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A Trace Formula for Discontinuous Eigenvalue Problem

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Abstract. In this paper, we deal with a Sturm-Liouville problem which has discontinuity at one point and contains an eigenparameter in a boundary condition. We obtain a regularized trace formula for the problem.

1. Introduction

Consider the boundary value problem

$$\tau(u) := -y'' + q(x)\,y = \lambda y, \quad x \in I, \tag{1.1}$$

with boundary conditions

$$y(0) = 0, \tag{1.2}$$

$$y'(\pi) - \lambda y(\pi) = 0, \tag{1.3}$$

and transmission conditions

$$y\left(\frac{\pi}{2}+0\right) = a_1 y\left(\frac{\pi}{2}-0\right), \tag{1.4}$$

$$y'\left(\frac{\pi}{2}+0\right) = a_1^{-1} y'\left(\frac{\pi}{2}-0\right) + a_2 y\left(\frac{\pi}{2}-0\right) = 0, \tag{1.5}$$

where $I := \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, λ is an eigenparameter, $q(x)$ is a real valued function which is continuous in $\left[0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right]$ and has finite limits $q\left(\frac{\pi}{2} \pm 0\right) := \lim_{x \to \frac{\pi}{2} \pm 0} q(x)$, a_1, a_2 are real numbers and $a_1 > 0$.

Gelfand and Levitan [9] first calculated the regularized trace for the classical Sturm-Liouville problem. After this work, developing trace formulas for continuous problems were investigated by many authors (see [1, 2, 4-8, 10-13, 17, 21]). The history and the current state of the theory of the regularized traces of the linear operators were presented in the survey paper [16]. As far as we know, there are a few works about the regularized trace of discontinuous eigenvalue problems (see [19, 20]). In [20], the author obtained some formulas for the regularized traces of similar problem that none of the boundary conditions contains an eigenparameter.

This paper is organized as follows: Firstly, the asymptotic formulas of the eigenvalues and eigenfunctions are derived. Then the regularized trace formula for the problem (1.1)-(1.5) is obtained similar to the techniques of [14,15].

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2. Preliminaries

The asymptotics formulas of the eigenvalues and eigenfunctions can be derived similar to the classical techniques of [3, 18].

We will define the solution of (1.1) by

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda), & x\in\left[0,\dfrac{\pi}{2}\right),\\ \phi_2(x,\lambda), & x\in\left(\dfrac{\pi}{2},\pi\right],\end{cases}\tag{2.1}$$

as follows: Let $\phi_1(x,\lambda)$ be the solution of (1.1) on $\left[0,\frac{\pi}{2}\right]$, which satisfies the initial conditions

$$y(0,\lambda)=0,\quad y'(0,\lambda)=1.\tag{2.2}$$

After defining this solution, we define the solution $\phi_2(x,\lambda)$ of (1.1) on $\left[\frac{\pi}{2},\pi\right]$ by means of the solution $\phi_1(x,\lambda)$ by the initial conditions

$$y\left(\frac{\pi}{2},\lambda\right)=a_1\phi_1\left(\frac{\pi}{2},\lambda\right),\quad y'\left(\frac{\pi}{2},\lambda\right)=a_1^{-1}\phi_1'\left(\frac{\pi}{2},\lambda\right)+a_2\phi_1\left(\frac{\pi}{2},\lambda\right).\tag{2.3}$$

Consequently, $\phi(x,\lambda)$ satisfies of (1.1) on I, the boundary condition (1.2) and the transmission conditions (1.4) and (1.5).

