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Rough Statistical Cluster Points

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Abstract. In this paper, we define the concepts of rough statistical cluster point and rough statistical limit point of a sequence in a finite dimensional normed space. Then we obtain an ordinary statistical convergence criteria associated with rough statistical cluster point of a sequence. Applying these definitions to the sequences of functions, we come across a new concept called statistical condensation point. Finally, we observe the relations between the sets of statistical condensation points, rough statistical cluster points and rough statistical limit points of a sequence of functions.

1. Introduction and Background

Let $x = (x_i)$ be a sequence in a finite dimensional normed space X and r be a nonnegative real number. Suppose that for every $\varepsilon > 0$ there exists a positive integer i_ε such that $\|x_i - x_*\| < r + \varepsilon$ for every $i \geq i_\varepsilon$. Then the sequence $x = (x_i)$ is said to be *r-convergent* to x_*, and we write $x_i \xrightarrow{r} x_*$. The set

$$\mathrm{LIM}^r x := \left\{ x_* \in X : x_i \xrightarrow{r} x_* \right\}$$

is called the *r-limit set* of the sequence $x = (x_i)$. A sequence $x = (x_i)$ is said to be *r-convergent* if $\mathrm{LIM}^r x \neq \emptyset$. In this case, r is called the *convergence degree* of the sequence $x = (x_i)$. For $r = 0$, we get the ordinary convergence.

Phu [16] and Burgin [3] introduced the notion of rough convergence independently with different titles. Here we will adopt the definitions and notations in [16]. Phu [16] showed that the set $\mathrm{LIM}^r x$ is bounded, closed and convex; introduced the notion of rough Cauchy sequence and also investigated the dependence of $\mathrm{LIM}^r x$ on the roughness degree r. In [17], the results given in [16] are extended to infinite dimensional normed spaces. Recently, Aytar [2] has given the relations between the ordinary core of a sequence $x = (x_i)$ of real numbers and the r-limit set of the sequence x. In [2], an ordinary convergence criterion is obtained, which states that a sequence is convergent if, and only if, its rough core is equal to its rough limit set for the same roughness degree. In [1], the concept of rough statistical convergence is defined, and by introducing the set of rough statistical limit points of a sequence, two statistical convergence criteria associated with this set are obtained. Also it is proved that this set is closed and convex. Moreover, the relations between the set of all statistical cluster points and the set of all rough statistical limit points of a sequence are investigated. Listán-García and Rambla-Barreno [13] studied on Chebyshev centers by using the rough convergence in Banach spaces. Recently, the rough statistical convergence theory have been studied by many authors (see [4], [6], [7], [14] and [15]).

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Now we briefly recall some of the basic concepts and notations of the theories of statistical convergence and rough statistical convergence, and we refer to [1, 8–10] for more details.

Let K be a subset of the set $\mathbb{N}$ of positive integers, and let us denote the set $\{k \in K : k \leq n\}$ by K_n. Then the *natural density* of K is defined by $\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n. Clearly, a finite subset has natural density zero and we have $\delta(K^c) = 1 - \delta(K)$ whenever $\delta(K)$ exists, where $K^c := \mathbb{N}\backslash K$ is the complement of $K \subset \mathbb{N}$. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$. In addition, $\delta(K) \neq 0$ means that $\limsup_{n\to\infty} \frac{|K_n|}{n} > 0$.

If a sequence $x = (x_i)$ satisfies some property P for all i except on a set of natural density zero, then we say that the sequence x satisfies the property P for *almost all i* and we abbreviate this by *a.a. i*. Let (x_{i_k}) be a subsequence of (x_i) and denote $K := \{i_k : k \in \mathbb{N}\}$. If $\delta(K) = 0$, then (x_{i_k}) is called *a subsequence of density zero* or *a thin subsequence*. We say that (x_{i_k}) is *a nonthin subsequence* of (x_i) if K does not have density zero.

A sequence $x = (x_i)$ in $\mathbb{R}^n$ is said to be *r-statistically convergent* to x_*, denoted by $x_i \overset{r-st}{\to} x_*$, provided that the set

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}$$

has natural density zero for every $\varepsilon > 0$; or equivalently, if the condition

$$\text{st-}\limsup \|x_i - x_*\| \leq r$$

is satisfied. In addition, we can write $x_i \overset{r-st}{\to} x_*$ if, and only if, the inequality

$$\|x_i - x_*\| < r + \varepsilon$$

holds for every $\varepsilon > 0$ and *a.a.i*.

Here r is called the *statistical convergence degree*. If we take $r = 0$, then we obtain the ordinary statistical convergence.

In a similar fashion to the idea of classical rough convergence, the idea of rough statistical convergence of a sequence can be interpreted as follows.

