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Minimax Fractional Programming with Nondifferentiable (G, β)-invexity

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Abstract. In this paper, we consider the minimax fractional programming Problem (FP) in which the functions are locally Lipschitz (G, β)-invex. With the help of a useful auxiliary minimax programming problem, we obtain not only G-sufficient but also G-necessary optimality conditions theorems for the Problem (FP). With G-necessary optimality conditions and (G, β)-invexity in the hand, we further construct dual Problem (D) for the primal one (FP) and prove duality results between Problems (FP) and (D). These results extend several known results to a wider class of programs.

1. Introduction

Recently, Antczak extended the invexity proposed by [2] to the G-invexity (see [4]) for scalar differentiable functions and introduced new necessary optimality conditions for differentiable mathematical programming problem. Antczak also applied the introduced G-invexity notion to develop sufficient optimality conditions and new duality results for differentiable mathematical programming problems. Furthermore, in the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions. In 2009, Antczak ([5]) defined vector G-invex (G-incave) functions with respect to η, and applied this vector G-invexity to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. He also established the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable vector optimization problems under the Kuhn-Tucker constraint qualification, see [5]. With this vector G-invexity concept, Antczak proved new duality results for nonlinear differentiable multiobjective programming problems in [6]. A number of new vector duality problems such as G-Mond-Weir, G-Wolfe and G-mixed dual vector problems to the primal one were also defined in [6].

In the last few years, many concepts of generalized convexity, which include (p, r)-invexity ([7]), (F, ρ)-convexity ([8]), (F, α, ρ, d)-convexity ([9]), (C, α, ρ, d)-convexity ([10]), (ϕ, ρ)-invexity ([11]), V-r-invexity ([12]) and their extensions, have been introduced and applied to different mathematical programming problems. In particular, they have also been applied to deal with minimax programming; see [13–17]

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for detail. Moreover, some researchers, for example [20] and [21], considered the minimax fractional programming involving higher order generalized convexity. However, we have not found paper which deal with minimax fractional programming Problem (FP) under assumptions of G-invexity or its generalizations.

Note that the function $G \circ f$ may be not differentiable even if the function G is differentiable. [18] introduced the (G_f, β_f)-invexity concept for locally Lipschtiz function f. This (G_f, β_f)-invexity extended Antczak's G-invexity concept to the nonsmooth case. Moreover, [19] deal with a generalized minimax programming under this nondifferentiable generalized invexity assumption. In this paper, we, under the assumption of the vector (G, β)-invexity proposed, further consider a nondifferentiable minimax programming Problem (FP), which includes the minimax problem considered in [19] as a special case. The generalized minimax fractional programming Problem (FP) considered in this paper is presented as follows.

$$\text{(FP)} \qquad \min \left\{ F(x) := \sup_{y \in Y} \left\{ \phi(x, y) := \frac{f(x, y)}{h(x, y)} \right\} \right\}$$
$$\text{subject to} \quad g_j(x) \leq 0, \ j = 1, \cdots, m,$$

where Y is a compact subset of $\mathbb{R}^p$, $f(\cdot, \cdot)$, $h(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g_j(\cdot) : \mathbb{R}^n \to \mathbb{R}$ $(j \in M)$. Let E_{FP} be the set of feasible solutions of Problem (FP); in other words, $E_{FP} = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in M\}$. For convenience, let us define the following sets for every $x \in E$.

$$J(x) = \{j \in M \,|\, g_j(x) = 0\}, \ Y(x) = \left\{ y \in Y \,|\, \phi(x, y) = \sup_{z \in Y} \phi(x, z) \right\}.$$

The rest of the paper is organized as follows. In Section 2, we present concepts regarding to nondifferentiable vector (G, β)-invexity and construct an auxiliary minimax programming problem which will be useful to help us to deal with the minimax fractional problem (FP). In Section 3, we present not only G-sufficient but also G-necessary optimality conditions for Problem (FP). When the G-necessary optimality conditions and the (G, β)-invexity concept are utilized, dual Problem (D) is formulated for the primal one (FP) and duality results between them are presented in Section 4.

