair of adjacent vertices have the same color. A subset of vertices assigned the same color is called a *color class*, every such class forms an independent set. A graph in which the vertex set can be partitioned into two independent sets is *bipartite*; three sets *tripartite*; k sets *k-partite* or *multipartite* with k independent sets. A k-partite graph is said to be complete when any two vertices are adjacent if and only if they belong to different partition classes. A complete k-partite graph is said to be *balanced*, and denoted by $T_k(n)$, if for any two partition classes V' and V'', $||V'| - |V''|| \leq 1$. It is also called *Turán's graph*.

As usual, we denote by P_n the path, by C_n the cycle, by S_n the star, by $K_{a,n-a}$, $1 \leq a \leq n-1$, the complete bipartite graph, and by K_n the complete graph, each on n vertices.

In the present paper we are interested in the study of the distance Laplacian eigenvalues of a connected graph with fixed order n and chromatic number χ. The origins of the study of the interconnections between the chromatic number and the eigenvalues of a graph goes back to 1967 when Wilf [48] proved an upper bound on the chromatic number χ using the (adjacency) spectral radius λ_1, namely $\chi \leq \lambda_1 + 1$, as an improvement of the well-known [8] Brooks' inequality $\chi \leq \Delta + 1$, where Δ denotes the maximum degree of a graph. Three years later, two papers were dedicated to studying the problem of bounding the chromatic number using the adjacency spectrum. Actually, Hoffman [32] and Hoffman and Howes [33] proved a series of bounds on the chromatic number χ using the adjacency eigenvalues of their frequencies, among which $\chi \geq 1 - \lambda_1/\lambda_n$, where λ_n denote the smallest adjacency eigenvalue of a graph. A tighter bound was given in [13] and in [22]: $\chi \geq n/(n - \lambda_1)$. For more details about the early results related to the interconnection between the chromatic number and the adjacency spectrum see [14], particularly Chapter 3.

Inequalities involving several eigenvalues and the chromatic number were also proved. For instance, we recall the following ones (see e.g. [9, 29, 32, 33]):

(a) $\lambda_1 + \lambda_{n-\chi+2} + \cdots + \lambda_n \leq 0$;
(b) If $n > \chi$ then $\lambda_2 + \cdots + \lambda_\chi + \lambda_{n-\chi+1} \geq 0$;
(c) If $n > t \cdot \chi$ then $\lambda_{t+1} + \cdots + \lambda_{t+\chi-1} + \lambda_{n-t(\chi-1)} \geq 0$,

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is the adjacency spectrum of the graph and χ denotes its chromatic number.

A Nordhaus–Gaddum type inequality (see [46], and [4] for a survey) for the spectral radius of a graph in terms of the order and chromatic number is proved by Hong and Shu [35]:

$$\lambda_1(G) + \lambda_1(\overline{G}) \leq \sqrt{n(n-1)\left(2 - \frac{1}{\chi(G)} - \frac{1}{\chi(\overline{G})}\right)},$$

where $\overline{G}$ denotes the complement of the graph G.

More bounds and results involving the chromatic number and the adjacency eigenvalues of graphs can be found in [13, 23–26, 28, 34, 42–44, 49].

Nikiforov [43] proved (as a corollary of a more general result involving difference between a diagonal matrix and a Hermitian matrix) an inequality involving both the adjacency and combinatorial Laplacian radii, λ_1 and μ_1 respectively, in addition to the chromatic number χ: $\chi \geq 1 + \lambda_1/(\mu_1 - \lambda_1)$.

For more results about interconnections between the chromatic number and the combinatorial Laplacian spectrum see [23, 47]. A few results relating the spectrum of the normalized Laplacian (see e.g. [12]) and the chromatic number of a graph can be found in [51].

Regarding the signless Laplacian spectral radius, denoted q_1, it is proved in at least three papers [10],[30] and [50] that $q_1 \le 2n(1 - 1/\chi)$. Among other results involving the signless Laplacian eigenvalues, we can cite the following inequalities proved in [36]:

$$q_1 - q_n \ge \chi \qquad \text{and} \qquad q_n \le \frac{2m}{n} - \frac{2m}{n(\chi - 1)};$$

where m denotes the number of edges of the graph.

For more results about interconnections between the chromatic number and the signless Laplacian spectrum see [16, 23, 31]

Concerning the interconnection between the distance spectrum and the chromatic number of a graph, only a few results are known. As an example, we recall the following lower bound from [38]: $\partial_1(G) \ge \partial_1(T_k(n))$ with equality if and only if $G \cong T_k(n)$.

Above, we gave a short overview of the research related to the interconnections between the chromatic number and different spectra of graphs: adjacency, Laplacian, signless Laplacian and distance. The present paper is a contribution to that topic. In the next section, we consider the extension of the topic to include, for the first time, the distance Laplacian spectrum. Actually, we prove results relating the chromatic number of a connected graph to its distance Laplacian eigenvalues and their frequencies. In order to prove those results, we need the following theorems from [3, 5].

The first theorem is about how the distance eigenvalues change under the deletion (or addition) of an edge.

