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Approximation by Generalized Positive Linear Kantorovich Operators

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Abstract. In the present article, we introduce generalized positive linear-Kantorovich operators depending on Pólya-Eggenberger distribution (PED) as well as inverse Pólya-Eggenberger distribution (IPED) and for these operators, we study some approximation properties like local approximation theorem, weighted approximation and estimation of rate of convergence for absolutely continuous functions having derivatives of bounded variation.

1. Introduction

Deo et al. [3] introduced the following new type generalization of positive linear operators based on Pólya-Eggenberger distribution (PED) as well as inverse Pólya-Eggenberger distribution (IPED)

$$M_n^{(\alpha)}(f;x) = \sum_k w_{n,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{n}\right), \quad x \in I, \quad n = 1, 2, ..., \tag{1}$$

where

$$w_{n,k}^{(\alpha)}(x) = \frac{n+p}{n+p+(\lambda+1)k} \binom{n+p+(\lambda+1)k}{k} \frac{\Phi_k^{\alpha}(x)\Phi_{n+p+\lambda k}^{\alpha}(1+\lambda x)}{\Phi_{n+p+(\lambda+1)k}^{\alpha}(1+(\lambda+1)x)}$$

and

$$\Phi_k^{\alpha}(x) = \prod_{i=0}^{k-1}(x+i\alpha),$$

with $0 \leqslant \alpha < 1$(may depend only on natural number n); k, p are nonnegative integers. In [16] Razi introduced a Kantorovich form of Stancu operators based on PED and obtained Voronvskaya formula as well as degree of approximation. Deo et al. [5] also established the asymptotic formula and other approximation results for Kantorovich variant of Stancu operators associated to IPED. Work of some other researchers in this direction can be seen in ([1], [4], [10], [11], [13], [14], [17], [18]).

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Now we consider a Kantorovich variant of operators (1) as:
Let f be a real valued continuous and bounded function on $[0,\infty)$ and $\lambda\in\{-1,0\}$, we define

$$\begin{aligned} V_n^{(\alpha)}(f;x) &= (n+p-\lambda)\sum_k \frac{n+p}{n+p+(\lambda+1)k}\begin{pmatrix} n+p+(\lambda+1)k \\ k\end{pmatrix} \\ &\quad\times\frac{\Phi_k^\alpha(x)\Phi_{n+p+\lambda k}^\alpha(1+\lambda x)}{\Phi_{n+p+(\lambda+1)k}^\alpha(1+(\lambda+1)x)}\int_{\frac{k}{n+p-\lambda}}^{\frac{k+1}{n+p-\lambda}} f(t)dt \\ &= (n+p-\lambda)\sum_k w_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+p-\lambda}}^{\frac{k+1}{n+p-\lambda}} f(t)\,dt,\quad x\in I, \end{aligned}\tag{2}$$

where

$$I=\begin{cases}[0,\infty),\ \lambda=0\\ [0,1],\ \lambda=-1.\end{cases}$$

The special cases of operators (1) have already been discussed in [3]. Likewise, special cases can be obtained for the operators (2). The purpose of this paper is to obtain approximation behaviour for operators (2) which includes local approximation theorem and weighted approximation. We also discuss rate of convergence for absolutely continuous functions having derivatives of bounded variation.

2. Basic Results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We recall that the monomials $e_k(x)=x^k$, for $k\in\mathbb{N}_0$ also called test functions, play an important role in uniform approximation by linear positive operators. Now we present the computation of the images of test functions by proposed operators (2).

Lemma 2.1. *[3] The generalized positive linear operators* (1) *satisfy*

$$M_n^{(\alpha)}(e_0(t);x)=1,\quad M_n^{(\alpha)}(e_1(t);x)=\left(\frac{n+p}{n}\right)\frac{x}{(1-(\lambda+1)\alpha)},$$

$$\begin{aligned} M_n^{(\alpha)}(e_2(t);x) &= \left(\frac{n+p}{n^2}\right)\frac{1}{(1-\lambda\alpha)(1-(\lambda+1)\alpha)} \\ &\quad\times\left[\frac{(n+p+\lambda+1)x(x+\alpha)}{1-2(\lambda+1)\alpha}+x(1+\lambda x)\right], \end{aligned}$$

$$\begin{aligned} &M_n^{(\alpha)}(e_3(t);x) \\ &= \frac{(n+p)x}{n^3(1-(\lambda+1)\alpha)}\left[\frac{(n+p+2\lambda+1)(n+p+2(2\lambda+1))(x+\alpha)(x+2\alpha)}{(1-(3\lambda+2)\alpha)(1-(5\lambda+3)\alpha)}\right. \\ &\quad\left.+\frac{3(n+p+2\lambda+1)(x+\alpha)}{(1-(3\lambda+2)\alpha)}+1\right], \end{aligned}$$

$$\begin{aligned} &M_n^{(\alpha)}(e_4(t);x) = \frac{(n+p)x}{n^4(1-(\lambda+1)\alpha)} \\ &\times\left[\frac{(n+p+2\lambda+1)(n+p+2(2\lambda+1))(n+p+3(2\lambda+1))(x+\alpha)(x+2\alpha)(x+3\alpha)}{(1-(3\lambda+2)\alpha)(1-(5\lambda+3)\alpha)(1-(7\lambda+4)\alpha)}\right. \\ &\quad+\frac{6(n+p+2\lambda+1)(n+p+2(2\lambda+1))(x+\alpha)(x+2\alpha)}{(1-(3\lambda+2)\alpha)(1-(5\lambda+3)\alpha)} \\ &\quad\left.+\frac{7(n+p+2\lambda+1)(x+\alpha)}{(1-(3\lambda+2)\alpha)}+1\right]. \end{aligned}$$