2. Notations and Preliminaries

In this section, we provide some definitions and results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper. For any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T$, we define:

$x > y$ if and only if $x_i > y_i$, for $i = 1, 2, \cdots, n$;
$x \geqq y$ if and only if $x_i \geq y_i$, for $i = 1, 2, \cdots, n$;
$x \geqslant y$ if and only if $x_i \geq y_i$, for $i = 1, 2, \cdots, n$, but $x \neq y$;
$x \not> y$ is the negation of $x > y$, $x \not\geqslant y$ is the negation of $x \geqslant y$.

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x \geqq 0\}$, $\dot{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n \,|\, x > 0\}$ and X be a subset of $\mathbb{R}^n$. For our convenience, denote $Q := \{1, \cdots, q\}$, $Q^* := \{1, \cdots, q^*\}$, $K := \{1, \cdots, k\}$, $M := \{1, \cdots, m\}$.

Definition 2.1. ([22]) *Let $d \in \mathbb{R}^n$, X be a nonempty set of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$. If*

$$f^0(x; d) := \lim_{\substack{y \to x \\ \mu \downarrow 0}} \sup \frac{1}{\mu} (f(y + \mu d) - f(y))$$

exists, then $f^0(x; d)$ is called the Clarke derivative of f at x in the direction d. If this limit superior exists for all $d \in \mathbb{R}^n$, then f is called Clarke differentiable at x. The set

$$\partial f(x) = \left\{ \zeta \mid f^0(x; d) \geq \langle \zeta, d \rangle, \ \forall \, d \in \mathbb{R}^n \right\}$$

is called the Clarke subdifferential of f at x.

Note that if a given function f is locally Lipschitz, then the Clarke subdifferential $\partial f(x)$ exists.

Lemma 2.2. ([18]) *Let ψ be a real-valued Lipschitz continuous function defined on X and denote the image of X under ψ by $I_\psi(X)$; let $\varphi : I_\psi(X) \to \mathbb{R}$ be a differentiable function such that $\varphi'(\gamma)$ is continuous on $I_\psi(X)$ and $\varphi'(\gamma) \geq 0$ for each $\gamma \in I_\psi(X)$. Then the chain rule*

$$(\varphi \circ \psi)^0(x, d) = \varphi'(\psi(x))\psi^0(x, d)$$

holds for each $d \in \mathbb{R}^n$. Therefore,

$$\partial(\varphi \circ \psi)(x) = \varphi'(\psi(x))\partial(\psi)(x).$$

Definition 2.3. *Let $f = (f_1, \cdots, f_k)$ be a vector-valued locally Lipschitz function defined on a nonempty set $X \subset \mathbb{R}^n$. Consider the functions $\eta : X \times X \to \mathbb{R}^n$, $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, and $\beta_i^f : X \times X \to \mathbb{R}_+$ for $i \in K$. Moreover, G_{f_i} is strictly increasing on its domain $I_{f_i}(X)$ for each $i \in K$. If*

$$G_{f_i} \circ f_i(x) - G_{f_i} \circ f_i(u) \geq (>) \beta_i^f(x, u) G'_{f_i}(f_i(u)) \langle \zeta_i, \eta(x, u)\rangle, \forall \zeta_i \in \partial f_i(u) \tag{1}$$

holds for all $x \in X$ ($x \neq u$) and $i \in K$, then f is said to be (strictly) nondifferentiable vector (G_f, β_f)-invex at u on X (with respect to η) (or shortly, (G_f, β_f)-invex at u on X), where $G_f = (G_{f_1}, \cdots, G_{f_k})$ and $\beta := (\beta_1^f, \beta_2^f, \cdots, \beta_k^f)$. If f is (strictly) nondifferentiable vector (G_f, β_f)-invex at u on X (with respect to η) for all $u \in X$, then f is (strictly) nondifferential vector (G_f, β_f)-invex on X (with respect to η).