Let $\lambda=s^2$. Then the following integral equations hold for $j=0$ and $j=1$:

$$\phi_1^{(j)}(x,\lambda)=\frac{1}{s}(\sin sx)^{(j)}+\frac{1}{s}\int_0^x(\sin s(x-t))^{(j)}q(t)\phi_1(t,\lambda)\,dt,\tag{2.4}$$

and

$$\begin{aligned}\phi_2^{(j)}(x,\lambda)&=\phi_2\left(\frac{\pi}{2},\lambda\right)\left(\cos s\left(x-\frac{\pi}{2}\right)\right)^{(j)}+\frac{1}{s}\phi_2'\left(\frac{\pi}{2},\lambda\right)\left(\sin s\left(x-\frac{\pi}{2}\right)\right)^{(j)}\\&+\frac{1}{s}\int_{\frac{\pi}{2}}^x(\sin s(x-t))^{(j)}q(t)\phi_2(t,\lambda)\,dt.\end{aligned}\tag{2.5}$$

Solving the equations (2.4) and (2.5) by the method of successive approximations, we obtain the following asymptotic representation for $|\lambda|\to\infty$:

$$\phi_1(x,\lambda)=\frac{1}{s}\sin sx-\frac{1}{s^2}Q_1(x)\cos sx+\frac{1}{s^3}\frac{q(x)+q(0)}{4}\sin sx+O\left(\frac{e^{|Im\,s|x}}{s^4}\right),\tag{2.6}$$

$$\begin{aligned}\phi_1'(x,\lambda)&=\cos sx+\frac{1}{s}Q_1(x)\sin sx-\frac{1}{s^2}\frac{q(x)-q(0)}{4}\cos sx+\frac{1}{s^3}\frac{q'(x)+q'(0)}{8}\sin sx\\&+O\left(\frac{e^{|Im\,s|x}}{s^4}\right),\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}\phi_2(x,\lambda)&=\frac{1}{s}\{A_1\sin sx-A_2\sin s(x-\pi)\}-\frac{1}{s^2}\left\{\left[A_1\left(Q_1\left(\tfrac{\pi}{2}\right)+Q_2(x)\right)+\frac{a_2}{2}\right]\cos sx\right.\\&+\left.\left[A_2\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2(x)\right)-\frac{a_2}{2}\right]\cos s(x-\pi)\right\}+\frac{1}{s^3}\left\{\left[A_1\left(\frac{q(x)+q(0)}{4}\right.\right.\right.\\&\left.-Q_1\left(\tfrac{\pi}{2}\right)Q_2(x)-\frac{a_2}{2}\left(Q_1\left(\tfrac{\pi}{2}\right)+Q_2(x)\right)\right]\sin sx-\left[A_2\left(\frac{q(x)+q(0)}{4}\right.\right.\\&\left.\left.+Q_1\left(\tfrac{\pi}{2}\right)Q_2(x)+\frac{a_2}{2}\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2(x)\right)\right]\sin s(x-\pi)\right\}+O\left(\frac{e^{|Im\,s|x}}{s^4}\right),\end{aligned}\tag{2.8}$$

$$\begin{aligned}\phi_2'(x,\lambda) &= A_1\cos sx - A_2\cos s(x-\pi) + \frac{1}{s}\left\{\left[A_1\left(Q_1\left(\tfrac{\pi}{2}\right)+Q_2(x)\right)+\frac{a_2}{2}\right]\sin sx\right.\\ &+\left.\left[A_2\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2(x)\right)-\frac{a_2}{2}\right]\sin s(x-\pi)\right\}+\frac{1}{s^2}\left\{\left[-A_1\left(\frac{q(x)-q(0)}{4}\right.\right.\right.\\ &+\left.\left.Q_1\left(\tfrac{\pi}{2}\right)Q_2(x)\right)-\frac{a_2}{2}\left(Q_1\left(\tfrac{\pi}{2}\right)+Q_2(x)\right)\right]\cos sx+\left[A_2\left(\frac{q(x)-q(0)}{4}\right.\right.\\ &-\left.\left.\left.Q_1\left(\tfrac{\pi}{2}\right)Q_2(x)\right)-\frac{a_2}{2}\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2(x)\right)\right]\cos s(x-\pi)\right\}+O\left(\frac{e^{|Im\,s|x}}{s^3}\right),\end{aligned} \tag{2.9}$$

where

$$Q_1(x)=\frac{1}{2}\int_0^x q(t)\,dt,\quad Q_2(x)=\frac{1}{2}\int_{\frac{\pi}{2}}^x q(t)\,dt,\ A_1=\frac{a_1+a_1^{-1}}{2},\ A_2=\frac{a_1-a_1^{-1}}{2}. \tag{2.10}$$