Assume that a sequence $y = (y_i)$ is statistically convergent and cannot be measured or calculated exactly, and one has to do with an approximated (resp., statistically approximated) sequence $x = (x_i)$ satisfying $\left\|x_i - y_i\right\| \leq r$ for all i (resp., for *a.a.i*, namely, $\delta\left(\left\{i \in \mathbb{N} : \left\|x_i - y_i\right\| > r\right\}\right) = 0$). Then the sequence x is not statistically convergent anymore, but since the inclusion

$$\left\{i \in \mathbb{N} : \left\|y_i - y_*\right\| \geq \varepsilon\right\} \supseteq \left\{i \in \mathbb{N} : \left\|x_i - y_*\right\| \geq r + \varepsilon\right\} \tag{1}$$

holds and we have $\delta\left(\left\{i \in \mathbb{N} : \left\|y_i - y_*\right\| \geq \varepsilon\right\}\right) = 0$, we get

$$\delta\left(\left\{i \in \mathbb{N} : \left\|x_i - y_*\right\| \geq r + \varepsilon\right\}\right) = 0,$$

i.e., the sequence x is r-statistically convergent.

In general, the rough statistical limit of a sequence $x = (x_i)$ may not be unique for the roughness degree $r > 0$. So we have to consider the so-called r-statistical limit set of the sequence x, which is defined by

$$\text{st-LIM}^r x := \left\{x_* \in X : x_i \overset{r-st}{\to} x_*\right\}.$$

The sequence x is said to be *r-statistically convergent* provided that $\text{st-LIM}^r x \neq \emptyset$.

The main purpose of this paper is to observe various aspects of the theory of rough statistical convergence on sequences of functions. To this end, first we define the set of rough statistical cluster points of a sequence in a normed space. Then we characterize this set by using the $r-$closed balls of ordinary statistical cluster points. Later, we explore some properties of the set of rough statistical cluster points of a sequence, and obtain an ordinary statistical convergence criterion associated with this set. In the next section, we also introduce the concept of rough statistical limit point of a sequence. When we apply these concepts to sequences of functions, we come across a new concept, called the statistical condensation point. Finally, we observe the relations between the set of statistical condensation points, the set of rough statistical cluster points and the set of rough statistical limit points of a sequence of functions.

2. Rough Statistical Cluster Points

First, we introduce the concept of rough statistical cluster point of a sequence. Throughout the rest, $x = (x_i)$ will denote a sequence in a finite dimensional normed space X.

Definition 2.1. *Let $r \geq 0$. The vector $\mu \in X$ is called the r-statistical cluster point of the sequence $x = (x_i)$ provided that*

$$\delta(\{i \in \mathbb{N} : \left\| x_i - \mu \right\| < r + \varepsilon\}) \neq 0$$

for every $\varepsilon > 0$. We denote the set of all r-statistical cluster points the sequence x by Γ_x^r.

Note that if we take $r = 0$, then we obtain the notion of ordinary statistical cluster point defined by Fridy [9]. It is easy to see that $\Gamma_x^{r_1} \subseteq \Gamma_x^{r_2}$ for $r_1 \leq r_2$.

Fridy [9] proved that the set $\Gamma_x = \Gamma_x^0$ is closed. We will show that the set Γ_x^r is closed for each $r > 0$.

Theorem 2.2. *For any sequence $x = (x_i)$, the set Γ_x^r is closed for every $r \geq 0$.*

Proof. Let $\Gamma_x^r \neq \emptyset$ and consider a sequence $y = (y_i) \subseteq \Gamma_x^r$ such that $\lim_{i\to\infty} y_i = y_*$. Let us show that

$$\delta(\{i \in \mathbb{N} : \left\| x_i - y_* \right\| < r + \varepsilon\}) \neq 0$$

for every $\varepsilon > 0$. Fix $\varepsilon > 0$. Since $\lim_{i\to\infty} y_i = y_*$, there exists an $i_0 = i_0(\varepsilon) \in \mathbb{N}$ such that $\left\| y_i - y_* \right\| < \frac{\varepsilon}{2}$ for all $i > i_0$. Fix j_0 such that $j_0 > i_0$. Then we have $\left\| y_{j_0} - y_* \right\| < \frac{\varepsilon}{2}$.