Theorem 1.1 ([3]). *Let G be a connected graph on n vertices and $m \ge n$ edges. Consider the connected graph $\tilde{G}$ obtained from G by the deletion of an edge. Let $\left(\partial_1^L, \partial_2^L, \ldots, w\partial_n^L\right)$ and $\left(\tilde{\partial}_1^L, \tilde{\partial}_2^L, \ldots \tilde{\partial}_n^L\right)$ denote the distance Laplacian spectra of G and $\tilde{G}$ respectively. Then $\tilde{\partial}_i^L \ge \partial_i^L$ for all $i = 1, \ldots n$.*

The second theorem shows the similarity between the second smallest distance Laplacian eigenvalue and the algebraic connectivity (the second smallest Laplacian eigenvalue, see e.g. [27]).

Theorem 1.2 ([3]). *Let G be a connected graph on n vertices. Then $\partial_{n-1}^L \ge n$ with equality holding if and only if $\overline{G}$ is disconnected. Furthermore, the multiplicity of n as an eigenvalue of $\mathcal{D}^L$ is one less than the number of components of $\overline{G}$.*

The next two theorem are about relationships between local structure of a graph and the distance Laplacian eigenvalues.

Theorem 1.3 ([5]). *Let G be a graph on n vertices. If $S = \{v_1, v_2, \ldots v_p\}$ is an independent set of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\partial + 2$ is an eigenvalue of $\mathcal{L}$ with multiplicity at least $p - 1$.*

Theorem 1.4 ([5]). *Let G be a graph on n vertices. If $K = \{v_1, v_2, \ldots v_p\}$ is a clique of G such that $N(v_i) - K = N(v_j) - K$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\partial + 1$ is an eigenvalue of $\mathcal{L}$ with multiplicity at least $p - 1$.*

2. Main Results

In this section we present our results about interconnections between the chromatic number and the distance Laplacian eigenvalues of a connected graph. Note that some of the results proved in this paper, were first obtained as conjectures using the AutoGraphiX system, a software devoted to conjecture-making in graph theory [1, 2, 11].

It was shown in [3] that $\partial_1^L(G) \geq n$ with equality if and only if $G \cong K_n$, that is, if and only if $\chi = n$. At that level, a question arises naturally: *what would the gap between ∂_1^L and n be whenever $\chi < n$?* We provide the answer to that question right after the following lemma, which will be used in the proof of that result as well as subsequent ones.

Lemma 2.1. *Let $l_1, l_2, \ldots, l_k$ and n be integers such that $l_1 + l_2 + \cdots + l_k = n$ and $l_i \geq 1$ for $i = 1, \ldots, k$. Let $p = |\{i : l_i \geq 2\}|$. The distance Laplacian spectrum of the complete k-partite graph $K_{l_1,\ldots,l_k}$ is $\left((n + l_1)^{(l_1-1)}, \ldots, (n + l_p)^{(l_p-1)}, n^{(k-1)}, 0\right)$.*

Proof :
Consider the complete k-partite graph $K_{l_1,\ldots,l_k}$ such that (without a loss of generality) $l_1 \geq l_2 \geq \cdots \geq l_k$, where at least $l_1 \geq 2$ (if $l_1 = 1$, the graph is K_n). Let p be the largest integer, $1 \leq p \leq k$, such that $l_p \geq 2$ and $l_{p+1} = 1$. Denote by $V_1, V_2, \ldots, V_p$ the maximal independent sets in $K_{l_1,\ldots,l_k}$ that contain at least 2 vertices each. For $i \in \{1, 2, \ldots, p\}$, the vertices of V_i have the same neighborhood and their common transmission is $n + l_i - 2$. Thus, using Theorem 1.3, $\partial^L = n + l_i$ is an distance Laplacian eigenvalue of $K_{l_1,\ldots,l_k}$ with multiplicity $l_i - 1$. This way, we just identified $\sum_{i=1}^{p}(l_i - 1) = n - k$ distance Laplacian eigenvalues of $K_{l_1,\ldots,l_k}$. Since complement of $K_{l_1,\ldots,l_k}$ contains k connected components, according to Theorem 1.2, n is a distance Laplacian eigenvalue of $K_{l_1,\ldots,l_k}$ with multiplicity $k - 1$. Therefore, the distance Laplacian spectrum of $K_{l_1,\ldots,l_k}$ is $\left((n + l_1)^{(l_1-1)}, \ldots, (n + l_p)^{(l_p-1)}, n^{(k-1)}, 0\right)$. □

Note that in the above lemma when $p = 0$, necessarily $k = n$ and $l_1 = \cdots = l_n = 1$. In that case the graph is K_n and the distance Laplacian spectrum is $(n^{(n-1)}, 0)$.

Theorem 2.2. *Let G be a connected graph on n vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$. If $G \ncong K_n$, then*

$$\partial_1^L \geq n + \left\lceil \frac{n}{\chi} \right\rceil.$$

Moreover, for $n \geq 3$ and every $\chi \in \{1, 2, \ldots, n-1\}$, there exists a graph $G_{n,\chi}$ with order n and chromatic number χ such that

$$\partial_1^L(G_{n,\chi}) = n + \left\lceil \frac{n}{\chi} \right\rceil.$$

Proof :
Let G be a graph with chromatic number χ. A χ-coloring of the vertices of G partitions the vertex set into χ independent sets. Let $l_1, l_2, \ldots, l_\chi$ be the cardinalities of those independent sets, and assume, without a loss of generality, that $l_1 \geq l_2 \geq \cdots \geq l_\chi$. Thus G is a spanning subgraph of the complete χ-partite graph $K_{l_1,\ldots,l_\chi}$. Using Lemma 2.1 and the fact that the addition of edges does not increase the value of the distance Laplacian spectral radius, we have

$$\partial_1^L \geq \partial_1^L\left(K_{l_1,\ldots,l_\chi}\right) = n + l_1.$$

Since l_1 is the largest among the independent sets cardinalities, it is at least equal to the average cardinality, *i.e.*, $l_1 \geq \lceil n/\chi \rceil$. This proves the bound.