Since the function $\phi(x,\lambda)$ satisfies the boundary condition (1.2) and the transmission conditions (1.4) and (1.5) to find the eigenvalues of the problem (1.1)-(1.5), we have to insert the function $\phi(x,\lambda)$ in the boundary condition (1.3) and find the roots of this equation. It is obvious that the characteristic function $\omega(\lambda)$ of the problem (1.1)-(1.5) is as follows

$$\omega(\lambda)=\phi_2'(\pi,\lambda)-s^2\phi_2(\pi,\lambda), \tag{2.11}$$

and the eigenvalues of the problem (1.1)-(1.5) coincide with the roots of $\omega(\lambda)$. Using equations (2.8) and (2.9), we obtain

$$\begin{aligned}\omega(\lambda) &= -sA_1\sin\pi s+\left[A_1\left(1+Q_1\left(\tfrac{\pi}{2}\right)+Q_2(\pi)\right)+\frac{a_2}{2}\right]\cos\pi s\\ &+\left[A_2\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2(\pi)-1\right)-\frac{a_2}{2}\right]+\frac{1}{s}\left\{A_1\left[Q_1\left(\tfrac{\pi}{2}\right)+Q_2(\pi)\right.\right.\\ &+\left.\left.Q_1\left(\tfrac{\pi}{2}\right)Q_2(\pi)-\frac{q(\pi)+q(0)}{4}\right]+\frac{a_2}{2}\left(1+Q_1\left(\tfrac{\pi}{2}\right)+Q_2(\pi)\right)\right\}\sin\pi s\\ &+O\left(\frac{e^{|Im\,s|x}}{s^2}\right).\end{aligned} \tag{2.12}$$

Using the Rouche theorem in (2.12), we obtain

$$s_n=n+\frac{C}{n\pi A_1}+O\left(\frac{1}{n^3}\right), \tag{2.13}$$

where

$$C=A_1\left(1+Q_1\left(\frac{\pi}{2}\right)+Q_2(\pi)\right)+\frac{a_2}{2}+(-1)^n\left(A_2\left(Q_1\left(\frac{\pi}{2}\right)-Q_2(\pi)-1\right)-\frac{a_2}{2}\right). \tag{2.14}$$

It follows from (2.13) that

$$\lambda_n=n^2+\frac{2C}{\pi A_1}+O\left(\frac{1}{n^2}\right). \tag{2.15}$$

3. Traces of the Problem

The series

$$\sum_{n=0}^{\infty}\left(\lambda_n-n^2-\frac{2C}{\pi A_1}\right)<\infty,$$

converges and is called the regularized first trace for the problem (1.1)-(1.5). The goal of this paper is to find its sum.

Theorem 3.1. *Suppose that $q(x)$ has a second-order piecewise integrable derivatives on $[0,\pi]$, then the following regularized trace formula holds*

$$s_\lambda = \sum_{n=0}^{\infty}\left(\lambda_n - n^2 - \frac{2C}{\pi A_1}\right) = Q_1\left(\tfrac{\pi}{2}\right) + Q_2(\pi) - \frac{q(\pi)+q(0)}{4} + \frac{a_2}{2A_1}\left(1+Q_1\left(\tfrac{\pi}{2}\right)\right) - \frac{C}{\pi A_1} - \frac{C^2}{2A_1^2}, \tag{3.1}$$

where A_i, $Q_i(x)$ $(i=1,2)$ and C satisfy the equations (2.10) and (2.14).