Lemma 2.2. *For Kantorovitch operators* (2) *hold*

$$V_n^{(\alpha)}\left(e_0(t);x\right)=1, \tag{3}$$

$$V_n^{(\alpha)}\left(e_1(t);x\right)=\frac{x}{1-(\lambda+1)\,\alpha}+\frac{1+2\lambda x}{2\,(n+p-\lambda)}, \tag{4}$$

$$\begin{aligned}V_n^{(\alpha)}\left(e_2(t);x\right)=&\frac{1}{(n+p-\lambda)^2}\left[\frac{1}{3}+\frac{(n+p)\,x}{1-(\lambda+1)\,\alpha}\right.\\&\left.+\frac{(n+p)}{(1-\lambda\alpha)\,(1-(\lambda+1)\,\alpha)}\left\{\frac{(n+p+\lambda+1)\,x\,(x+\alpha)}{1-2\alpha\,(\lambda+1)}+x\,(1+\lambda x)\right\}\right],\end{aligned} \tag{5}$$

$$\begin{aligned}V_n^{(\alpha)}\left(e_3(t);x\right)=&\frac{1}{(n+p-\lambda)^3}\left[\frac{1}{4}+\frac{2(n+p)x}{(1-(\lambda+1)\alpha)}\right.\\&+\frac{3(n+p)}{2\,(1-(\lambda+1)\alpha)}\left\{\frac{(n+p+\lambda+1)}{(1-\alpha\lambda)\,(1-2(\lambda+1)\alpha)}+\frac{2(n+p+2\lambda+1)}{(1-(3\lambda+2)\alpha)}\right\}x(x+\alpha)\\&+\frac{3(n+p)}{2(1-\lambda\alpha)\,(1-(\lambda+1)\alpha)}x(1+\lambda x)\\&\left.+\frac{(n+p)(n+p+2\lambda+1)\,(n+p+2(2\lambda+1))}{(1-(\lambda+1)\alpha)\,(1-(3\lambda+2)\alpha)\,(1-(5\lambda+3)\alpha)}x(x+\alpha)(x+2\alpha)\right],\end{aligned} \tag{6}$$

and

$$\begin{aligned}V_n^{(\alpha)}\left(e_4(t);x\right)=&\frac{1}{(n+p-\lambda)^4}\left[\frac{1}{5}+\frac{4(n+p)}{(1-(\lambda+1)\alpha)}x\right.\\&+\frac{(n+p)}{(1-(\lambda+1)\alpha)}\left\{\frac{2(n+p+\lambda+1)}{(1-\lambda\alpha)\,(1-2(\lambda+1)\alpha)}+\frac{13(n+p+2\lambda+1)}{(1-(3\lambda+2)\alpha)}\right\}x(x+\alpha)\\&+\frac{2(n+p)}{(1-\lambda\alpha)\,(1-(\lambda+1)\alpha)}x(1+\lambda x)\\&+\frac{8(n+p)(n+p+2\lambda+1)\,(n+p+2(2\lambda+1))}{(1-(\lambda+1)\alpha)\,(1-(3\lambda+2)\alpha)\,(1-(5\lambda+3)\alpha)}x(x+\alpha)(x+2\alpha)\\&+\frac{(n+p)(n+p+2\lambda+1)\,(n+p+2(2\lambda+1))\,(n+p+3(2\lambda+1))}{(1-(\lambda+1)\alpha)\,(1-(3\lambda+2)\alpha)\,(1-(5\lambda+3)\alpha)\,(1-(7\lambda+4)\alpha)}\\&\left.\times x(x+\alpha)(x+2\alpha)(x+3\alpha)\right].\end{aligned} \tag{7}$$

Proof. From Lemma 2.1 and using the definition of operators $V_n^{(\alpha)}$, we obtain the required result. □

Lemma 2.3. *The generalized operators given by* (2) *satisfy*

$$V_n^{(\alpha)}(e_1-x;x)=\frac{1}{2\,(n+p-\lambda)}+\left[\frac{\alpha\,(\lambda+1)}{1-(\lambda+1)\alpha}+\frac{\lambda}{n+p-\lambda}\right]x,$$

$$
\begin{aligned}
&V_n^{(\alpha)}\left((e_1-x)^2;x\right)\\
&=\frac{1}{3(n+p-\lambda)^2}+\left[\frac{n+p}{(n+p-\lambda)^2\,(1-(\lambda+1)\alpha)}\right.\\
&\quad-\frac{1}{(n+p-\lambda)}+\frac{(n+p)\,(n+p+\lambda+1)}{(n+p-\lambda)^2\,(1-\lambda\alpha)\,(1-(\lambda+1)\alpha)\,(1-2(\lambda+1)\alpha)}\\
&\quad\left.+\frac{(n+p)}{(n+p-\lambda)^2\,(1-\lambda\alpha)\,(1-(\lambda+1)\alpha)}\right]x\\
&\quad+\left[\frac{(n+p)\,(n+p+\lambda+1)}{(n+p-\lambda)^2\,(1-\lambda\alpha)\,(1-(\lambda+1)\alpha)\,(1-2(\lambda+1)\alpha)}\right.\\
&\quad\left.\frac{\lambda\,(n+p)}{(n+p-\lambda)^2\,(1-\lambda\alpha)\,(1-(\lambda+1)\alpha)}-\frac{1+(\lambda+1)\alpha}{1-(\lambda+1)\alpha}-\frac{2\lambda}{n+p-\lambda}\right]x^2.
\end{aligned}
$$

Proof. Using Lemma 2.2, it is easy to obtain above central moments. □

Now we recall a result from [9], with the help of which we shall obtain a result for higher order central moments.

Lemma 2.4. *[9] Let V be any linear operators then*

$$V\left((e_1-x)^j;x\right)=V\left(e_j;x\right)-\sum_{i=0}^{j-1}\binom{j}{i}x^{j-i}V\left((e_1-x)^i;x\right),$$

and in the case when $V\left(e_j;x\right)=e_j\,(x)$, *for* $j=0,1$, *then we get*

$$V\left((e_1-x)^3;x\right)=V\,(e_3;x)-x^3-3xV\left((e_1-x)^2;x\right),$$

and

$$V\left((e_1-x)^4;x\right)=V\,(e_4;x)-x^4-4xV\left((e_1-x)^3;x\right)+6x^2V\left((e_1-x)^2;x\right).$$

Remark 2.5. *For sufficiently large n, we can write the following inequalities by using Lemma 2.3 and Lemma 2.4:*

1. $V_n^{(\alpha)}\left((e_1-x)^2;x\right)\le A\frac{(1+x^2)}{n}$,
2. $V_n^{(\alpha)}\left((e_1-x)^4;x\right)\le V_n^{(\alpha)}\left(e_4\,(x)\,;x\right)\le B\frac{(1+x^2)^2}{n}$,

where A and B are some positive constants.