For the case of equality, according to Lemma 2.1, it suffices to take $G_{n,\chi} \cong K_{l_1,l_2,\ldots,l_\chi}$ where $l_1 = \lceil n/\chi \rceil$ and $l_1 + l_2 + \cdots + l_\chi = n$, which is always possible. □

The following corollary completes in some way that result (from [3]) we mentioned above as the motivation of the previous theorem.

Corollary 2.3. *Let G be a connected graph on $n \geq 3$ vertices with distance Laplacian spectral radius ∂_1^L. If $G \ncong K_n$, then $\partial_1^L \geq n + 2$. Moreover the bound is the best possible as shown by, among others, the complete $n-1$-partite graph (which is obtained from K_n by the deletion of an edge).*

Next, we provide a lower bound on the sum of the distance Laplacian spectral radius ∂_1^L and the chromatic number χ over the class of connected graphs of order n. We also prove the sharpness of the bound by providing a family of extremal graphs.

Corollary 2.4. *Let G be a connected graph on n vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$. Then $\partial_1^L + \chi \geq n + \left\lceil 2\sqrt{n} \right\rceil$, and the bound is the best possible as shown by a complete k-partite graph $K_{l_1,\ldots,l_k}$ with $k = \left\lceil \sqrt{n} \right\rceil$ and $l_1 = \left\lceil 2\sqrt{n} \right\rceil - \left\lceil \sqrt{n} \right\rceil$.*

Proof :
From Theorem 2.2,

$$\partial_1^L + \chi \geq n + \frac{n}{\chi} + \chi.$$

The latest expression, as a continuous function in χ, reaches its maximum for $\chi = \sqrt{n}$ and therefore the bound follows.

Using Lemma 2.1, we show that the bound is reached at least for a complete k-partite graph $K_{l_1,\ldots,l_k}$ with $k = \left\lceil \sqrt{n} \right\rceil$ and $l_1 = \left\lceil 2\sqrt{n} \right\rceil - \left\lceil \sqrt{n} \right\rceil$. □

Note that the family of extremal graphs provided for the above bound is not unique. For instance, if $n = 5$ there are three graphs for which the bound is reached: $K_n \cong K_{1,1,1,1,1}$ for which $\chi = 5$ and $\partial_1^L = 5$, $K_{2,2,1}$ for which $\chi = 3$ and $\partial_1^L = 7$, and $K_n \cong K_{3,2}$ (belongs to the mentioned family) for which $\chi = 2$ and $\partial_1^L = 8$. All three graphs are illustrated in Figure 1.

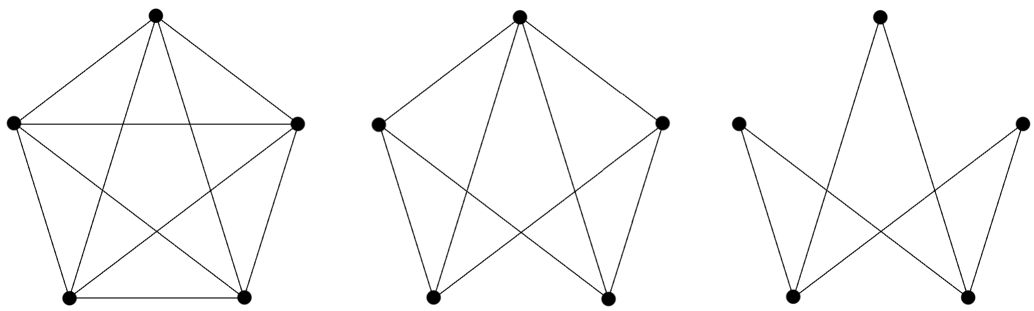

Figure 1: Three extremal graphs on 5 vertices for the bound of Theorem 2.4.

For a real interval I let $\mu(I) = \mu_G(I)$ denote the number of distance Laplacian eigenvalues that belong to I, i.e., $\mu(I) = |\{\partial_i^L \ : \ \partial_i^L \in I\}|$. When the interval is reduced to one number, say $I = \{a\}$, we write $\mu(a)$ instead of $\mu(\{a\})$. Note that in the evaluation of $\mu(I)$ an eigenvalue is counted as many times as it appears in the spectrum. We next focus on the number of distance Laplacian eigenvalues that are less than the bound of Theorem 2.2. The first result, stated in the next proposition, characterizes the graphs distance Laplacian eigenvalues of which are less than that bound.

Proposition 2.5. *Let G be a connected graph on n vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$. Let $b_\chi = n + \left\lceil \frac{n}{\chi} \right\rceil$, then $\mu\left([0, b_\chi)\right) = n$ if and only if G is the complete graph K_n.*

Proof :
First, note that $\partial_1^L(K_n) = n < b_n = n + 1$.
If $\mu_G\left([0, b_\chi)\right) = n$, then $\partial_1^L(G) < b_\chi$. According to Theorem 2.2, the only graph satisfying the inequality is the complete graph K_n. □

We next characterize all graphs for which ∂_1^L is the only distance Laplacian eigenvalue not less than the bound of Theorem 2.2.