Proof. Since $\omega(\lambda)$ is an entire function from Hadamard's theorem (see [11]), using (2.11) we have

$$\omega(\lambda) = A\Phi(\lambda), \tag{3.2}$$

where $\Phi(\lambda) = \prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right)$ and A is a certain constant to be determined below.

Let $\lambda = -\mu^2$. We calculate the sum of the series (3.1) by comparing the asymptotic expressions obtained from (3.2) on the right and left.

Put

$$\Phi\left(-\mu^2\right) = \prod_{n=0}^{\infty}\left(1 + \frac{\mu^2}{\lambda_n}\right) = \left(\frac{\lambda_0 + \mu^2}{\mu\pi}\right) B\Psi\left(\mu^2\right)\sinh\mu\pi, \tag{3.3}$$

where

$$B = \frac{1}{\lambda_0}\prod_{n=1}^{\infty}\left(\frac{n^2}{\lambda_n}\right), \quad \Psi\left(\mu^2\right) = \prod_{n=1}^{\infty}\left(1 - \frac{n^2-\lambda_n}{\mu^2+n^2}\right). \tag{3.4}$$

To study the asymptotic behaviour of $\Psi\left(\mu^2\right)$ as $\mu \to \infty$, we consider

$$\begin{aligned}
\ln\Psi\left(\mu^2\right) &= \sum_{n=1}^{\infty}\ln\left(1 - \frac{n^2-\lambda_n}{\mu^2+n^2}\right) \\
&= -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{n^2-\lambda_n}{\mu^2+n^2}\right)^k \\
&= -\sum_{n=1}^{\infty}\frac{n^2-\lambda_n}{\mu^2+n^2} - \sum_{k=2}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{n^2-\lambda_n}{\mu^2+n^2}\right)^k \\
&= \sum_{n=1}^{\infty}\frac{C_1}{\mu^2+n^2} + \frac{1}{\mu^2}\sum_{n=1}^{\infty}\left(\lambda_n - n^2 - C_1\right) - \frac{1}{\mu^2}\sum_{n=1}^{\infty}\frac{\left(\lambda_n - n^2 - C_1\right)n^2}{\mu^2+n^2} \\
&\quad - \sum_{k=2}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{n^2-\lambda_n}{\mu^2+n^2}\right)^k
\end{aligned} \tag{3.5}$$

where $C_1 = \frac{2C}{\pi A_1}$.

Asymptotic expressions can be obtained according to the following lemma (similar to [15, Ch5]).

Lemma 3.2. *If $\left|n^2 - \lambda_n\right| \le \rho$, then*

$$\sum_{n=1}^{\infty}\frac{\left|n^2-\lambda_n\right|^k}{\left(\mu^2+n^2\right)^k} \le \frac{\pi}{2}\frac{\rho^k}{\mu^{2k-1}}. \tag{3.6}$$

It follows from (3.6) that

$$\sum_{k=2}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\frac{\left|n^2-\lambda_n\right|^k}{(\mu^2+n^2)^k}\leq\frac{\pi}{2}\sum_{k=2}^{\infty}\frac{\rho^k}{\mu^{2k-1}}=\frac{\pi}{2}\frac{\rho^2}{\mu^3}\sum_{k=0}^{\infty}\left(\frac{\rho}{\mu^2}\right)^k=O\left(\frac{1}{\mu^3}\right), \tag{3.7}$$

and since $\sup_n\left|\lambda_n-n^2-C_1\right|n^2<\infty$, it follows from (3.6) that

$$\frac{1}{\mu^2}\sum_{n=1}^{\infty}\frac{\left(\lambda_n-n^2-C_1\right)n^2}{\mu^2+n^2}=O\left(\frac{1}{\mu^3}\right). \tag{3.8}$$