Remark 2.4. *In order to define (strictly) nondifferential vector (G_f, β_f)-incave function with respect to η for a given f, the direction of the inequalities (1) in Definition 2.3 should be changed to the opposite one.*

Remark 2.5. (a) *Let $f : X \to \mathbb{R}$ be a differential (G_f, β_f)-invex function, then $G_f(f)$ is α-invex by Definition 2.3 in this paper and α-invexity as defined in [23], where $\alpha = \beta_f$.*

(b) *Let $f : X \to \mathbb{R}$ be a differential (G_f, β_f)-invex function and $G_f(a) = a$ for $a \in \mathbb{R}$, then f is α-invex as defined in [23], where $\alpha = \beta_f$.*

(c) *Let $f = (f_1, \cdots, f_k)$ be a differential vector (G_f, β_f)-invex function and $\beta_i^f(x, u) = 1$ for all $x, u \in X$ ($i \in K$), then f is vector G-invex as defined in [5]. Further, if $|K| = 1$, then f is G-invex as defined in [4].*

For fixed $e \in \mathbb{R}$, we construct the following auxiliary minimax programming Problem (G-Pe) for Problem (FP).

$$(G\text{-P}e) \quad \min \left\{ \Phi(x, e) := \sup_{y \in Y} (G \circ f - G \circ (eh))(x, y) \right\}$$
$$s.t.\ G_g \circ g(x) := \left(G_{g_1} \circ g_1(x), G_{g_2} \circ g_2(x), \cdots, G_{g_m} \circ g_m(x) \right) \leqq G_g(0),$$

where $G_g(0) := \left(G_{g_1}(0), G_{g_2}(0), \cdots, G_{g_m}(0) \right)$. We denote by $X_{G\text{-}Pe} = \{x \in \mathbb{R}^n \mid G_g \circ g(x) \leqq G_g(0)\}$, $J'(\bar{x}) := \{j \in M : G_{g_j} \circ g_j(\bar{x}) = G_{g_j}(0)\}$. If the function G_{g_j} is strictly increasing on $I_{g_j}(X)$ for each $j \in M$, then $X_{FP} = X_{(G\text{-}Pe)}$ and $J(\bar{x}) = J'(\bar{x})$. So, we represent the set of all feasible solutions and the set of constraint active indices for either (FP) or (G-Pe) by the notations E and $J(\bar{x})$, respectively. Moreover, it is easy to see that the following lemma holds.

Lemma 2.6. *Let x^* be an optimal solution for (FP) and $v^* := F(x^*)$. If the function G is strictly increasing in $\mathbb{R}$, then*

(*i*) *x^* is an optimal solution to Problem (G-Pv^*) and $\Phi(x^*, v^*) = 0$;*

(*ii*) *$G \circ f(x^*, y) - G \circ (v^* h)(x^*, y) = 0$ whenever $y \in Y(x^*)$.*

3. Optimality Conditions

In this section, we establish firstly the G-necessary optimality conditions for Problem (FP) involving functions which are locally Lipschitz with respect to the variable x. For this purpose, we will need some additional assumptions with respect to Problem (FP).

Condition 3.1. *Assume that:* (a) *the set Y is compact;*
(b) *$f(x,y)$ and $h(x,y)$ are locally Lipschitz in x for fixed $y \in Y$;*
(c) *$f(x,y)$ is regular at x; $f(x,y)$ is strictly differentiable at x for fixed $y \in Y$;*
(d) *$f(x,y)$ and $\partial f_x(x,y)$ are upper semicontinuous at (x,y); $h(x,y)$ and $\partial h_x(x,y)$ are upper semicontinuous at (x,y);*
(e) *g_j, $j \in M$, are regular and locally Lipschitz at x.*

Condition 3.2. *For each $\eta = (\eta_1, \cdots, \eta_m) \in \mathbb{R}^m$ satisfying the conditions*

$$\eta_j = 0, \forall\, j \in M \setminus J(x^*);\ \eta_j \geq 0,\ \forall\, j \in J(x^*),$$

the following implication holds:

$$z_j^* \in \partial g_j(x^*)\ (\forall\, j \in M), \sum_{j=1}^{m} \eta_j z_j^* = 0 \ \Rightarrow \eta_j = 0,\ j \in M.$$

The following necessary optimality conditions are presented in [24].