Theorem 2.6. *Let G be a connected graph on n vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$. Let $b_\chi = n + \left\lceil \frac{n}{\chi} \right\rceil$. If $G \ncong K_n$ then $\mu([0, b_\chi)) \leq n-1$ with equality if and only if $G \cong \overline{S_{n-p} \cup pK_1}$, where $1 \leq p \leq n-2$.*

Proof :
The inequality follows immediately from Theorem 2.2. For the characterization of the extremal graphs, we first prove that if $G \cong \overline{S_{n-p} \cup pK_1}$ (p fixed and satisfying $1 \leq p \leq n-2$) then equality holds. It is obvious that $\chi(\overline{S_{n-p} \cup pK_1}) = n-1$. Therefore, the only complete multipartite graph containing $\overline{S_{n-p} \cup pK_1}$ is $K_{2,1,\cdots,1}$ whose distance Laplacian spectrum is $\left(n+2, n^{n-2}, 0\right)$. Thus $\partial_1^L(\overline{S_{n-p} \cup pK_1}) \geq n+2$. In addition, $S_{n-p} \cup pK_1$ contains $n-1$ connected components, and thus n occurs $n-2$ times in the distance Laplacian of $\overline{S_{n-p} \cup pK_1}$. In conclusion, if $G \cong \overline{S_{n-p} \cup pK_1}$ then $\mu_G([0, b_\chi)) \leq n-1$. See Figure 2 for examples of those graphs.

Conversely, it suffices to prove that if $G \ncong \overline{S_{n-p} \cup pK_1}$, for some p ($1 \leq p \leq n-2$), G has at least two distance Laplacian eigenvalues not less than b_χ. So consider a connected graph G, such that $G \ncong \overline{S_{n-p} \cup pK_1}$, for some p ($1 \leq p \leq n-2$) *i.e.*, $\chi(G) \leq n-2$. Let $K_{l_1,l_2,\ldots,l_\chi}$ be a complete χ-partite graph containing G, where $l_1 \geq l_2 \geq \cdots \geq l_\chi$ and $K_{l_1,l_2,\ldots,l_\chi} \ncong K_{2,1,\ldots,1}$. We have two cases:

- If $l_1 \geq 3$, then at least two distance Laplacian eigenvalues of $K_{l_1,l_2,\ldots,l_\chi}$ are not less than b_χ. Using Theorem 1.1, it is easy to see that G contains at least two distance Laplacian eigenvalues not less than b_χ.
- If $l_1 = 2$, then also $l_2 = 2$. Therefore, at least two distance Laplacian eigenvalues, $n + l_1$ and $n + l_2$, of $K_{l_1,l_2,\ldots,l_\chi}$ are not less than b_χ. Again using Theorem 1.1, G contains at least two distance Laplacian eigenvalues not less than b_χ. □

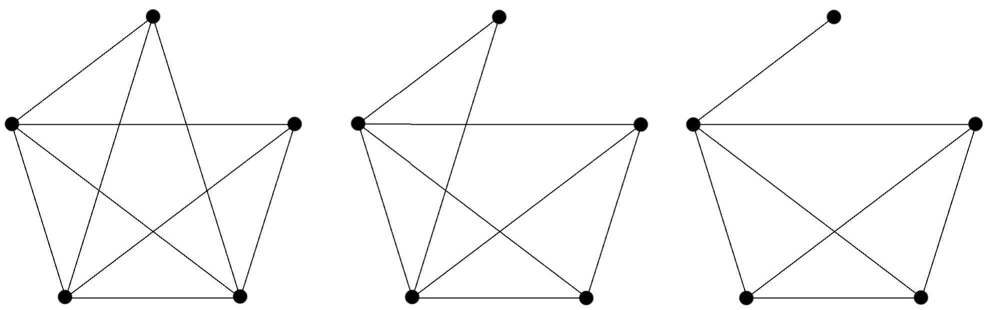

Figure 2: All graphs for the bound of Theorem 2.6 on 5 vertices.

We continue along those lines and study how large the number of distance Laplacian eigenvalues less than the bound of Theorem 2.2 when the diameter of the graph is at least 3. For the extremal graphs we need the following definition. A *double star* S_{Δ_1,Δ_2} is the tree obtained from a K_2 by attaching $\Delta_1 - 1$ pendant vertices to one vertex and $\Delta_2 - 1$ pendant vertices to the other vertex (See Figure 3 for the double star $S_{6,4}$). The double star S_{Δ_1,Δ_2} contains $n = \Delta_1 + \Delta_2$ vertices among which $n-2$ are pendant, and the two non pendant vertices have degrees Δ_1 and Δ_2 respectively. The complement of a double star S_{Δ_1,Δ_2}, $\overline{S_{\Delta_1,\Delta_2}}$, can be seen as the graph obtained from a complete graph $K_{\Delta_1+\Delta_2-2}$ and isolated vertices, say u and v, by adding edges between u and $\Delta_1 - 1$ from the clique and edges between v and the other $\Delta_2 - 1$ vertices from the clique.