As we know,

$$\sum_{n=1}^{\infty}\frac{1}{\mu^2+n^2}=\frac{\pi\coth\mu\pi}{2\mu}-\frac{1}{2\mu^2}=\frac{\pi}{2\mu}-\frac{1}{2\mu^2}+O\left(e^{-2\mu\pi}\right). \tag{3.9}$$

Therefore, substituting (3.7)-(3.9) into (3.5) then, we obtain

$$\ln\Psi\left(\mu^2\right)=\frac{\pi C_1}{2\mu}+\frac{1}{\mu^2}\left(s_\lambda-\lambda_0+\frac{C_1}{2}\right)+O\left(\frac{1}{\mu^3}\right),$$

where

$$s_\lambda=\sum_{n=0}^{\infty}\left(\lambda_n-n^2-C_1\right). \tag{3.10}$$

Therefore, we get

$$\begin{aligned}\Psi\left(\mu^2\right)&=\exp\left\{\frac{\pi C_1}{2\mu}+\frac{1}{\mu^2}\left(s_\lambda-\lambda_0+\frac{C_1}{2}\right)+O\left(\frac{1}{\mu^3}\right)\right\}\\&=1+\frac{\pi C_1}{2\mu}+\frac{1}{\mu^2}\left(s_\lambda-\lambda_0+\frac{C_1}{2}+\frac{\pi^2C_1^2}{8}\right)+O\left(\frac{1}{\mu^3}\right).\end{aligned} \tag{3.11}$$

Relying on (3.11), then we derive from (3.3) that

$$\Phi\left(-\mu^2\right)=\frac{1}{2\pi}Be^{\mu\pi}\left\{\mu+\frac{\pi C_1}{2}+\frac{1}{\mu}\left(s_\lambda+\frac{C_1}{2}+\frac{\pi^2C_1^2}{8}\right)+O\left(\frac{1}{\mu^2}\right)\right\}. \tag{3.12}$$

We now study the asymptotic behaviour of the function

$$\omega\left(-\mu^2\right)=\phi_2'\left(\pi,-\mu^2\right)+\mu^2\phi_2\left(\pi,-\mu^2\right), \tag{3.13}$$

using the Liouville equation. Then according to formula (2.8) and (2.9), we have

$$\begin{aligned}\phi_2\left(x,-\mu^2\right)=&\frac{1}{\mu}\left\{A_1\sinh\mu x-A_2\sinh\mu\left(x-\pi\right)\right\}+\frac{1}{\mu^2}\left\{\left[A_1\left(Q_1\left(\tfrac{\pi}{2}\right)+Q_2\left(x\right)\right)\right.\right.\\&\left.+\frac{a_2}{2}\right]\cosh\mu x+\left[A_2\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2\left(x\right)\right)-\frac{a_2}{2}\right]\cosh\mu\left(x-\pi\right)\Big\}\\&+O\left(\frac{1}{\mu^3}\right),\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\phi_2'\left(x,-\mu^2\right) &= A_1\cosh\mu x - A_2\cosh\mu\left(x-\pi\right) + \frac{1}{\mu}\left\{\left[A_1\left(Q_1\left(\tfrac{\pi}{2}\right)+Q_2\left(x\right)\right)+\frac{a_2}{2}\right]\sinh\mu x\right.\\ &+\left[A_2\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2\left(x\right)\right)-\frac{a_2}{2}\right]\sinh\mu\left(x-\pi\right)\Big\} + \frac{1}{\mu^2}\left\{\left[A_1\frac{q\left(x\right)+q\left(0\right)}{4}\right.\right.\\ &\left.+\frac{a_2}{2}Q_1\left(\tfrac{\pi}{2}\right)\right]\cosh\mu x + \left[\frac{a_2}{2}Q_1\left(\tfrac{\pi}{2}\right) - A_2\frac{q\left(x\right)-q\left(0\right)}{4}\right]\cosh\mu\left(x-\pi\right)\Big\}\\ &+O\left(\frac{1}{\mu^3}\right).\end{aligned} \tag{3.15}$$