Theorem 3.3 (Necessary optimality conditions). *Let x^* be an optimal solution to Problem (P) as considered in [24]. We also assume that Conditions 1 and 2 as defined in [24] are satisfied for Problem (P). Then there exist positive integer q^* and vectors $y_i \in Y(x^*)$ together with scalars λ_i^* ($i \in Q^*$) and μ_j^* ($j \in M$) such that*

$$0 \in \sum_{i=1}^{q^*} \lambda_i^* \partial_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* \partial g_j(x^*), \tag{2}$$

$$\mu_j^* g_j(x^*) = 0, \mu_j^* \geq 0, j \in M, \tag{3}$$

$$\sum_{i=1}^{q^*} \lambda_i^* = 1, \lambda_i^* > 0, i \in Q^*. \tag{4}$$

Furthermore, if α is the number of nonzero λ_i^*, and β is the number of nonzero μ_j^*, then

$$1 \leq \alpha + \beta \leq n + 1.$$

Making use of the above Theorem 3.3, we can derive the following G-necessary conditions theorem for Problem (FP); see Theorem 3.4, here we require the scalars λ_i^* ($i = 1, \ldots, q^*$) are positive.

Theorem 3.4 (G-necessary optimality conditions). *Let Problem (FP) satisfy Conditions 3.1 and 3.2 as defined in this paper; let x^* be an optimal solution of Problem (FP). Assume that G is both continuously differentiable and strictly increasing on $\mathbb{R}$. If G_{g_j} is both continuously differentiable and strictly increasing function on $I_{g_j}(X)$ with $G'_{g_j}(g_j(x^*)) > 0$ for each $j \in M$, then there exist positive integer q^* ($1 \leq q^* \leq n+1$) and vectors $y_i \in Y(x^*)$ together with scalars λ_i^* ($i \in Q^*$) and μ_j^* ($j \in M$) such that*

$$0 \in \sum_{i=1}^{q^*} \lambda_i^* \left[G'(f(x^*, y_i))\partial_x f(x^*, y_i) - v^* G'(v^* h(x^*, y_i))\partial_x h(x^*, y_i)\right] + \sum_{j=1}^{m} \mu_j^* G'_{g_j}(g_j(x^*))\partial g_j(x^*), \tag{5}$$

$$G \circ f(x^*, y_i) - G \circ (v^* h)(x^*, y_i) = 0, i \in Q^*, \tag{6}$$

$$\mu_j^* (G_{g_j} \circ g_j(x^*) - G_{g_j}(0)) = 0,\ \mu_j^* \geq 0, j \in M, \tag{7}$$

$$\sum_{i=1}^{q^*} \lambda_i^* = 1, \lambda_i^* \geq 0,\ i \in Q^*. \tag{8}$$

Proof. Since x^* is an optimal solution to Problem (FP), by Lemma 2.6, x^* is an optimal solution to Problem (G-Pν^*) and Eq. (6) holds for all $i \in Q^*$. Moreover, we can deduce from Conditions 3.1 and 3.2 that Conditions 1 and 2 as defined in [24] are satisfied for Problem G-Pν^*. Therefore, by Theorem 3.3, there exist positive integer q^* and vectors $y_i \in Y(x^*)$ together with scalars λ_i^* ($i \in Q^*$) and μ_j^* ($j \in M$) such that (7), (8) and

$$0 \in \sum_{i=1}^{q^*} \lambda_i^* \partial_x (G \circ f - G \circ (\nu^* h))(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* \partial (G_{g_j} \circ g_j)(x^*) \tag{9}$$

hold.