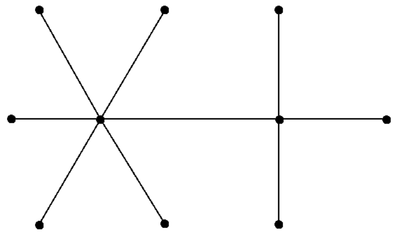

Figure 3: The double star $S_{6,4}$.

Theorem 2.7. *Let G be a connected graph on $n \geq 5$ vertices with chromatic number χ, diameter D and distance Laplacian spectrum $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$. Let $b_\chi = n + \left\lceil \frac{n}{\chi} \right\rceil$. If $D \geq 3$, then $\mu([0, b_\chi)) \leq n-2$ and the bound is the best possible as shown by the complement of the double star $S_{n-2,2}$.*

Proof :
Since the diameter of $\overline{S_{n-p} \cup pK_1}$ is 2 for every p with $1 \leq p \leq n-2$, the inequality follows from the previous corollary. So it remains to prove that equality is reached at least for $\overline{S_{n-2,2}}$, the complement of the double star $S_{n-2,2}$. Assume without loss of generality that the vertices of $\overline{S_{n-2,2}}$ are labeled such that v_1 and v_2 are respectively the vertices with degrees 2 and $n-2$ in $S_{n-2,2}$, v_3 is the pending neighbor of v_1 in $S_{n-2,2}$, and $v_4, \ldots, v_n$ are the pending neighbors of v_2 in $S_{n-2,2}$. Using that labeling, we can write the value of the characteristic polynomial of $\mathcal{D}^L\left(\overline{S_{n-2,2}}\right)$ as follows.

$$p(t) = det\left(\begin{bmatrix} R & N \\ N^T & M \end{bmatrix}\right),$$

with $M = (t-(n+1))I_{n-2} + J_{n-2}$, where J_{n-2} is the all 1's square matrix of order $n-2$ and I_{n-2} is the $(n-2)\times(n-2)$-identity,

$$N = \begin{bmatrix} 1 & \overbrace{2 \ \ldots \ 2}^{n-3 \text{ times}} \\ 2 & 1 \ \ldots \ 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} t-(n+2) & 3 \\ 3 & t-2(n-1) \end{bmatrix}.$$

The determinant of M is $det(M) = (t-3)(t-(n+1))^{(n-3)}$, and the inverse of M is $((t-3)(t-(n+1)))^{-1} \cdot M'$, where M' is the $(n-2)\times(n-2)$–matrix, all diagonal entries of which are equal to $t-4$ and all non diagonal entries are equal to -1. Now, using the properties of the determinants (see for example [14, Lemma 2.2.])

$$\begin{aligned}
p(t) &= det(M) \cdot det(R - NM^{-1}N^T) \\
&= (t-3)(t-(n+1))^{(n-3)} \cdot det\left(\begin{bmatrix} t-(n+2) & 3 \\ 3 & t-2(n-1) \end{bmatrix} - \right. \\
&\quad \left. \frac{1}{(t-3)(t-(n+1))}\begin{bmatrix} (n+1)t-(n^2+n+4) & 2(n-2)t-(2n^2-n-7) \\ 2(n-2)t-(2n^2-n-7) & (4n-11)t-4(n^2-2n-2) \end{bmatrix}\right) \\
&= \frac{(t-(n+1))^{(n-5)}}{t-3} \cdot det\left((t-(n+1))(t-3)\begin{bmatrix} (t-(n+2)) & 3 \\ 3 & (t-2(n-1)) \end{bmatrix}\right. \\
&\quad \left. - \begin{bmatrix} (n+1)t-(n^2+n+4) & 2(n-2)t-(2n^2-n-7) \\ 2(n-2)t-(2n^2-n-7) & (4n-11)t-4(n^2-2n-2) \end{bmatrix}\right) \\
&= (t-(n+1))^{(n-4)}(t^3-4nt^2-4t^2+5n^2t+12nt-2n^3-8n^2-2n)t
\end{aligned}$$

Thus the distance Laplacian spectrum of $\overline{S_{n-2,2}}$ is $\left(t_1, t_2, (n+1)^{(n-4)}, t_3, 0\right)$ where t_1, t_2 and t_3 are the solutions of the equation $f(t) = t^3-4nt^2-4t^2+5n^2t+12nt-2n^3-8n^2-2n = 0$ with $t_1 \geq t_2 \geq t_3$. Now, evaluating

$f(t)$ for a few values of t, we get $f(2n+2) = 2n^2 - 2n - 8 > 0$, $f(2n) = -2n < 0$, $f(n+2) = 2n - 8 > 0$, $f(n+1) = n - 3 > 0$, and $f(n) = -2n < 0$. Therefore, we have $2n \le t_1 \le 2n+2$, $n+2 \le t_2 \le 2n$, and $n \le t_1 \le n+1$. Since $\chi\left(\overline{S_{n-2,2}}\right) = n-2$, $b_\chi = n+2$ and the result follows. □

In the above results we were interested in the maximum possible values from the distance Laplacian eigenvalues that lie in the interval $[0, b_\chi)$. The next proposition is about the minimum number of those values lying in that interval.