Let j be any point of the set $\{i \in \mathbb{N} : \left\| x_i - y_{j_0} \right\| < r + \frac{\varepsilon}{2}\}$. Since $\left\| x_j - y_{j_0} \right\| < r + \frac{\varepsilon}{2}$, we get

$$\begin{aligned} \left\| x_j - y_* \right\| &\leq \left\| x_j - y_{j_0} \right\| + \left\| y_{j_0} - y_* \right\| \\ &< r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon, \end{aligned}$$

which shows that $j \in \{i \in \mathbb{N} : \left\| x_i - y_* \right\| < r + \varepsilon\}$. Hence we have

$$\{i \in \mathbb{N} : \left\| x_i - y_{j_0} \right\| < r + \frac{\varepsilon}{2}\} \subseteq \{i \in \mathbb{N} : \left\| x_i - y_* \right\| < r + \varepsilon\}. \tag{2}$$

Since the natural density of the set on the left-hand side of the inclusion relation (2) is not equal to zero, the natural density of the set on the right-hand side is different from zero. Therefore we have $y_* \in \Gamma_x^r$. □

We note that if $\mu \in \Gamma_x^r$ then $\delta\left(\left\{i : \left\| x_i - \mu \right\| < r + \varepsilon\right\}\right) \neq 0$. By the statistical analogue of Bolzano-Weierstrass Theorem (see [18, Theorem 2]), the subsequence $(x_i)_{i\in K}$ has a statistical cluster point, where $K = \left\{i : \left\| x_i - \mu \right\| \leq r\right\}$. If we denote this statistical cluster point by ν then we have $\left\| \mu - \nu \right\| \leq r$. Therefore we have that if $\mu \in \Gamma_x^r$ then there exists a vector $\nu \in \Gamma_x$ such that $\left\| \mu - \nu \right\| \leq r$.

Theorem 2.3. *Let $r > 0$. For any sequence $x = (x_i)$, we have $x_* \in \Gamma_x^r$ if and only if there exists a sequence $y = (y_i)$ such that $x_* \in \Gamma_y$ and $\left\| x_i - y_i \right\| \leq r$ for a.a.i.*

Proof. Necessity. Fix r and ε. Assume that $x_* \in \Gamma_x^r$. Hence we have $\delta(K) \neq 0$, where $K := \{i \in \mathbb{N} : \|x_i - x_*\| < r + \varepsilon\}$. Define

$$y_i := \begin{cases} x_* & , \|x_i - x_*\| \leq r \text{ and } i \in K \\ x_i + r\frac{x_* - x_i}{\|x_i - x_*\|} & , \|x_i - x_*\| > r \text{ and } i \in K \\ z_i & , i \notin K \end{cases}, \tag{3}$$

where the sequence $z = (z_i)$ is arbitrary. It is clear that

$$\left\|y_i - x_*\right\| = \begin{cases} 0 & , \text{ if } \|x_i - x_*\| \le r \\ \|x_i - x_*\| - r & , \text{ otherwise} \end{cases} \tag{4}$$

and

$$\left\|x_i - y_i\right\| \le r$$

for every $i \in K$. Now let us show that the inclusion

$$K \subseteq \left\{i \in \mathbb{N} : \left\|y_i - x_*\right\| < \varepsilon\right\} \tag{5}$$

holds. If $i_0 \in K$, then we have $\left\|x_{i_0} - x_*\right\| < r + \varepsilon$. Hence the following two cases are possible:

(i) If $\left\|x_{i_0} - x_*\right\| \le r$, then from (4), we get $\left\|y_{i_0} - x_*\right\| = 0$, i.e., $i_0 \in \left\{i \in \mathbb{N} : \left\|y_i - x_*\right\| < \varepsilon\right\}$.

(ii) If $\left\|x_{i_0} - x_*\right\| > r$, then from (4), we get $\left\|y_{i_0} - x_*\right\| = \left\|x_{i_0} - x_*\right\| - r < r + \varepsilon - r = \varepsilon$, i.e., $i_0 \in \left\{i \in \mathbb{N} : \left\|y_i - x_*\right\| < \varepsilon\right\}$.

Since $\delta(K) \neq 0$, by the inclusion relation (5), we have $\delta\left(\left\{i \in \mathbb{N} : \left\|y_i - x_*\right\| < \varepsilon\right\}\right) \neq 0$.

Sufficiency. Suppose that $x_* \in \Gamma_y$ and fix $\varepsilon > 0$. Then we have $\delta\left(\left\{i \in \mathbb{N} : \left\|y_i - x_*\right\| < \varepsilon\right\}\right) \neq 0$. Take $j \in \left\{i \in \mathbb{N} : \left\|y_i - x_*\right\| < \varepsilon\right\}$. We can write

$$\begin{aligned} \left\|x_j - x_*\right\| &\le \left\|x_j - y_j\right\| + \left\|y_j - x_*\right\| \\ &< r + \varepsilon. \end{aligned}$$

Therefore we get $j \in \{i \in \mathbb{N} : \|x_i - x_*\| < r + \varepsilon\}$, which shows that the inclusion

$$\left\{i \in \mathbb{N} : \left\|y_i - x_*\right\| < \varepsilon\right\} \subseteq \{i \in \mathbb{N} : \|x_i - x_*\| < r + \varepsilon\}$$

holds. From this inclusion, we have $\delta(\{i \in \mathbb{N} : \|x_i - x_*\| < r + \varepsilon\}) \neq 0$. □

The following theorem presents a simple way to find the set Γ_x^r.