Lemma 2.6. *There holds the following inequality for operators* (2)*,*

$$\left|V_n^{(\alpha)}\,(f;x)\right|\leqslant\left\|f\right\|.$$

Proof. From operators (2), we have

$$
\begin{aligned}
\left|V_n^{(\alpha)}\,(f;x)\right|&=\left|(n+p-\lambda)\sum_{k=0}^{\infty}w_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+p-\lambda}}^{\frac{k+1}{n+p-\lambda}}f\,(t)-f(x)dt\right|\\
&\le(n+p-\lambda)\sum_{k=0}^{\infty}w_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+p-\lambda}}^{\frac{k+1}{n+p-\lambda}}\left|f\,(t)-f(x)\right|dt\\
&\le(n+p-\lambda)\sum_{k=0}^{\infty}w_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+p-\lambda}}^{\frac{k+1}{n+p-\lambda}}\sup\left|f\,(t)-f(x)\right|dt\\
&\le\|f\|.
\end{aligned}
$$

□

3. Direct Results

3.1. Local Approximation

Let $C_B(I)$ be the space of all the real valued continuous and bounded functions f on the interval I, endowed with the norm

$$\left\|f\right\| = \sup_{x\in I}\left|f(x)\right|.$$

For the function $f \in C_B(I)$, let us consider following Peetre's $K-$functional:

$$K_2(f;\delta) := \inf_{x\in C_B^2(I)}\left\{\left\|f-g\right\| + \delta\left\|g''\right\|\right\},\ \delta > 0,$$

where $C_B^2(I) = \{g \in C_B(I) : g', g'' \in C_B(I)\}$. By DeVore and Lorentz([6], p.177. Theorem 2.4) there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leqslant C\omega_2\left(f;\sqrt{\delta}\right), \tag{8}$$

where $\omega_2\left(f;\sqrt{\delta}\right)$ is second order modulus of continuity defined by

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<h<\sqrt{\delta}}\sup_{x\in I}\left|f(x+2h) - 2f(x+h) + f(x)\right|.$$

The usual modulus of smoothness(or simply modulus of continuity) for $f \in C_B(I)$ is given by

$$\omega(f;\delta) = \sup_{0<h\leqslant\delta}\sup_{x\in I}\left|f(x+h) - f(x)\right|.$$

Theorem 3.1. *Let $f \in C_B(I)$ then for all $x \in I$, we have the following inequality:*

$$\left|V_n^{(\alpha)}(f;x) - f(x)\right| \leq C\omega_2\left(f,\frac{\sqrt{\phi_{n,\lambda}^{(\alpha)}(x)}}{2}\right) + \omega\left(f,\frac{(\lambda+1)x\alpha}{(1-(\lambda+1)\alpha)} + \frac{1+2\lambda x}{2(n+p-\lambda)}\right),$$

where C is a positive constant and

$$\phi_{n,\lambda}^{(\alpha)}(x) = V_n^{(\alpha)}\left((t-x)^2;x\right) + \left\{\frac{(\lambda+1)x\alpha}{(1-(\lambda+1)\alpha)} + \frac{1+2\lambda x}{2(n+p-\lambda)}\right\}^2.$$

Proof. Let us define the auxiliary operators:

$$\hat{V}_n^{(\alpha)}(f;x) = V_n^{(\alpha)}(f;x) + f(x) - f\left(\frac{x}{1-(\lambda+1)\alpha} + \frac{1+2\lambda x}{2(n+p-\lambda)}\right). \tag{9}$$

For these auxiliary operators and for all $x \in I$ we may observe that $\hat{V}_n^{(\alpha)}(f;x)$ are linear such that

$$\hat{V}_n^{(\alpha)}(1;x) = 1 \ \ \textit{and}\ \ \hat{V}_n^{(\alpha)}(t;x) = x,$$

i. e., preserve linear functions. Therefore

$$\hat{V}_n^{(\alpha)}(t-x;x) = 0. \tag{10}$$

Consider $g \in C_B^2(I)$ and for $t, x \in I$, Taylor's theorem implies

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Using linearity of operators $\hat{V}_n^{(\alpha)}$ and taking into account (9) and (10), we obtain

$$\begin{aligned}
&\hat{V}_n^{(\alpha)}(g;x) - g(x)\\
&= g'(x)\hat{V}_n^{(\alpha)}((t-x);x) + \hat{V}_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&= \hat{V}_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&= V_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&\quad - \int_x^{\frac{x}{1-(\lambda+1)\alpha}+\frac{1+2\lambda x}{2(n+p-\lambda)}}\left(\frac{x}{1-(\lambda+1)\alpha}+\frac{1+2\lambda x}{2(n+p-\lambda)}-u\right)g''(u)du.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|\hat{V}_n^{(\alpha)}(g;x) - g(x)\right|\\
&\leqslant V_n^{(\alpha)}\left(\left|\int_x^t (t-u)g''(u)du\right|;x\right)\\
&\quad + \left|\int_x^{\frac{x}{1-(\lambda+1)\alpha}+\frac{1+2\lambda x}{2(n+p-\lambda)}}\left(\frac{x}{1-(\lambda+1)\alpha}+\frac{1+2\lambda x}{2(n+p-\lambda)}-u\right)g''(u)du\right|. \qquad (11)
\end{aligned}$$

Since $\left|\int_x^t (t-u)g''(u)du\right| \leqslant (t-x)^2\left\|g''\right\|$, therefore (11) gives

$$\begin{aligned}
&\left|\hat{V}_n^{(\alpha)}(g;x) - g(x)\right|\\
&\leqslant \left[V_n^{(\alpha)}\left((t-x)^2;x\right) + \left(\frac{x}{1-(\lambda+1)\alpha}+\frac{1+2\lambda x}{2(n+p-\lambda)}-x\right)^2\right]\left\|g''\right\|\\
&\leqslant \left[V_n^{(\alpha)}\left((t-x)^2;x\right) + \left\{\frac{(\lambda+1)x\alpha}{(1-(\lambda+1)\alpha)}+\frac{1+2\lambda x}{2(n+p-\lambda)}\right\}^2\right]\left\|g''\right\|\\
&= \phi_{n,\lambda}^{(\alpha)}(x)\left\|g''\right\|. \qquad (12)
\end{aligned}$$

Again using definition of auxiliary operators and from Lemma 2.6, we get

$$\begin{aligned}
&\left|V_n^{(\alpha)}(f;x) - f(x)\right|\\
&\leqslant \left|\hat{V}_n^{(\alpha)}((f-g);x)\right| + \left|g(x)-f(x)\right| + \left|\hat{V}_n^{(\alpha)}(g;x) - g(x)\right|\\
&\quad + \left|f\left(\frac{x}{1-(\lambda+1)\alpha}+\frac{1+2\lambda x}{2(n+p-\lambda)}\right) - f(x)\right|\\
&\leqslant 4\left\|f-g\right\| + \phi_{n,\lambda}^{(\alpha)}(x)\left\|g''\right\| + \omega\left(f, \frac{(\lambda+1)x\alpha}{(1-(\lambda+1)\alpha)}+\frac{1+2\lambda x}{2(n+p-\lambda)}\right).
\end{aligned}$$