Proposition 2.8. *Let G be a connected graph on $n \ge 6$ vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \ge \partial_2^L \ge \cdots \ge \partial_n^L = 0$. Let $b_\chi = n + \left\lceil \frac{n}{\chi} \right\rceil$, then $\mu([0, b_\chi)) \ge 1$. The bound is the best possible as shown by the path P_n.*

Proof :
The inequality is trivial since $\partial_n^L = 0$ for any connected graph (see *e.g.*, [3]). Thus to be done it suffices to prove that for a path of order $n \ge 6$, $\partial_{n-1}^L(P_n) \ge b_\chi = b_2 = n + \lceil n/2 \rceil$.
Since the deletion of an edge does not decrease the value of any of the the distance Laplacian eigenvalues (see [3]), we have $\partial_{n-1}^L(P_n) \ge \partial_{n-1}^L(C_n)$. From the distance Laplacian spectrum of C_n calculated in [3], $\partial_{n-1}^L \ge \lfloor n^2/4 \rfloor \ge n + \lceil n/2 \rceil$, for $n \ge 6$. This completes the proof. □

In the above four results we were interested in the number of distance Laplacian eigenvalues on each side of the bound of Theorem 2.2, now we turn our attention to the gap between ∂_1^L and that bound. The next result is a statement about how large the gap is when there is only one distance Laplacian eigenvalue not less than the bound.

Proposition 2.9. *Let G be a connected graph on n vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \ge \partial_2^L \ge \cdots \ge \partial_n^L = 0$, such that $\partial_2^L < b_\chi$, where $b_\chi = n + \lceil n/\chi \rceil$. Then $\partial_1^L - b_\chi \le n - 3$ with equality if and only if $G \cong \overline{S_{n-1} \cup K_1}$.*

Proof :
The condition $\partial_2^L < b_\chi$ implies that $\mu([0, b_\chi)) = n - 1$, thus by Theorem 2.6, there exists $p \in \{1, \ldots, n-2\}$ such that $G \cong \overline{S_{n-p} \cup pK_1}$. Since the deletion of an edge does not decrease the value of ∂_1^L, we have $\partial_1^L - b_{n-1} = \partial_1^L - n - 2 \le \partial_1^L(\overline{S_{n-1} \cup K_1}) - n - 2$. Now, using Theorem 1.4, we can easily show that the distance Laplacian spectrum of $\overline{S_{n-1} \cup K_1}$ is $\left(2n-1, (n+1)^{(n-3)}, n, 0\right)$. Thus $\partial_1^L - b_\chi \le n - 3$ with equality at least for $\overline{S_{n-1} \cup K_1}$. To show the uniqueness of the graph for which equality holds, it suffices to prove that $\partial_1^L(\overline{S_{n-2} \cup 2 \times K_1}) - b_{n-1} < n - 3$. Using Theorem 1.3 and Theorem 1.4, we can easily show that the distance Laplacian spectrum of $\overline{S_{n-2} \cup 2 \times K_1}$ is $\left(2n-2, (n+1)^{(n-3)}, n, 0\right)$. Thus $\partial_1^L(\overline{S_{n-2} \cup 2 \times K_1}) - b_{n-1} = n - 4 < n - 3$. □

In [3] it was pointed out that the order n is a special distance Laplacian eigenvalue (whenever it is) and Lemma 2.1 states that n appears $\chi - 1$ times in the distance Laplacian spectrum of a complete χ-partite graph. Is there any relation between the number of occurrences of n in a distance Laplacian spectrum and the chromatic number in a graph? the next proposition answers the question.

Proposition 2.10. *Let G be a connected graph on n vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \ge \partial_2^L \ge \cdots \ge \partial_n^L = 0$. Then $\mu(n) \le \chi - 1$, with equality if and only if G is a complete χ-partite graph.*

Proof :
If $\overline{G}$ is connected, then $\mu(n) = 0$ and the inequality is strict. Assume that $\overline{G}$ is disconnected and let p be the number of its connected component. Denote by $n_1, n_2, \ldots, n_p$ the numbers of vertices of the p components of $\overline{G}$. Thus G contains (all the edges of) the complete p-partite graph $K_{n_1,\ldots,n_p}$. Since adding edges does not decrease the chromatic number of a graph, $p \le \chi$ with equality if and only if $G \cong K_{n_1,\ldots,n_p}$. According to Theorem 1.2, $\mu(n) = p - 1$, therefore the result follows.

The above result states that for a connected graph on n vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$, at most $\partial_{n-\chi+1}^L$ can be equal to n. So what the least possible value of $\partial_{n-\chi+1}^L$ is? The answer is as follows.

Proposition 2.11. *Let G be a connected graph on n vertices with chromatic number χ and distance Laplacian spectrum $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$. Then $\partial_{n-\chi}^L \geq n+2$. The bound is the best possible as shown by any complete χ-partite graph $K_{n_1,\ldots,n_\chi}$ such that $2 \in \{n_1, \ldots, n_\chi\}$.*

Proof :
Let G be a connected graph on n vertices with chromatic number χ. Denote $n_1, n_2 \ldots n_\chi$ the numbers of vertices of the χ independent sets of G defined by the χ-coloring such that $n_1 \geq n_2 \geq \cdots \geq n_\chi$. Then G is a spanning subgraph of $K_{n_1,n_2\ldots n_\chi}$ Let $p = \max\{i \;:\; n_i \geq 2\}$. Combining Theorem 1.1 and Lemma 2.1, we have $\partial_{n-\chi}^L(G) \geq \partial_{n-\chi}^L(K_{n_1,n_2\ldots n_\chi}) = n + n_p \geq n+2$. The case of equality follows from Lemma 2.1 and the fact that $n_p = 2$. □

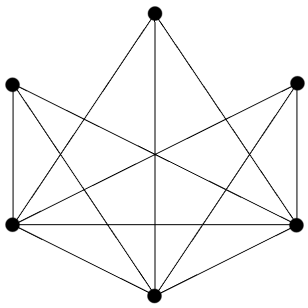

Figure 4: The complete 4-partite graph $K_{3,1,1,1}$.