Theorem 2.4.

$$\Gamma_x^r = \bigcup_{c \in \Gamma_x} \overline{B}_r(c), \tag{6}$$

where $\overline{B}_r(c) := \left\{y \in X : \left\|y - c\right\| \le r\right\}$.

Proof. Let $\mu \in \bigcup_{c \in \Gamma_x} \overline{B}_r(c)$. Then there exists a vector $c \in \Gamma_x$ such that $\mu \in \overline{B}_r(c)$, i.e., $\left\|c - \mu\right\| \le r$. Fix $\varepsilon > 0$. Since $c \in \Gamma_x$, there exists a set $K = K(\varepsilon) := \{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\}$ with $\delta(K) \neq 0$. We have

$$\begin{aligned} \left\|x_i - \mu\right\| &\le \|x_i - c\| + \left\|c - \mu\right\| \\ &< \varepsilon + r \end{aligned}$$

for every $i \in K$. Therefore we get $\delta\left(\left\{i \in \mathbb{N} : \left\|x_i - \mu\right\| < \varepsilon + r\right\}\right) \neq 0$, which completes the first part of the proof.

For the converse inclusion, take $\mu \in \Gamma_x^r$. Then we have

$$\delta\left(\left\{i \in \mathbb{N} : \left\|x_i - \mu\right\| < \varepsilon + r\right\}\right) \neq 0 \tag{7}$$

for every $\varepsilon > 0$. Let us show that $\mu \in \bigcup_{c\in\Gamma_x} \overline{B}_r(c)$. Suppose that this is not satisfied. Then we get $\mu \notin \overline{B}_r(c)$, i.e., $\left\|\mu - c\right\| > r$ for every $c \in \Gamma_x$. Since the set Γ_x is closed, there exists a vector $\widetilde{c} \in \Gamma_x$ such that $\left\|\mu - \widetilde{c}\right\| = \min\left\{\left\|\mu - c\right\| : c \in \Gamma_x\right\}$. We can write $t := \left\|\mu - \widetilde{c}\right\| > r$, because $\left\|\mu - c\right\| > r$ for all $c \in \Gamma_x$. Define $\widetilde{\varepsilon} := \frac{t-r}{3}$. Then we get

$$X \backslash B_{\widetilde{\varepsilon}}(\Gamma_x) \supseteq \left\{y \in X : \left\|\mu - y\right\| < \widetilde{\varepsilon} + r\right\}, \tag{8}$$

where $B_{\widetilde{\varepsilon}}(\Gamma_x) = \left\{y \in X : \min\left\{\left\|y - c\right\| : c \in \Gamma_x\right\} < \widetilde{\varepsilon}\right\}$. By definition of Γ_x we can say that the set $\{i : x_i \notin B_{\widetilde{\varepsilon}}(\Gamma_x)\}$ has density zero. Then by the inclusion (8) , we have

$$\{i : x_i \notin B_{\widetilde{\varepsilon}}(\Gamma_x)\} \supseteq \left\{i : \left\|x_i - \mu\right\| < \widetilde{\varepsilon} + r\right\}. \tag{9}$$

Therefore, from the inclusion (9) , we have that the set $\left\{i : \left\|x_i - \mu\right\| < \widetilde{\varepsilon} + r\right\}$ has natural density zero, which contradicts to (7). □

Now we present an ordinary statistical convergence criterion associated with the set Γ_x^r.

Theorem 2.5. *The sequence $x = (x_i)$ is statistically convergent if and only if Γ_x^r =st-LIM$^r x$.*

Proof. Necessity. Suppose that the sequence x converges statistically to x_*. Then we have $\Gamma_x = \{x_*\}$. By Theorem 2.4, we can write $\Gamma_x^r = \overline{B}_r(x_*)$. Therefore, from [1, Theorem 2.10], we get $\Gamma_x^r = \overline{B}_r(x_*)$ =st-LIM$^r x$.

Sufficiency. By Theorem 2.4 and [1, Theorem 2.12 (b)], we have

$$\bigcup_{c\in\Gamma_x} \overline{B}_r(c) = \bigcap_{c\in\Gamma_x} \overline{B}_r(c). \tag{10}$$

The equality (10) is valid if, and only if, either the set Γ_x is empty or it is a singleton. Since st-LIM$^r x = \bigcap_{c\in\Gamma_x} \overline{B}_r(c) = \overline{B}_r(x_*)$ (see [1, Theorem 2.10]), we have $st - \lim x_i = x_*$. □

We note that in Theorem 2.5, the sequence $x = (x_i)$ need not be statistically convergent in order that the inclusion $\Gamma_x^r \subseteq$st-LIM$^r x$ holds, but this sequence must be statistically convergent in order that the converse inclusion holds.