Putting $\phi_2\left(x,-\mu^2\right)$ and $\phi_2'\left(x,-\mu^2\right)$ at π and also using formulas $\sinh\mu\pi = \frac{e^{\mu\pi}}{2}+O\left(\frac{1}{e^{2\mu\pi}}\right)$, $\cosh\mu\pi = \frac{e^{\mu\pi}}{2}+O\left(\frac{1}{e^{2\mu\pi}}\right)$ into (3.13), we have

$$\begin{aligned}\omega\left(-\mu^2\right) &= \frac{e^{\mu\pi}}{2}\left\{\mu A_1 + A_1\left(1+Q_1\left(\tfrac{\pi}{2}\right)+Q_2\left(\pi\right)\right)+\frac{a_2}{2}\right.\\ &+(-1)^n\left(A_2\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2\left(\pi\right)-1\right)-\frac{a_2}{2}\right)+\frac{1}{\mu}\left\{A_1\left[Q_1\left(\tfrac{\pi}{2}\right)+Q_2\left(\pi\right)\right.\right.\\ &\left.\left.\left.-\frac{q\left(\pi\right)+q\left(0\right)}{4}+\frac{a_2}{2}\left(1+Q_1\left(\tfrac{\pi}{2}\right)\right)\right]\right\}+O\left(\frac{1}{\mu^2}\right)\right\}.\end{aligned} \tag{3.16}$$

It follows from the equalities (3.2), (3.12), (3.16) and comparing the coefficients of μ, we obtain

$$\begin{aligned}&\frac{AB}{\pi} = A_1,\\ &\frac{AB}{\pi}\frac{\pi C_1}{2} = A_1\left(1+Q_1\left(\tfrac{\pi}{2}\right)+Q_2\left(\pi\right)\right)+\frac{a_2}{2}+(-1)^n\left(A_2\left(Q_1\left(\tfrac{\pi}{2}\right)-Q_2\left(\pi\right)-1\right)-\frac{a_2}{2}\right),\\ &s_\lambda = Q_1\left(\tfrac{\pi}{2}\right)+Q_2\left(\pi\right)-\frac{q\left(\pi\right)+q\left(0\right)}{4}+\frac{a_2}{2A_1}\left(1+Q_1\left(\tfrac{\pi}{2}\right)\right)-\frac{C}{\pi A_1}-\frac{C^2}{2A_1^2}.\end{aligned} \tag{3.17}$$

Therefore, we obtain

$$\begin{aligned}s_\lambda &= \sum_{n=0}^{\infty}\left(\lambda_n - n^2 - \frac{2C}{\pi A_1}\right)\\ &= Q_1\left(\frac{\pi}{2}\right)+Q_2\left(\pi\right)-\frac{q\left(\pi\right)+q\left(0\right)}{4}+\frac{a_2}{2A_1}\left(1+Q_1\left(\frac{\pi}{2}\right)\right)-\frac{C}{\pi A_1}-\frac{C^2}{2A_1^2},\end{aligned}$$

where

$$C = A_1\left(1+Q_1\left(\frac{\pi}{2}\right)+Q_2\left(\pi\right)\right)+\frac{a_2}{2}+(-1)^n\left(A_2\left(Q_1\left(\frac{\pi}{2}\right)-Q_2\left(\pi\right)-1\right)-\frac{a_2}{2}\right),$$

$$A_1 = \frac{a_1+a_1^{-1}}{2},\ A_2 = \frac{a_1-a_1^{-1}}{2},\ Q_1\left(\frac{\pi}{2}\right) = \frac{1}{2}\int_0^{\frac{\pi}{2}} q\left(t\right)dt,\ Q_2\left(\pi\right) = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} q\left(t\right)dt,$$

completing the proof of Theorem 3.1. □

References

[1] E. E. Adıguzelov, The trace formula for the difference of two Sturm Liouville differential operators with operator coefficients, Izv. Akad. Nauk Azerb. SSR, Ser. Fiz. Tekhn. Mat. Nauk., 5(1976) 20-24.