Note that the condition play an important in the characterization of the extremal graphs. Indeed, there exist complete χ-partite graphs with $\mu(n) \leq \chi - 1$, *i.e.* $\partial_{n-\chi+1}^L = \partial_{n-\chi+2}^L = \cdots = \partial_{n-1}^L = n$, and $\partial_{n-\chi}^L > n+2$, such as the complete 4-partite graph $K_{3,1,1,1}$ (see Figure 4) for which $\partial_2^L = 9 > 8$ and $\partial_3^L = \partial_4^L = \partial_5^L = 6$.

The family of extremal graphs provided in the above proposition is not the only one for which the bound is reached. Actually there exist graphs of order n with chromatic number χ such that $\partial_2^L = \partial_3^L = \cdots = \partial_{n-\chi}^L = n+2$ and $\partial_{n-\chi+1}^L = \partial_{n-\chi+2}^L = \cdots = \partial_{n-1}^L = n$. For instance, the distance Laplacian spectrum of $\overline{C_4 \cup (n-4)K_1}$ is $\left(n+4, (n+2)^{(2)}, n^{(n-4)}, 0\right)$, for $n \geq 5$. Figure 5 shows the graph $\overline{C_4 \cup 4 \times K_1}$ on 8 vertices with chromatic number 5 whose distance Laplacian spectrum is $\left(12, 10^{(2)}, 8^{(4)}, 0\right)$.

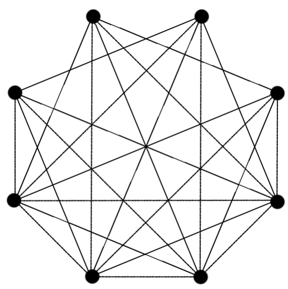

Figure 5: The graph $\overline{C_4 \cup 4 \times K_1}$.

References

[1] M. Aouchiche, J.-M. Bonnefoy, A. Fidahoussen, G. Caporossi, P. Hansen, L. Hiesse, J. Lacheré, A. Monhait, *Variable neighborhood search for extremal graphs. 14. The AutoGraphiX 2 system*, In L. Liberti and N. Maculan (editors), Global Optimization: From Theory to Implementation, Springer (2006) 281–310.

[2] M. Aouchiche, G. Caporossi, P. Hansen, *Variable neighborhood search for extremal graphs. 20. Automated comparison of graph invariants*, MATCH Commun. Math. Comput. Chem. **58** (2007) 365–384.

[3] M. Aouchiche, P. Hansen, *Two Laplacians for the distance matrix of a graph*, Linear Algebra Appl. 439 (2013) 21–33.

[4] M. Aouchiche, P. Hansen, *A survey of Nordhaus–Gaddum type relations*, Discrete Appl. Math. 161 (2013) 466–546.

[5] M. Aouchiche, P. Hansen, *Some properties of the distance Laplacian eigenvalues of a graph*, Czech. Math. Journal 64 (2014) 751–761.

[6] M. Aouchiche, P. Hansen, *Distance spectra of graphs: a survey*, Linear Algebra Appl. 458 (2014) 301–386.

[7] M. Aouchiche, P. Hansen, C. Lucas, *On the extremal values of the second largest Q-eigenvalue*, Linear Algebra Appl., 435 (2011) 2591–2606.

[8] R. L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc. 37 (1941) 194–197.

[9] A. E. Brouwer, W. H. Haemers,*Spectra of Graphs*, Springer, New York, 2012.

[10] G. X. Cai, Y.–Z. Fan, *The signless Laplacian spectral radius of graphs with given chromatic number*, (in Chinese), Mathematica Applicata 22 (2009) 161–167.

[11] G. Caporossi, P. Hansen, *Variable neighborhood search for extremal graphs: I. The AutoGraphiX system*, Disc. Math, **212** (2000) 29–44.

[12] F.R.K. Chung, *Spectral Graph Theory*, CBMS Lecture Notes 92, AMS Publication, 1997.

[13] D. Cvetković, *Chromatic number and the spectrum of a graph*, Publ. Inst. Math. (Beograd), 14(28) (1972) 25–38.

[14] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs—Theory and Applications*, 3rd edition, Johann Ambrosius Barth Verlag, Heidelberg–Leipzig, 1995.

[15] D. Cvetković, M. Doob, I. Gutman, A. Torgašev, *Recent Results in the Theory of Graph Spectra.* North Holland, Amsterdam, 1988.

[16] D. Cvetković, P. Rowlinson, S. K. Simić, *Signless Laplacians of finite graphs*, Linear Algebra Appl. 423 (2007) 155–171.

[17] D. Cvetković, P. Rowlinson, S. Simić, *Eigenvalue bounds for the signless Laplacian*, Publ. Inst. Math. (Beograd), 81(95) (2007) 11–27.

[18] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, 2010.

[19] D. Cvetković, S. K. Simić, *Towards a spectral theory of graphs based on the signless Laplacian. I*, Publ. Inst. Math. (Beograd) 85(99) (2009) 19–33.