3. Rough Statistical Limit Points

Definition 3.1. *Let $r \geq 0$. The vector $\nu \in X$ is called the r-statistical limit point of the sequence $x = (x_i)$, provided that there is a nonthin subsequence (x_{i_k}) of (x_i) such that for every $\varepsilon > 0$ there exists a number $k_0 = k_0(\varepsilon) \in \mathbb{N}$ with*

$$\left\|x_{i_k} - \nu\right\| < r + \varepsilon$$

for all $k \geq k_0$. We denote the set of all r-statistical limit points the sequence x by Λ_x^r.

Note that if we take $r = 0$, then we obtain the notion of ordinary statistical limit point defined by Fridy [9]. Now we present a result which characterizes the set Λ_x^r. The proof is immediate by definitions.

Proposition 3.2. *We have $\nu \in \Lambda_x^r$ if and only if there exists a nonthin subsequence (x_{i_k}) of (x_i) such that $\limsup_{k\to\infty} \left\|x_{i_k} - \nu\right\| \leq r$.*

Example 3.3. *Let us consider the uniformly distributed sequence (see [12]) $x = (x_i)$ defined by*

$$x = \left(0, 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, ...\right).$$

We know that $L_x = [0,1]$, $\Gamma_x = [0,1]$ and $\Lambda_x = \Lambda_x^0 = \emptyset$ (see [9, Example 4]), where L_x denotes the set of all ordinary limit points of the sequence x. The density of the indices of (x_i) on any subinterval of length d is d itself.

Let $r = 1$. By Theorem 2.4, we have $\Gamma_x^1 = \bigcup_{c\in\Gamma_x} \overline{B}_1(c) = [-1, 2]$.

Now we show that $1 \in \Lambda_x^1$ and $2 \notin \Lambda_x^1$.

Let (x_{i_k}) be a subsequence of (x_i) such that $x_{i_k} \in \left[\frac{1}{2}, 1\right]$ for every $k \in \mathbb{N}$. The density of the indices of this subsequence is $\frac{1}{2}$, i.e., (x_{i_k}) is a nonthin subsequence. Take $k_0 = 1$. Then we have $\left|x_{i_k} - 1\right| < 1 + \varepsilon$ for all $k \geq k_0$. Hence we get $1 \in \Lambda_x^1$. Using the same subsequence we obtain $a \in \Lambda_x^1$ for every $a \in (1, 2)$. Similarly, we get $a \in \Lambda_x^1$ for every $a \in (-1, 2)$.

Let us show that $2 \notin \Lambda_x^1$. If (x_{i_k}) is an arbitrary subsequence of (x_i) that $1-$converges to the number 2, we show that this subsequence is thin. Let $\varepsilon > 0$. We have

$$\begin{aligned}
\{k \in \mathbb{N} : x_{i_k}\} &= \left\{k \in \mathbb{N} : \left|x_{i_k} - 2\right| < 1 + \varepsilon\right\} \cup \left\{k \in \mathbb{N} : \left|x_{i_k} - 2\right| \geq 1 + \varepsilon\right\} \\
\delta\left(\{k \in \mathbb{N} : x_{i_k}\}\right) &= \delta\left(\left\{k \in \mathbb{N} : \left|x_{i_k} - 2\right| < 1 + \varepsilon\right\} \cup \left\{k \in \mathbb{N} : \left|x_{i_k} - 2\right| \geq 1 + \varepsilon\right\}\right) \\
&\leq \delta\left(\left\{k \in \mathbb{N} : \left|x_{i_k} - 2\right| < 1 + \varepsilon\right\}\right) + \delta\left(\left\{k \in \mathbb{N} : \left|x_{i_k} - 2\right| \geq 1 + \varepsilon\right\}\right).
\end{aligned}$$

Since the subsequence (x_{i_k}) is $1-$convergent to 2, we get $\delta\left(\left\{k \in \mathbb{N} : \left|x_{i_k} - 2\right| \geq 1 + \varepsilon\right\}\right) = 0$. Since the density of x_i 's in any interval of length l is l itself, the density of x_{i_k}'s in any interval is less than l. Hence we have $\delta\left(\left\{k \in \mathbb{N} : \left|x_{i_k} - 2\right| < 1 + \varepsilon\right\}\right) \leq \varepsilon$. Therefore we get $\delta\left(\{k \in \mathbb{N} : x_{i_k}\}\right) \leq \varepsilon$. Since ε is arbitrary, we get $\delta\left(\{k \in \mathbb{N} : x_{i_k}\}\right) = 0$ which shows that the subsequence (x_{i_k}) is thin. Hence the proof is complete since the subsequence (x_{i_k}) is arbitrary.

Similar calculations can be made for the number -1.