Taking infimum on both the sides over $g \in C_B^2(I)$,

$$\left|V_n^{(\alpha)}(f;x) - f(x)\right| \leqslant 4K_2\left(f; \frac{\phi_{n,\lambda}^{(\alpha)}(x)}{4}\right) + \omega\left(f, \frac{(\lambda+1)x\alpha}{(1-(\lambda+1)\alpha)}+\frac{1+2\lambda x}{2(n+p-\lambda)}\right).$$

Hence by (8), we get

$$\left|V_n^{(\alpha)}(f;x) - f(x)\right| \leqslant C\omega_2\left(f; \frac{\sqrt{\phi_{n,\lambda}^{(\alpha)}(x)}}{2}\right) + \omega\left(f, \frac{(\lambda+1)x\alpha}{(1-(\lambda+1)\alpha)}+\frac{1+2\lambda x}{2(n+p-\lambda)}\right).$$

This completes the proof. □

Now consider the following Lipschitz type space [15], a two parameter family:
For fixed $\beta, \gamma > 0$, we have

$$Lip_M^{\beta,\gamma}(r) := \left\{ f \in C_B(I) : \left| f(t) - f(x) \right| \leq M \frac{|t-x|^r}{(t+\beta x^2+\gamma x)^{r/2}}; x, t \in (0,\infty) \right\},$$

where M is any positive constant and $r \in (0,1]$.

Theorem 3.2. *Let $f \in Lip_M^{\beta,\gamma}(r)$ and $r \in (0,1]$ then for all $x \in (0,\infty)$, the following inequality holds:*

$$\left| V_n^{(\alpha)}(f;x) - f(x) \right| \leq M \left(\frac{\eta_n^{\alpha}(x)}{\beta x^2 + \gamma x} \right)^{r/2},$$

where

$$\eta_n^{\alpha}(x) = V_n^{(\alpha)}\left((t-x)^2; x\right).$$

Proof. First we prove the result for $r = 1$. Then for $f \in Lip_M^{\beta,\gamma}(r)$ and for $x \in (0,\infty)$, we have

$$\begin{aligned} \left| V_n^{(\alpha)}(f;x) - f(x) \right| &\leq V_n^{(\alpha)}\left(\left| f(t) - f(x) \right|; x \right) \\ &\leq M V_n^{(\alpha)}\left(\frac{|t-x|}{(t+\beta x^2+\gamma x)^{1/2}}; x \right). \end{aligned}$$

Applying Cauchy-Schwarz inequality for sum and $\frac{1}{(t+\beta x^2+\gamma x)^{1/2}} \leq \frac{1}{(\beta x^2+\gamma x)^{1/2}}$, we obtain

$$\begin{aligned} \left| V_n^{(\alpha)}(f;x) - f(x) \right| &\leqslant \frac{M}{(\beta x^2+\gamma x)^{1/2}} \left[V_n^{(\alpha)}\left((t-x)^2; x\right) \right]^{1/2} \\ &= M \left\{ \frac{\eta_n^{(\alpha)}(x)}{\beta x^2 + \gamma x} \right\}^{1/2}. \end{aligned}$$

Therefore result holds for $r = 1$.

Further we prove the result for $0 < r < 1$. Again for $f \in Lip_M^{\beta,\gamma}(r)$, using the same argument as in case for $r = 1$ and applying Holder's inequality with $p = 2/r, q = 2/2 - r$, we finally get

$$\begin{aligned} \left| V_n^{(\alpha)}(f;x) - f(x) \right| &\leqslant \frac{M}{(\beta x^2+\gamma x)^{r/2}} \left[V_n^{(\alpha)}\left((t-x)^2; x\right) \right]^{r/2} \\ &= M \left\{ \frac{\eta_n^{(\alpha)}(x)}{\beta x^2 + \gamma x} \right\}^{r/2}. \end{aligned}$$

Hence the result. □

Let $\varphi(x) = 1 + x^2$ be a weight function and consider

$$B_\varphi(I) := \left\{ f : \left| f(x) \right| \leqslant M_f \varphi(x) \right\},$$

endowed with the norm

$$\left\| f \right\|_\varphi = \sup_{x \in I} \frac{\left| f(x) \right|}{\varphi(x)},$$

Also let

$$C_\varphi(I) := \left\{ f : f \in B_\varphi(I) \text{ and } f \text{ is continuous} \right\},$$

and

$$C^*_\varphi(I) := \left\{ f \in C_\varphi(I) : \lim_{x\to\infty} \frac{\left|f(x)\right|}{\varphi(x)} < \infty \right\}.$$

The usual modulus of continuity of f on $[0,b]$ is defined as:

$$\omega_b\left(f,\delta\right) = \sup_{|t-x|\leqslant\delta} \sup_{x,t\in[0,b]} \left|f(t) - f(x)\right|.$$

Theorem 3.3. *Let $f \in C_\varphi(I)$ then for operators $V_n^{(\alpha)}\left(f;x\right)$ we have*

$$\left\|V_n^{(\alpha)}\left(f;x\right) - f(x)\right\|_{C[0,b]} \leq 4M_f\left(1+b^2\right)\eta_n^{(\alpha)}(b) + 2\omega_{b+1}\left(f, \sqrt{\eta_n^{(\alpha)}(b)}\right), \tag{13}$$

where $\eta_n^{(\alpha)}(x) = L_n^{(\alpha)}\left((t-x)^2;x\right)$.