[20] D. Cvetković, S. K. Simić, *Towards a spectral theory of graphs based on the signless Laplacian. II*, Linear Algebra Appl. 432 (2010) 2257–2272.

[21] D. Cvetković, S. K. Simić, *Towards a spectral theory of graphs based on the signless Laplacian. III*, Appl. Anal. Discrete Math. 4 (2010) 156–166.

[22] C. S. Edwards, C. H. Elphick, *Lower bounds for the clique and the chromatic numbers of a graph.* Discrete Applied Mathematics, 5 (1983) 51–64.

[23] C. Elphick, P. Wocjan, *Unified spectral bounds on the chromatic number*, arXiv:1210.7844v5, [math.CO], 2014.

[24] Y. Z. Fan, G. D. Yu, Y. Wang, *The chromatic number and the least eigenvalue of a graph*, Electronic Journal of Combinatorics, 19 (2012) P39.

[25] O. Favaron, M. Mahéo, J. F. Saclé, *Some eigenvalue properties in graphs (conjectures of Graffiti. II)*, Discrete Mathematics 111 (1993) 197–220.

[26] L. Feng, Q. Li, X. D. Zhang, *Spectral radii of graphs with given chromatic number*, Applied mathematics letters, 20 (2007) 158–162.

[27] M. Fiedler, *Algebraic connectivity of graphs*, Czechoslovak Math. J. 23(98) (1973) 298–305.

[28] W. H. Haemers, *Eigenvalue techniques in design and graph theory*, Math. Centre Tract 121, Mathematical Centre, Amsterdam, 1980.

[29] W. H. Haemers, *Interlacing eigenvalues and graphs*, Linear Algebra Appl. 226 (1995) 593–616.

[30] P. Hansen, C. Lucas, *An inequality for the signless Laplacian index of a graph using the chromatic number*, Graph Theory Notes of New York LVII (2009) 39–42.

[31] P. Hansen, C. Lucas, *Bounds and conjectures for the signless Laplacian index of graphs*, Linear Algebra Appl. 432 (2010) 3319–3336.

[32] A. J. Hoffman, *On eigenvalues and colorings of graphs*, In: Graph Theory and its Applications (ed. B. Harris), Academic Press, New York – London, 1970, pp. 79–91.

[33] A. J. Hoffman, L. Howes, *On eigenvalues and colorings of graphs, II*, Annals of the New York Academy of Sciences, 175 (1970) 238–242.

[34] Y. Hong, *Bounds of eigenvalues of a graph*, Acta Math. Appl. Sinica (English Ser.) 4 (1988) 165–168.

[35] Y. Hong, J. L. Shu, *A sharp upper bound for the spectral radius of the Nordhaus–Gaddum type*, Discrete Mathematics 211 (2000) 229–232.

[36] L. S. de Lima, C. S. Oliveira, N. M. M. de Abreu, V. Nikiforov, *The smallest eigenvalue of the signless Laplacian*, Linear Algebra Appl. 435 (2011) 2570–2584.

[37] H. Lin, B. Zhou, *On the distance Laplacian spectral radius of graphs*, Linear Algebra Appl. 475 (2015) 265–275.

[38] Z. Liu, *On the spectral radius of the distance matrix*, Appl. Anal. Discrete Math. 4 (2010) 269–277.

[39] R. Merris, *Laplacian matrices of graphs: a survey*, Linear Algebra Appl. 197/198 (1994) 143–176.

[40] B. Mohar, *Graph Laplacians*, In: *Topics in Algebraic Graph Theory* (eds, L. W. Beineke, R. J. Wilson), Cambridge University Press, 2004, pp. 113–136.

[41] M. Nath, S. Paul, *On the distance Laplacian spectra of graphs*, Linear Algebra Appl. 460 (2014) 97–110.

[42] V. Nikiforov, *Some inequalities for the largest eigenvalue of a graph*, Combinatorics, Probability & Computing, 11 (2002) 179–189.

[43] V. Nikiforov, *Chromatic number and spectral radius*, Linear Algebra Appl. 426 (2007) 810–814.

[44] V. Nikiforov, *Bounds on graph eigenvalues II*, Linear Algebra Appl. 427 (2007) 183–189.

[45] A. Niu, D. Fan, G. Wang, *On the distance Laplacian spectral radius of bipartite graphs*, Discrete Applied Mathematics, 186 (2015) 207–213.

[46] E. A. Nordhaus, J. Gaddum, *On complementary graphs*, Amer. Math. Monthly 63 (1956) 175–177.
[47] J. Tan, *Laplacian spectra and chromatic numbers of graphs*, Kyushu Journal of Mathematics, 54 (2000) 165–170.
[48] H. S. Wilf, *The eigenvalues of a graph and its chromatic number*, J. London Math. Soc, 42 (1967) 330–332.
[49] P. Wocjan, C. Elphick, *New spectral bounds on the chromatic number encompassing all eigenvalues of the adjacency matrix*, Electron. J. Combin. 20 (2013), no. 3, Paper 39, 18 pp.
[50] G. Yu, Y. Wu, J. Shu, *Signless Laplacian spectral radii of graphs with given chromatic number*, Linear Algebra Appl. 435 (2011) 1813–1822.
[51] X. D. Zhang, *On the Laplacian spectra of graphs*, Ars Combinatoria, 72 (2004) 191–198.