As a direct consequence of this example we have the following:

1. Theorem 2.4 is not valid if we replace the set Γ_x^r with Λ_x^r.
2. We know that the set Λ_x may not be closed (see [9, Proposition 1]). Also the set Λ_x^r may not be closed for $r > 0$.
3. Let $x = (x_i)$ be a sequence of real numbers. We know that Λ_x may be empty although the sequence x is bounded (see Example 3.3). Here we claim that if the sequence x has a bounded nonthin subsequence, then there exists an $r \geq 0$ such that the set Λ_x^r is nonempty. The proof of this claim is straightforward if we replace the role of the sequence in [16, Proposition 2.2] with its subsequence.

Theorem 3.4. *For a sequence $x = (x_i)$, we have $\Lambda_x^r \subseteq \Gamma_x^r$.*

The proof of the theorem above is similar to that of [9, Proposition 1]. As can be seen in Example 3.3, the inclusion in Theorem 3.4 may be strict.

4. Applications to Sequences of Functions

Throughout this section, A will denote a subset of $\mathbb{R}$ and $f = (f_i)$ will denote a sequence of real functions defined on A.

Definition 4.1. *Let $r \geq 0$. A function μ is called an r-statistical cluster point of the sequence $f = (f_i)$ provided that*

$$\delta(\{i \in \mathbb{N} : \left|f_i(x) - \mu(x)\right| < r + \varepsilon\}) \neq 0$$

for each $x \in A$ and for every $\varepsilon > 0$. Note that the set $\{i \in \mathbb{N} : \left|f_i(x) - \mu(x)\right| < r + \varepsilon\}$ depends on x and ε. We denote the set of all statistical cluster points and all r-statistical cluster points of the sequence $f = (f_i)$ by Γ_f and Γ_f^r, respectively.

Definition 4.2. *Let $r \geq 0$. A function ν is called an r-statistical limit point of the sequence $f = (f_i)$ provided that there is a nonthin subsequence of (f_{i_k}) of (f_i) such that for every $\varepsilon > 0$ and for each $x \in A$ there exists a number $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ with*

$$\left| f_{i_k}(x) - \nu(x) \right| < r + \varepsilon$$

for all $k \geq k_0$. We denote the set of all statistical limit points and all r-statistical limit points of the sequence $f = (f_i)$ by Λ_f and Λ_f^r, respectively.

We note that the definitions given above are pointwise convergence versions of Definitions 2.1 and 3.1. Now we define a new concept of statistical limit point, which is called the *statistical condensation point*. In the next subsection, we obtain the relations between the definitions given in this section. Classical (non-statistical) versions of Definitions 4.3 and 4.4 are due to Hančl et al. [11].

Definition 4.3. *Let $r > 0$. A function ν is called an r-statistical condensation point of the sequence $f = (f_i)$ if for every $x \in A$ there is a nonthin index set $I(x)$ such that*

$$B_r\left(x, \nu(x)\right) \cap G\left(f_i\right) \neq \varnothing$$

for every $i \in I(x)$, where the set $B_r\left(x, \nu(x)\right)$ is an open disk with radius r and center $\left(x, \nu(x)\right)$, and $G\left(f_i\right)$ is the set of graph f_i for every $i \in \mathbb{N}$. We denote the set of all r-statistical condensation points of the sequence $f = (f_i)$ by Ψ_f^r.

Definition 4.4. *A function μ is said to be a statistical condensation point of the sequence $f = (f_i)$ if it is an r-statistical condensation point of f for every $r > 0$. We denote the set of all statistical condensation points of the sequence $f = (f_i)$ by Φ_f.*

4.1. Relations between the sets Γ_f^r, Λ_f^r, Ψ_f^r, Γ_f, Λ_f and Φ_f

First we recall that a sequence $f = (f_i)$ converges statistically pointwise to a function μ provided that

$$\delta\left(\left\{i \in \mathbb{N} : \left|f_i(x) - \mu(x)\right| \geq \varepsilon\right\}\right) = 0$$

for every $\varepsilon > 0$ and every $x \in A$. In this case we write $st - \lim f_i = \mu$ on A [5]. The next example shows that there are no inclusion relations between the sets Ψ_f^r and Γ_f^r, and between the sets Ψ_f^r and Λ_f^r for a sequence f.

Example 4.5. *Define*

$$f_i(x) := \begin{cases} \begin{array}{ll} x^{2i} & , x \in [-1,1] \\ 0 & , x \in (-\infty,-1) \cup (1,\infty) \end{array} & , \ i \text{ is a nonsquare} \\ x^{2i} + 1 & , \ i \text{ is a square} \end{cases}.$$

Then we have $st - \lim f_i = \begin{cases} 1 & , x \in \{-1,1\} \\ 0 & , \text{otherwise} \end{cases}$, i.e., this function is the statistical pointwise limit of the sequence $f = (f_i)$.