By Lemma 2.2 and the continuity of G and G_{g_j}, we obtain

$$\partial_x (G \circ f - G \circ (\nu^* h))(x^*, y_i) = G'(f(x^*, y_i))\partial_x f(x^*, y_i) - \nu^* G'(\nu^* h(x^*, y_i))\partial_x h(x^*, y_i), i \in Q^*,$$
$$\partial (G_{g_j} \circ g_j)(x^*) = G'_{g_j}(g_j(x^*))\partial g_j(x^*), j \in M.$$

The above two equations, together with (9), follow that (5). The proof is complete. □

Next, we derive G-sufficient optimality conditions for Problem (FP) under the assumptions of (G, β)-invexity concept as proposed in [18], see also Definition 2.3 in this paper.

Theorem 3.5 (G-sufficient optimality conditions). *Let $(x^*, \mu^*, \nu^*, q^*, \lambda^*, \bar{y})$ satisfy conditions (5)-(8); let G be both continuously differentiable and strictly increasing on $\mathbb{R}$; let G_{g_j} be both continuously differentiable and strictly increasing on $I_{g_j}(X)$ for each $j \in M$. Assume that $f(\cdot, y_i)$ and $\nu^* h(\cdot, y_i)$ are (G, β_i)-invex and (G, β_i)-incave at x^* on E, respectively, for each $i \in Q^*$. If g_j is (G_j^g, β_j^g)-invex at x^* on E for each $j \in M$, then x^* is an optimal solution to (FP).*

Proof. Suppose, contrary to the result, that x^* is not an optimal solution for Problem (FP). Hence, there exists $x_0 \in E$ such that

$$\sup_{y \in Y} \frac{f(x_0, y)}{h(x_0, y)} < \nu^* = \frac{f(x^*, y_1)}{h(x^*, y_1)} = \cdots = \frac{f(x^*, y_{q^*})}{h(x^*, y_{q^*})}.$$

By the monotonicity of G, we have

$$G \circ f(x_0, y_i) < G \circ (\nu^* h)(x_0, y_i), i \in Q^*.$$

Thus,

$$G \circ f(x_0, y_i) - G \circ f(x^*, y_i) - (G \circ (\nu^* h)(x_0, y_i) - G \circ (\nu^* h)(x^*, y_i)) < 0, i \in Q^*,$$

here the identities $G \circ f(x^*, y_i) = G \circ (\nu^* h)(x^*, y_i)$, $i \in Q^*$, are used. Employing (6), (7) and the fact that

$$G_{g_j} \circ g_j(x_0) \leq G_{g_j}(0) = G_{g_j} \circ g_j(x^*), j \in J(x^*),$$

we can write the following statement.

$$\sum_{i=1}^{q^*} \lambda_i^* \frac{G \circ f(x_0, y_i) - G \circ f(x^*, y_i) - (G \circ (\nu^* h)(x_0, y_i) - G \circ (\nu^* h)(x^*, y_i))}{\beta_i(x_0, x^*)} + \sum_{j=1}^{m} \mu_j^* \frac{G_{g_j} \circ g_j(x_0) - G_{g_j} \circ g_j(x^*)}{\beta_j^g(x_0, x^*)} < 0. \tag{10}$$

By the generalized invexity assumptions of $f(\cdot, y_i)$, $\nu^* h(\cdot, y_i)$ and g_j, we have

$$G \circ f(x_0, y_i) - G \circ f(x^*, y_i) \geq (>) \beta_i(x_0, x^*) G'(f(x^*, y_i)) \langle \xi_i^f, \eta(x_0, x^*) \rangle, \ \forall \xi_i^f \in \partial_x f(x^*, y_i), \tag{11}$$

$$G \circ (\nu^* h)(x_0, y_i) - G \circ (\nu^* h)(x^*, y_i) \leq (<) \beta_i(x_0, x^*) \nu^* G'(\nu^* h(x^*, y_i)) \langle \xi_i^h, \eta(x_0, x^*) \rangle, \forall \xi_i^h \in \partial_x h(x^*, y_i), \tag{12}$$