[2] I. Albayrak, M. Bayramoğlu, E. Adıgüzelov, The second regularized trace formula for the Sturm-Liouville problem with spectral parameter in a boundary condition, Methods Funct. Anal. Topology, 6(3) (2000)1-8.

[3] N. Altınışık, M. Kadakal M, O.Sh. Mukhtarov, Eigenvalues and eigenfunctions of discontinuous Sturm Liouville problems with eigenparameter dependent boundary conditions, Acta Mathematica Hungarica, 102(2004) 159-175.
[4] N.M. Aslanova, A trace formula of a boundary value problem for the operator Sturm Liouville equation, Siberian Math J., 49(2008) 959-967.
[5] D. Borisov, P. Freitas, Eigenvalue asymptotics, inverse problems and a trace formula for the linear damped wave equation. Journal of Differential Equations, 247(2009) 3028-3039.
[6] R.Z. Chalilova, On arranging Sturm Liouville operator equations trace, Funktsional Anal, Teor. Funktsi I Pril. Mahaçkala, 3(1)(1976) 154-161.
[7] L.A. Dikii, On a formula of Gel'fand-Levitan, Uspekhi Mat. Nauk., 8(1953)119-123(Russian).
[8] L.A. Dikii, Trace formulas for Sturm-Liouville differential operators, Uspehi Mat. Nauk., 12(3) (1958) 111-143.
[9] M. Gelfand, B. M. Levitan, On a simple identity for eigenvalues of the differential operator of second order. Dokl. Akad. Nauk SSSR, 88(1953) 593-596.
[10] E. Gul, On the regularized trace of a second order differential operator, Appl. Math. Comput., 198 (2008) 471-480.
[11] N.J. Guliyev, The regularized trace formula for the Sturm-Liouville equation with spectral parameter in the boundary conditions, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., 22(2005) 99-102.
[12] K. Koklu, I. Albayrak, A. Bayramov, A regularized trace formula for second order differential operator equations, Mathematica Scandinavica, 107(2010) 123-138.
[13] B. M. Levitan, Calculation of a regularized trace for the Sturm-Liouville operator. Uspekhi Mat. Nauk., 19(1964) 161-165 (Russian).
[14] B.M. Levitan, Regularized traces and smooth periodicity conditions for the potential of the Sturm Liouville equation, Sib. Mat. Zhurn., 22(2)(1981) 137-148 (Russian).
[15] B.M. Levitan, I.S. Sargsjan, Sturm-Liouville and Dirac Operators, Kluwer, Dordrecht, 59, 1991.
[16] V.A. Sadovnichii, V.E. Podol'skii, Traces of operators, Uspekhi Mat. Nauk., 61(5)(2006) 89-156, Transl:Russian Math. Surveys 61(2006) 885-953.
[17] E. Şen, A. Bayramov, K. Oruçoğlu, The regularized trace formula for a differential operator with unbounded operator coefficient, Adv. Stud. Contemp. Math., 25(4)(2015) 583-590.
[18] E.C. Titchmarsh, Eigenfunctions Expansion Associated with Second Order Differential Equations I, 2 nd, Oxford Univ. Press. London, 1962.
[19] C.F. Yang, Trace and inverse problem of a discontinuous Sturm Liouville operator with retarded argument, J. Math. Anal. Appl., 395(2012) 30-41.
[20] C.F. Yang, Traces of Sturm Liouville operators with discontinuities, Inverse Problem Science Engineering, 22(5)(2013) 803-813.
[21] C.F. Yang, Trace formula for differential pencils with spectral parameter dependent boundary conditions, Math. Meth. Appl. Sci., 37(2014) 1325-1332.