Take $r = \frac{1}{2}$. Then we have

$$\Lambda_f^{\frac{1}{2}} = \Gamma_f^{\frac{1}{2}} = \left\{ \begin{array}{c} \mu : \mu(k, t(x)) = \begin{cases} k & , x = 1, -1 \\ t(x) & , \text{otherwise} \end{cases}, \text{ where } k \in \left[\frac{1}{2}, \frac{3}{2}\right] \\ \text{and } t \text{ is any function from } \mathbb{R} \text{ to } \left[-\frac{1}{2}, \frac{1}{2}\right] \end{array} \right\}$$

and

$$\Psi_f^{\frac{1}{2}} = \left\{ \begin{array}{c} \mu : \mu(k, t(x)) = \begin{cases} k & , x = 1, -1 \\ t(x) & , \text{otherwise} \end{cases}, \text{ where } k \in \left(-\frac{1}{2}, \frac{3}{2}\right) \\ \text{and } t \text{ is any function from } \mathbb{R} \text{ to } \left(-\frac{1}{2}, \frac{1}{2}\right] \end{array} \right\}.$$

Let $\mu(x) := \begin{cases} \frac{1}{2} & , x = 1, -1 \\ -\frac{1}{2} & , otherwise \end{cases}$. *Then we get* $\mu \in \Lambda_f^{\frac{1}{2}} = \Gamma_f^{\frac{1}{2}}$ *but* $\mu \notin \Psi_f^{\frac{1}{2}}$.

Let $v(x) := \begin{cases} 0 & , x = 1, -1 \\ \frac{1}{2} & , otherwise \end{cases}$. *Then we have* $v \in \Psi_f^{\frac{1}{2}}$ *but* $v \notin \Lambda_f^{\frac{1}{2}} = \Gamma_f^{\frac{1}{2}}$. *Consequently, neither the inclusion* $\Psi_f^r \subseteq \Gamma_f^r$ *nor the inclusion* $\Gamma_f^r \subseteq \Psi_f^r$ *holds.*

On the other hand, we obtain

$$\Lambda_f^0 = \Gamma_f^0 = \left\{ h(x) = \begin{cases} 1 & , x = 1, -1 \\ 0 & , otherwise \end{cases} \right\}$$

and

$$\Phi_f = \left\{ \mu : \mu(k, x) = \begin{cases} k & , x = 1, -1 \\ 0 & , otherwise \end{cases}, \text{ where } k = 1 \text{ or } k = 0 \right\}.$$

Hence we have $\Gamma_f \subset \Phi_f$ *and* $\Lambda_f \subset \Phi_f$ *for this sequence.*

Theorem 4.6. *We have* $\Gamma_f \subseteq \Phi_f$ *for the sequence* $f = (f_i)$.

Proof. Let $\mu \in \Gamma_f$. Fix $x \in A$ and $\varepsilon > 0$. Then we have $\delta(K) \neq 0$ where $K = K(\varepsilon, x) := \{i \in \mathbb{N} : |f_i(x) - \mu(x)| < \varepsilon\}$. Let $r := \varepsilon$ and take $(x, f_i(x)) \in G(f_i)$. Since $|f_i(x) - \mu(x)| < r$ for $i \in K$, we have $(x, f_i(x)) \in B_r(x, \mu(x))$ for every $i \in K$, which shows that

$$B_r(x, \mu(x)) \cap G(f_i) \neq \emptyset, \tag{11}$$

that is, $\mu \in \Phi_f$. □

Since we have $\Lambda_f \subseteq \Gamma_f$, we get $\Lambda_f \subseteq \Phi_f$ by Theorem 4.6.

The origin of the rest of this study is due to Hančl et al. [11]. In the paper [11], the following results for sequences of fuzzy numbers are obtained. Here we will show that these results are also valid for any sequence of functions.

It is easy to see that Φ_f contains all functions being statistical pointwise limits of all subsequences of $f = (f_i)$. The following example shows that the opposite of this claim is not true in general.

Example 4.7. *Let us define*

$$f_i(x) := \begin{cases} \begin{cases} 2 & , x \in (3, 4) \\ 0 & , otherwise \end{cases} & , i \text{ is a square} \\ g_1(x) & , i \text{ is a nonsquare} \end{cases},$$

where $g_1(x) := \begin{cases} 2 & , x \in (0, 2) \\ 0 & , otherwise \end{cases}$. *Then we have* $\Phi_f = \{g_1, g_2, g_3, g_4\}$, *where* $g_2(x) := \begin{cases} 2 & , x \in [0, 2) \\ 0 & , otherwise \end{cases}$, $g_3(x) := \begin{cases} 2 & , x \in (0, 2] \\ 0 & , otherwise \end{cases}$ *and* $g_4(x) := \begin{cases} 2 & , x \in [0, 2] \\ 0 & , otherwise \end{cases}$. *Note that the sequence* f *is statistically pointwise convergent to the functions* g_1, g_2, g_3 *and* g_4, *if we omit the set of discontinuity points.*

As we noted above, none of the elements g_2, g_3, g_4 *in* Φ_f *is a statistical pointwise limit of a subsequence of* f. *In fact, they are not the elements of the sets* Γ_f *and* Λ_f. *It is clear that* $\Gamma_f = \Lambda_f = \{g_1\}$.