Proof. Let $x \in [0,b]$, $t \in (b+1,\infty)$, and $t-x>1$ we have

$$\begin{aligned}
\left|f(t) - f(x)\right| &\leqslant M_f\varphi(t-x) = M_f\left\{1+(t-x)^2\right\} = M_f\left(1+t^2+x^2-2xt\right)\\
&\leqslant M_f\left(2+t^2+x^2\right) = M_f\left\{2+2x^2+(t-x)^2+2x(t-x)\right\}\\
&\leqslant M_f(t-x)^2\left\{3+2x+2x^2\right\} \leqslant 4M_f(t-x)^2\left(1+x^2\right)\\
&\leqslant 4M_f(t-x)^2\left(1+b^2\right).
\end{aligned} \tag{14}$$

If $x \in [0,b]$ and $t \in [0,b+1]$ then we have

$$\left|f(t) - f(x)\right| \leqslant \omega_{b+1}\left(|t-x|\right) \leqslant \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}\left(f,\delta\right), \delta > 0. \tag{15}$$

Combining (14) and (15), we obtain

$$\left|f(t) - f(x)\right| \leqslant 4M_f\left(1+b^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}\left(f,\delta\right).$$

Taking Cauchy Schwartz inequality we get

$$\begin{aligned}
\left|V_n^{(\alpha)}(t) - f(x)\right| &\leqslant 4M_f\left(1+b^2\right)V_n^{(\alpha)}\left((t-x)^2;x\right)\\
&\quad + \left(1+\frac{1}{\delta}V_n^{(\alpha)}\left(|t-x|\,;x\right)\right)\omega_{b+1}\left(f,\delta\right)\\
&\leqslant 4M_f\left(1+b^2\right)V_n^{(\alpha)}\left((t-x)^2;x\right)\\
&\quad + \left[1+\frac{1}{\delta}\left\{V_n^{(\alpha)}\left((t-x)^2;x\right)\right\}^{1/2}\right]\omega_{b+1}\left(f,\delta\right)\\
&\leqslant 4M_f\left(1+b^2\right)\eta_n^{(\alpha)}(b) + 2\omega\left(f, \sqrt{\eta_n^{(\alpha)}(b)}\right).
\end{aligned}$$

□

3.2. *Weighted Approximation*

Theorem 3.4. *Let $f \in C^*_\varphi(I)$ then, we have*

$$\lim_{n\to\infty} \left\| V_n^{(\alpha)}(f;x) - f(x) \right\|_\varphi = 0. \tag{16}$$

Proof. To prove equation (16), it is sufficient to verify the following three conditions (as in papers [7, 8]):

$$\lim_{n\to\infty} \left\| V_n^{(\alpha)}\left(t^i;x\right) - t^i \right\|_\varphi = 0,\ i = 0,1,2. \tag{17}$$

It is very clear that equation (17) is true for $i = 0$ as $V_n^{(\alpha)}(1;x) = 1$. Now using Lemma 2.2 we have

$$\begin{aligned}
\left\| V_n^{(\alpha)}(t;x) - x \right\|_\varphi &= \sup_{x\in I} \frac{1}{1+x^2} \left| \frac{x}{1-(\lambda+1)\alpha} + \frac{1+2\lambda x}{2(n+p-\lambda)} - x \right| \\
&= \sup_{x\in I} \frac{1}{1+x^2} \left| \frac{1}{2(n+p-\lambda)} + x\left(\frac{1}{1-(\lambda+1)\alpha} + \frac{\lambda}{n+p-\lambda} - 1 \right) \right| \\
&\leq \frac{1}{2(n+p-\lambda)} + \left| \frac{1}{1-(\lambda+1)\alpha} + \frac{\lambda}{n+p-\lambda} - 1 \right|
\end{aligned}$$

which implies that $\lim_{n\to\infty} \left\| V_n^{(\alpha)}(t;x) - x \right\|_\varphi = 0$. Similarly we have

$$\begin{aligned}
\left\| V_n^{(\alpha)}\left(t^2;x\right) - x^2 \right\|_\varphi &= \sup_{x\in I} \frac{1}{(n+p-\lambda)^2} \left| \frac{1}{3} + \frac{(n+p)x}{1-(\lambda+1)\alpha} + \frac{(n+p)}{(1-\lambda\alpha)(1-(\lambda+1)\alpha)} \right. \\
&\quad \left. \times \left\{ \frac{(n+p+\lambda+1)x(x+\alpha)}{1-2(\lambda+1)\alpha} + x(1+\lambda x) \right\} - x^2 \right| \frac{1}{(1+x^2)} \\
&\leq \frac{1}{3(n+p-\lambda)^2} + \frac{(n+p)}{(n+p-\lambda)^2(1-(\lambda+1)\alpha)} \\
&\quad + \left| \frac{(n+p)(n+p+\lambda+1)}{(1-\lambda\alpha)(1-(\lambda+1)\alpha)(1-2(\lambda+1)\alpha)(n+p-\lambda)^2} \right| \\
&\quad + \frac{(n+p)(n+p+\lambda+1)\alpha}{(1-\lambda\alpha)(1-(\lambda+1)\alpha)(n+p-\lambda)^2} \frac{1}{|1-2(\lambda+1)\alpha|} \\
&\quad + \frac{(n+p)(1+\lambda)}{(n+p-\lambda)^2(1-\lambda\alpha)(1-(\lambda+1)\alpha)}
\end{aligned}$$

Hence $\lim_{n\to\infty} \left\| V_n^{(\alpha)}\left(t^2;x\right) - x^2 \right\|_\varphi = 0$.
The proof is complete. □

Yüksel and Ispir in [19] gave the rate of convergence by using weighted modulus of continuity. Inspired by them, we are also finding the rate of convergence for our operators (2).

For each $f \in C^*_\varphi(I)$, the weighted modulus of continuity is given by:

$$\Omega(f;\delta) = \sup_{x\in I, 0<h\leqslant\delta} \frac{\left|f(x+h) - f(x)\right|}{1+(x+h)^2}.$$

Following lemma gives the properties of the weighted modulus of continuity:

Lemma 3.5. *[19] Let $f \in C^*_\varphi(I)$ then we have* *(i)* $\Omega(f;\delta)$ *is a monotone increasing function of* δ;

(ii) $\lim_{\delta\to 0+}\Omega(f;\delta)=0$;

(iii) *for each* $m\in N$, $\Omega(f;m\delta)=m\Omega(f;\delta)$;

(iv) *for each* $\mu\in I$, $\Omega(f;\mu\delta)=(1+\mu)\Omega(f;\delta)$.