$$G_{g_j} \circ g_j(x_0) - G_{g_j} \circ g_j(x^*) \geq (>) \beta_j^g(x_0, x^*) G'_{g_j}(g_j(x^*)) \langle \xi_j^g, \eta(x_0, x^*) \rangle, \forall \ \xi_j^g \in \partial g_j(x^*) \tag{13}$$

for $i \in Q^*$ and $j \in M$. Employing (11), (12) and (13) to (10), we deduce that

$$\sum_{i=1}^{q^*} \lambda_i^* \Big(G'(f(x^*, y_i)) \langle \xi_i^f, \eta(x_0, x^*) \rangle - \nu^* G'(\nu^* h(x^*, y_i)) \langle \xi_i^h, \eta(x_0, x^*) \rangle \Big) + \sum_{j=1}^{m} \mu_j^* G'_{g_j}(g_j(x^*)) \langle \xi_j^g, \eta(x_0, x^*) \rangle < 0$$

or

$$\left\langle \sum_{i=1}^{q^*} \lambda_i^* \Big(G'(f(x^*, y_i)) \xi_i^f - \nu^* G'(\nu^* h(x^*, y_i)) \xi_i^h \Big) + \sum_{j=1}^{m} \mu_j^* G'_{g_j}(g_j(x^*)) \xi_j^g, \eta(x_0, x^*) \right\rangle < 0,$$

which implies that

$$0 \notin \sum_{i=1}^{q^*} \lambda_i^* \left[G'(f(x^*, y_i)) \partial_x f(x^*, y_i) - \nu^* G'(\nu^* h(x^*, y_i)) \partial_x h(x^*, y_i) \right] + \sum_{j=1}^{m} \mu_j^* G'_{g_j}(g_j(x^*)) \partial g_j(x^*).$$

This is a contradiction to condition (5). □.

Similar to the proof of Theorem 3.5, we can establish Theorem 3.6. Therefore, we simply state it here.

Theorem 3.6 (G-sufficient optimality conditions). *Let $(x^*, \mu^*, \nu^*, q^*, \lambda^*, \bar{y})$ satisfy conditions (5)-(8); let G be both continuously differentiable and strictly increasing on $\mathbb{R}$; let G_{g_j} be both continuously differentiable and strictly increasing on $I_{g_j}(X)$ for each $j \in M$. If $(G \circ f - G \circ (\nu^* h))(\cdot, y_i)$ is β_i-invex at x^* on E for each $i \in Q^*$ and g_j is (G_j^g, β_j^g)-invex at x^* on E for each $j \in M$, then x^* is an optimal solution to (FP).*

4. Duality

Making use of the optimality conditions of the preceding section, we present dual Problem (D) to the primal one (FP), and establish G-weak, G-strong and G-strict converse duality theorems. For convenience, we use the following notations.

$$K(x) = \{(q, \lambda, \bar{y}) \in \mathrm{N} \times \mathbb{R}_+^q \times \mathbb{R}^{pq} \mid 1 \le q \le n+1, \lambda = (\lambda_1, \cdots, \lambda_q) \in \mathbb{R}_+^q \text{with} \textstyle\sum_{i=1}^q \lambda_i = 1,$$
$$\bar{y} = (y_1, \cdots y_q) \text{ with } y_i \in Y(x), i = 1, \cdots q\}.$$

$H_1(q, \lambda, \bar{y})$ denotes the set of all triplets $(z, \mu, \nu) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+$ satisfying

$$0 \in \sum_{i=1}^{q} \lambda_i \left[G'(f(z, y_i)) \partial_x f(z, y_i) - \nu G'(\nu h(z, y_i)) \partial_x h(z, y_i) \right] + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(z)) \partial g_j(z), \tag{14}$$

$$f(z, y_i) \ge \nu h(z, y_i), i \in Q, \tag{15}$$

$$\mu_j g_j(z) \ge 0, j \in M, \tag{16}$$

$$y_i \in Y(z), \ (q, \lambda, \bar{y}) \in K(z). \tag{17}$$

Our dual problem (D) can be stated as follows.

$$\text{(D)} \max_{(q, \lambda, \bar{y}) \in K(z)} \sup_{(z, \mu, \nu) \in H_1(q, \lambda, \bar{y})} \nu.$$

Note that if $H_1(q, \lambda, \bar{y})$ is empty for some triplet $(q, \lambda, \bar{y}) \in K(z)$, then define $\sup_{(z,\mu,\nu) \in H_1(q, \lambda, \bar{y})} \nu = -\infty$.