By making small changes in the example above, we may observe that even a constant sequence of functions may have more than one statistical condensation point.

Example 4.8. *Define the sequence* $f_i := g_1$ *for all* $i \in \mathbb{N}$. *Then we have* $\Phi_f = \{g_1, g_2, g_3, g_4\}$.

Although the functions g_1, g_2, g_3 and g_4 defined above are different, if we omit the points of discontinuity, they are identical. Hence we reach the following equivalence relation. The origin of this definition can be found in [11, Definition 3.1].

Definition 4.9. *For two functions μ and ν, define $\mu \sim \nu$ if they have the same sets of continuity points and*

$$\lim_{x\to x_0^+} \mu(x) = \lim_{x\to x_0^+} \nu(x) \text{ and } \lim_{x\to x_0^-} \mu(x) = \lim_{x\to x_0^-} \nu(x) \tag{12}$$

hold for every $x_0 \in A$.

It is easy to see that the relation $\sim$ is an equivalence relation. Note that, if the relation $\sim$ is defined only by the condition (12), then the functions

$$h_1(x) := \begin{cases} 2 & , x = 1 \\ 0 & , \text{otherwise} \end{cases} \quad \text{and} \quad h_2(x) := \begin{cases} 3 & , x = 5 \\ 0 & , \text{otherwise} \end{cases}$$

become equivalent with respect to this relation, and therefore the equivalence classes become very large. For instance, the functions having only one point of discontinuity become equivalent.

Theorem 4.10. *$\mu \sim \nu$ if and only if they have the same sets of continuity points C, and μ and ν agree on some dense set $D \subset C$, i.e., there exists a dense set $D \subset C$ such that $\mu|_D = \nu|_D$.*

The proof of this theorem is the same as in [11, Lemma 3.1].

Now we give an example which shows that there is a sequence $f = (f_i)$ which is statistically pointwise convergent to μ at each continuity point of μ such that $\mu \notin \Phi_f$.

Example 4.11. *Define $f_i(x) := 0$, and $\mu(x) := \begin{cases} 1 & , \text{if } x = 0 \\ 0 & , \text{otherwise} \end{cases}$. Then the sequence f is statistically pointwise convergent to μ except at $x = 0$. On the other hand, take $r = \frac{1}{2}$. Then we have $B_r(0, \mu(0)) \cap G(f_i) = \varnothing$ for every $i \in \mathbb{N}$. Since $\delta(\{i \in \mathbb{N} : B_r(0, \mu(0)) \cap G(f_i) = \varnothing\}) = 1$, we get $\mu \notin \Psi_f^{\frac{1}{2}}$, i.e., $\mu \notin \Phi_f$.*

Similarly, the fact $\mu \in \Phi_f$ does not guarantee that the sequence $f = (f_i)$ is statistically pointwise convergent to μ at each continuity point of μ.

Example 4.12. *Define*

$$f_i(x) := \begin{cases} \mu(x) & , i \text{ is an odd number} \\ \left.\begin{matrix} i & , x \in [1, 2] \\ 0 & , \text{otherwise} \end{matrix}\right\} & , \text{otherwise} \end{cases},$$

where $\mu(x) := \begin{cases} 1 & , x \in [1, 2] \\ 0 & , \text{otherwise} \end{cases}$. Since $\delta(\{i \in \mathbb{N} : B_r(x, \mu(x)) \cap G(f_i) \neq \varnothing\}) \neq 0$ for every $x \in \mathbb{R}$ and every $r > 0$, we have $\mu \in \Psi_f^r$ for every $r > 0$, i.e., we get $\mu \in \Phi_f$. But there are so many continuity points of μ such that the sequence f is not statistically pointwise convergent to μ at those points. For example, the function μ is continuous at the point $\frac{3}{2}$, but we have

$$\delta\left(\left\{i \in \mathbb{N} : \left|f_i\left(\frac{3}{2}\right) - \mu\left(\frac{3}{2}\right)\right| \geq \varepsilon\right\}\right) = \delta(\{2i : i \in \mathbb{N}\}) \neq 0,$$

for every $\varepsilon \in (0, 1)$, that is, the sequence f is not statistically convergent to μ at the point $\frac{3}{2}$.

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