Theorem 3.6. *Let* $f\in C_{\varphi}^{*}(I)$ *the operators* $V_n^{(\alpha)}(f;x)$ *satisfy*

$$\sup_{x\in I}\frac{\left|V_n^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{5/2}}\leq K\Omega(f;\delta_n),$$

where K is positive constant and $\delta_n=\frac{1}{\sqrt{n}}$

Proof. Let $x,t\in[0,\infty)$, $\delta>0$ then by definition of $\Omega(f;\delta)$ and property given by Lemma 3.5, we have

$$\begin{aligned}\left|f(t)-f(x)\right| &\leqslant \left(1+(x+|t-x|)^2\right)\Omega(f;|t-x|)\\ &\leqslant \left(1+(x+|t-x|)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta)\\ &\leqslant 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta).\end{aligned}$$

Since $V_n^{(\alpha)}(f;x)$ are positive linear operators, therefore

$$\begin{aligned}&\left|V_n^{(\alpha)}(f;x)-f(x)\right|\\ &\leqslant (n+p-\lambda)\sum_{k=0}^{\infty}w_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+p-\lambda}}^{\frac{k+1}{n+p-\lambda}}\left|f(t)-f(x)\right|dt\\ &\leq 2\Omega(f;\delta)\left(1+x^2\right)(n+p-\lambda)\sum_{k=0}^{\infty}w_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+p-\lambda}}^{\frac{k+1}{n+p-\lambda}}\left\{1+(t-x)^2\right\}\left(1+\frac{|t-x|}{\delta}\right)dt\\ &\leqslant 2\Omega(f;\delta)\left(1+x^2\right)\Bigg[1+V_n^{(\alpha)}\left((t-x)^2;x\right)\\ &\quad +V_n^{(\alpha)}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\Bigg].\end{aligned}\tag{18}$$

Using Cauchy Schwarz inequality and Remark 2.5, we obtain

$$\begin{aligned}\left|V_n^{(\alpha)}(f;x)-f(x)\right| &\leq 2\Omega(f;\delta)\left(1+x^2\right)\Big[1+V_n^{(\alpha)}\left((t-x)^2;x\right)\\ &\quad+\frac{1}{\delta}\left\{\left(V_n^{(\alpha)}\left((t-x)^2;x\right)\right)^{1/2}+\left(V_n^{(\alpha)}\left((t-x)^2;x\right)\right)^{1/2}\left(V_n^{(\alpha)}\left((t-x)^2;x\right)\right)^{1/4}\right\}\Big]\\ &\leq K\left(1+x^2\right)^{5/2}\Omega(f;\delta_n),\end{aligned}$$

where $\delta_n=\frac{1}{\sqrt{n}}$ and $K=2\left(1+A+\sqrt{A}+\sqrt{AB}\right)$. Hence the result. □

3.3. *Rate of convergence*

Consider the class $DBV_\gamma I$, $\gamma\geq 0$ of functions f defined on I such that each function f of this class has a derivative of bounded variation on every finite subinterval of I and $|f(t)|\leq M_f t^\gamma$, $\forall t>0$.

It can be noticed that for $f \in DBV_\gamma I$, we can write

$$f(x) = \int_0^x g(t)\,dt + f(0),$$

where $g(t)$ is a function of bounded variation on each finite subinterval of I.

Now, in this section we shall estimate the rate of convergence for generalized positive linear Kantorovich operators $V_n^{(\alpha)}$ for functions belonging to the class $DBV_\gamma I$. Several researchers have studied rate of convergence for different operators like ([2], [12], [20]).

Before studying the approximation of functions having a derivative of bounded variation, we need following results:
First, we write the operators $V_n^{(\alpha)}$ given by (2) in alternate form as:

$$V_n^{(\alpha)}(f;x) = \int_0^\infty K_n^{(\alpha)}(x,t)\, f(t)\,dt, \tag{19}$$

where

$$K_n^{(\alpha)}(x,t) = (n+p-\lambda)\sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x)\,\chi_{n,k}(t)$$

and $\chi_{n,k}(t)$ is the characteristic function of the interval $\left[\frac{k}{n+p-\lambda}, \frac{k+1}{n+p-\lambda}\right]$ w.r.t I.

Also put $\beta_n(x,y) = \int_0^y K_n^{(\alpha)}(x,t)dt;\; y \geq 0$.

Remark 3.7. *For sufficiently large n, there exists positive constants K_1 and K_2 such that*

(i) $V_n^{(\alpha)}\left((e_1-x)^2;x\right) \leq \frac{K_1(1+x)^2}{n+p-\lambda}$,

(ii) $V_n^{(\alpha)}\left((e_1-x);x\right) \leq \frac{K_2(1+x)}{n+p-\lambda}$,

(iii) By Cauchy Schwarz inequality,

$$V_n^{(\alpha)}(|e_1-x|;x) \leq \left[V_n^{(\alpha)}\left((e_1-x)^2;x\right)\right]^{1/2} \leq \sqrt{\frac{K_1}{n+p-\lambda}}\,(1+x).$$

Lemma 3.8. *For $x \in (0,\infty)$ and sufficiently large n, we have*

(i) Since $0 \leq y < x$, therefore

$$\beta_n(x,y) = \int_0^y K_n^{(\alpha)}(x,t)dt \leq \frac{K_1(1+x)^2}{(n+p-\lambda)(x-y)^2},$$

(ii) If $x < z < \infty$ then we get

$$1-\beta_n(x,z) = \int_z^\infty K_n^{(\alpha)}(x,t)dt \leq \frac{K_1(1+x)^2}{(n+p-\lambda)(z-x)^2}.$$

Theorem 3.9. *Let $f \in DBV_\gamma(0,\infty), \gamma > 0$ then for all $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}
&\left|V_n^{(\alpha)}(f;x) - f(x)\right| \\
&\leq \frac{K_2(1+x)}{(n+p-\lambda)}\left|\frac{f'(x+)+f'(x-)}{2}\right| \\
&\quad + \sqrt{\frac{K_1}{n+p-\lambda}}(1+x)\left|\frac{f'(x+)-f'(x-)}{2}\right| + \sqrt{\frac{K_1}{n+p-\lambda}}(1+x)f'(x+) \\
&\quad + M(r,\gamma,x) + \frac{\left|f(x)\right|}{n+p-\lambda}K_1(1+x)^2 \\
&\quad + \frac{K_1}{(n+p-\lambda)}\left(\frac{1+x}{x}\right)^2\left|f(2x)-f(x)-xf'(x+)\right| \\
&\quad + \frac{x}{\sqrt{n}}\vee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} f_x' + \frac{K_1(1+x)^2}{x(n+p-\lambda)}\sum_{k=1}^{[\sqrt{n}]}\vee_{x-\frac{x}{k}}^{x+\frac{x}{k}} f_x',
\end{aligned} \tag{20}$$

where f_x is an auxiliary operator defined by

$$f_x(t) = \begin{cases} f(t) - f(x-), & 0 \leq t < x \\ 0, & t = x \\ f(t) - f(x+), & x < t < \infty \end{cases}$$

and moreover $\vee_a^b f(x)$ denotes the total variation of f on $[a,b]$.

Proof. Since $V_n^{(\alpha)}(1;x) = 1$, for all $x \in (0,\infty)$, we obtain

$$\begin{aligned}
V_n^{(\alpha)}(f;x) - f(x) &= \int_0^\infty (f(t) - f(x))K_n^{(\alpha)}(x,t)dt \\
&= \int_0^\infty K_n^{(\alpha)}(x,t)\int_x^t f'(u)\,du dt.
\end{aligned} \tag{21}$$

For $f \in DBV_\gamma(0,\infty)$, we can write

$$\begin{aligned}
f'(u) &= \frac{1}{2}(f'(x+)+f'(x-)) + f_x{}'(u) + \frac{1}{2}(f'(x+)-f'(x-))\,\mathrm{sgn}(u-x) \\
&\quad + \delta_x(u)\left(f'(u) - \frac{1}{2}(f'(x+)+f'(x-))\right),
\end{aligned} \tag{22}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$$

It is obvious that

$$\int_0^\infty\left(\int_x^t\left(f'(u) - \frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)\,du\right)K_n^{(\alpha)}(x,t)\,dt = 0. \tag{23}$$

Using (19), we get

$$\begin{aligned}
&\int_0^\infty\left(\int_x^t \frac{1}{2}(f'(x+)+f'(x-))\,du\right)K_n^{(\alpha)}(x,t)\,dt \\
&= \frac{1}{2}(f'(x+)+f'(x-))\,V_n^{(\alpha)}((t-x);x).
\end{aligned} \tag{24}$$

Also

$$\begin{aligned}
&\int_0^\infty \left(\int_x^t \frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(u-x)\,du\right)K_n^{(\alpha)}(x,t)\,dt\\
&=\int_0^\infty \frac{1}{2}\left(f'(x+)-f'(x-)\right)(t-x)\,K_n^{(\alpha)}(x,t)\,dt\\
&\le \frac{1}{2}\left|f'(x+)-f'(x-)\right|\int_0^\infty |t-x|\,K_n^{(\alpha)}(x,t)\,dt\\
&=\frac{1}{2}\left|f'(x+)-f'(x-)\right|V_n^{(\alpha)}\left(|t-x|;x\right). \qquad (25)
\end{aligned}$$

Taking into consideration Lemma 3.8 as well as equations (21), (23), (24), and (25), we have

$$\begin{aligned}
&V_n^{(\alpha)}(f;x)-f(x)\\
&\le \frac{1}{2}\left(f'(x+)+f'(x-)\right)V_n^{(\alpha)}\left((t-x);x\right)+\frac{1}{2}\left|f'(x+)-f'(x-)\right|V_n^{(\alpha)}\left(|t-x|;x\right)\\
&\quad+\int_0^\infty\left(\int_x^t f_x'(u)\,du\right)K_n^{(\alpha)}(x,t)\,dt\\
&\le \frac{1}{2}\left(f'(x+)+f'(x-)\right)\frac{K_2(1+x)}{(n+p-\lambda)}+\frac{1}{2}\left|f'(x+)-f'(x-)\right|\sqrt{\frac{K_1}{n+p-\lambda}}(1+x)\\
&\quad+\int_0^\infty\left(\int_x^t f_x'(u)\,du\right)K_n^{(\alpha)}(x,t)\,dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|V_n^{(\alpha)}(f;x)-f(x)\right|\\
&\le \frac{K_2}{2}\left|\left(f'(x+)+f'(x-)\right)\right|\frac{(1+x)}{(n+p-\lambda)}\\
&\quad+\frac{1}{2}\left|f'(x+)-f'(x-)\right|\sqrt{\frac{K}{n+p-\lambda}}(1+x)+J_{1,n,x}+J_{2,n,x}, \qquad (26)
\end{aligned}$$

where

$$J_{1,n,x}=\left|\int_0^x\left(\int_x^t f_x'(u)\,du\right)K_n^{(\alpha)}(x,t)\,dt\right|$$

and

$$J_{2,n,x}=\int_x^\infty\left(\int_x^t f_x'(u)\,du\right)K_n^{(\alpha)}(x,t)\,dt.$$

Applying Lemma 3.8, integration by parts and take $y=x-\frac{x}{\sqrt{n}}$, we obtain

$$\begin{aligned}
J_{1,n,x}&=\left|\int_0^x\left(\int_x^t f_x'(u)\,du\right)d_t\beta_n(x,t)\right|=\left|\int_0^x \beta_n(x,t)\,f_x'(t)\,dt\right|.\\
&\le \int_0^y \left|\beta_n(x,t)\right|\left|f'_x(t)\right|dt+\int_y^x\left|\beta_n(x,t)\right|\left|f'_x(t)\right|dt
\end{aligned}$$

Since ${f_x}'(x) = 0$ and $\beta_n(x,t) \leq 1$, it implies

$$\int_{x-\frac{x}{\sqrt{n}}}^{x} \left|{f_x}'(t)\right| \beta_n(x,t)\,dt = \int_{x-\frac{x}{\sqrt{n}}}^{x} \left|{f_x}'(t) - {f_x}'(x)\right| \beta_n(x,t)\,dt$$
$$\leq \int_{x-\frac{x}{\sqrt{n}}}^{x} \vee_t^x ({f_x}')\,dt \leq \frac{x}{\sqrt{n}} \vee_{x-\frac{x}{\sqrt{n}}}^{x} ({f_x}').$$

Again applying Lemma 3.8 and put $t = x - \frac{x}{u}$,

$$\begin{aligned}
&\int_0^{x-\frac{x}{\sqrt{n}}} \left|{f_x}'(t)\right| \beta_n(x,t)\,dt \\
&\leq \frac{K_1(1+x)^2}{(n+p-\lambda)} \int_0^{x-\frac{x}{\sqrt{n}}} \frac{\left|{f_x}'(t)\right|}{(x-t)^2} dt \\
&\leq \int_0^{x-\frac{x}{\sqrt{n}}} \left|{f_x}'(t)\right| \beta_n(x,t)\,dt \leq \frac{K_1(1+x)^2}{(n+p-\lambda)} \int_0^{x-\frac{x}{\sqrt{n}}} \frac{\vee_t^x ({f_x}')}{(x-t)^2} dt \\
&\leq \frac{K_1(1+x)^2}{x(n+p-\lambda)} \int_1^{\sqrt{n}} \vee_{x-\frac{x}{u}}^{x} ({f_x}')\,du \\
&\leq \frac{K_1(1+x)^2}{x(n+p-\lambda)} \sum_{k=1}^{[\sqrt{n}]} \vee_{x-\frac{x}{k}}^{x} ({f_x}').
\end{aligned}$$

Thus,

$$J_{1,n,x} \leq \frac{x}{\sqrt{n}} \vee_{x-\frac{x}{\sqrt{n}}}^{x} ({f_x}') + \frac{K_1(1+x)^2}{x(n+p-\lambda)} \sum_{k=1}^{[\sqrt{n}]} \vee_{x-\frac{x}{k}}^{x} ({f_x}')$$

By using Lemma 3.8, we can write

$$J_{2,n,x} \leq J_{2,n,x}^{(1)} + J_{2,n,x}^{(2)}$$

where

$$J_{2,n,x}^{(1)} = \left| \int_x^{2x} \left(\int_x^t {f_x}'(u)\,du \right) d_t\left(1 - \beta_n(x,t)\right) \right|$$

and

$$J_{2,n,x}^{(2)} = \left| \int_{2x}^{\infty} \left(\int_x^t {f_x}'(u)\,du \right) K_n^{(\alpha)}(x,t)\,dt \right|.$$

Applying integration by parts as well as using Lemma 3.8, (22), $1 - \beta_n(x,t) \leq 1$ and putting $t = x + \frac{x}{u}$

successively,

$$\begin{aligned}
J^{(1)}_{2,n,x} &= \left| \int_x^{2x} f_x{}'(u)\,du\,(1-\beta_n(x,2x)) - \int_x^{2x} f_x{}'(t)\,(1-\beta_n(x,t))\,dt \right| \\
&\le \left| \int_x^{2x} (f'(u) - f'(x+))\,du \right| \left|(1-\beta_n(x,2x))\right| + \int_x^{2x} \left|f_x{}'(t)\right| \left|1-\beta_n(x,t)\right| dt \\
&\le \frac{K_1(1+x)^2}{(n+p-\lambda)\,x^2} \left| f(2x) - f(x) - xf'(x+) \right| \\
&\quad + \int_x^{x+\frac{x}{\sqrt{n}}} \left|f_x{}'(t)\right| \left|1-\beta_n(x,t)\right| dt + \int_{x+\frac{x}{\sqrt{n}}}^{2x} \left|f_x{}'(t)\right| \left|1-\beta_n(x,t)\right| dt \\
&\le \frac{K_1(1+x)^2}{(n+p-\lambda)\,x^2} \left| f(2x) - f(x) - xf'(x+) \right| \\
&\quad + \frac{K_1(1+x)^2}{(n+p-\lambda)} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\vee_x^t(f_x{}')}{(t-x)^2} dt + \int_x^{x+\frac{x}{\sqrt{n}}} \vee_x^t(f_x{}')\,dt \\
&\le \frac{K_1(1+x)^2}{(n+p-\lambda)\,x^2} \left| f(2x) - f(x) - xf'(x+) \right| \\
&\quad + \frac{K_1(1+x)^2}{(n+p-\lambda)} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\vee_x^t(f_x{}')}{(t-x)^2} dt + \frac{x}{\sqrt{n}} \vee_x^{x+\frac{x}{\sqrt{n}}}(f_x{}') \\
&\le \frac{K_1(1+x)^2}{(n+p-\lambda)\,x^2} \left| f(2x) - f(x) - xf'(x+) \right| \\
&\quad + \frac{K_1(1+x)^2}{(n+p-\lambda)\,x} \sum_{k=1}^{[\sqrt{n}]} \vee_x^{x+\frac{x}{k}}(f_x{}') + \frac{x}{\sqrt{n}} \vee_x^{x+\frac{x}{\sqrt{n}}}(f_x{}').
\end{aligned}$$

Finally, Remark 3.7 implies

$$\begin{aligned}
J^{(2)}_{2,n,x} &= \left| \int_{2x}^{\infty} \left(\int_x^t (f'(u) - f'(x+))\,du \right) K_n^{(\alpha)}(x,t)\,dt \right| \\
&\le \int_{2x}^{\infty} \left| f(t) - f(x) \right| K_n^{(\alpha)}(x,t)\,dt + \int_{2x}^{\infty} |t-x|\, f'(x+)\, K_n^{(\alpha)}(x,t)\,dt \\
&\le M \int_{2x}^{\infty} t^{\gamma} K_n^{(\alpha)}(x,t)\,dt + \left| f(x) \right| \int_{2x}^{\infty} K_n^{(\alpha)}(x,t)\,dt \\
&\quad + \sqrt{\frac{K_1}{n+p-\lambda}}\,(1+x)\,f'(x+).
\end{aligned}$$

As it is obvious that $t \le 2(t-x)$ and $x \le t-x$ when $t \ge 2x$ therefore applying Holder's inequality,

$$\begin{aligned}
J^{(2)}_{2,n,x} &\le M2^{\gamma} \left(\int_0^{\infty} (t-x)^{2r} K_n^{(\alpha)}(x,t)\,dt \right)^{\frac{\gamma}{2r}} + \frac{K_1 \left| f(x) \right|}{n+p-\lambda} (1+x)^2 \\
&\quad + \sqrt{\frac{K}{n+p-\lambda}}\,(1+x)\,f'(x+). \\
&= M(\gamma, r, x) + \frac{K_1 \left| f(x) \right|}{(n+p-\lambda)} (1+x)^2 + \sqrt{\frac{K_1}{n+p-\lambda}}\,(1+x)\,f'(x+).
\end{aligned}$$

Estimates of $J^{(1)}_{2,n,x}$ and $J^{(2)}_{2,n,x}$ gives

$$
\begin{aligned}
J_{2,n,x} \leq & \frac{K_1(1+x)^2}{(n+p-\lambda)\,x^2}\left|f(2x)-f(x)-xf'(x+)\right| \\
& + \frac{K_1(1+x)^2}{(n+p-\lambda)\,x}\sum_{k=1}^{[\sqrt{n}]}\vee_x^{x+\frac{x}{k}}\left(f_x{}'\right) + \frac{x}{\sqrt{n}}\vee_x^{x+\frac{x}{\sqrt{n}}}\left(f_x{}'\right) \\
& + M(\gamma,r,x) + \frac{K_1\left|f(x)\right|}{(n+p-\lambda)}(1+x)^2 + \sqrt{\frac{K_1}{n+p-\lambda}}\,(1+x)\,f'(x+)\,.
\end{aligned}
$$

Using $J_{1,n,x}$ and $J_{2,n,x}$ in (26), we get the required result. □

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