Theorem 4.1 (G-weak duality). *Let x and $(z, \mu, \nu, q, \lambda, \bar{y})$ be (FP)-feasible and (D)-feasible, respectively; let G be both continuously differentiable and strictly increasing on $\mathbb{R}$; let G_{g_j} be both continuously differentiable and strictly increasing on $I_{g_j}(X)$ for each $j \in M$. Suppose that $f(\cdot, y_i)$ and $\nu h(\cdot, y_i)$ are (G, β_i)-invex and (G, β_i)-incave at z on E, respectively, for each $i \in Q$. If g_j is (G_j^g, β_j^g)-invex at z on E for each $j \in M$, then*

$$\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geqslant \nu.$$

Proof. Suppose to the contrary that $\sup_{y\in Y} \frac{f(x,y)}{h(x,y)} < v$. Therefore, we obtain

$$\frac{f(x,y)}{h(x,y)} < v, \forall\, y \in Y.$$

Thus, we obtain from the monotonicity assumption of G that

$$G \circ f(x,y) - G \circ (vh)(x,y) < 0,\ \ y \in Y.$$

This, together with (15), follows that

$$G \circ f(x,y_i) - G \circ f(z,y_i) - (G \circ (vh)(x,y_i) - G \circ (vh)(z,y_i)) < 0,$$

where $y_i \in Y(z)$. Again, we obtain from the monotonicity assumptions of G_{g_j} for $j \in M$ and the fact

$$g_j(x) \leq 0,\ \ \mu_j g_j(z) \geq 0,\ \ \mu_j \geq 0,\ \ j \in M$$

that

$$G_{g_j} \circ g_j(x) \leq G_{g_j} \circ g_j(z), j \in M.$$

Hence

$$\sum_{i=1}^{q} \lambda_i \frac{G \circ f(x,y_i) - G \circ f(z,y_i) - (G \circ (vh)(x,y_i) - G \circ (vh)(z,y_i))}{\beta_i(x,z)} + \sum_{j=1}^{m} \mu_j \frac{G_{g_j} \circ g_j(x) - G_{g_j} \circ g_j(z)}{\beta_j^g(x,z)} < 0. \tag{18}$$

Similar to the proof of Theorem 3.5, by (18) and the generalized invexity assumptions of $f(\cdot, y_i)$, $vh(\cdot, y_i)$ and g_j, we have

$$\left\langle \sum_{i=1}^{q} \lambda_i (G'(f(z,y_i))\xi_i^f - vG'(vh(z,y_i))\xi_i^h) + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(z))\xi_j^g, \eta(x,z) \right\rangle < 0,$$

which follows that

$$0 \notin \sum_{i=1}^{q} \lambda_i \left[G'(f(z,y_i))\partial_x f(z,y_i) - vG'(vh(z,y_i))\partial_x h(z,y_i)\right] + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(z))\partial g_j(z).$$

Thus, we have a contradiction to (14). So $\sup_{y\in Y} \frac{f(x,y)}{h(x,y)} \geqslant v$. □

Theorem 4.2 (G-strong duality). *Let Problem (FP) satisfy Conditions 3.1 and 3.2 as defined in this paper; let x^* be an optimal solution of Problem (FP). Suppose that G is both continuously differentiable and strictly increasing on $\mathbb{R}$, G_{g_j} is both continuously differentiable and strictly increasing on $I_{g_j}(X)$ with $G'_{g_j}(g_j(x^*)) > 0$ for each $j \in M$. If the hypothesis of Theorem 4.1 holds for all (D)-feasible points $(z, \mu, v, q, \lambda, \bar{y})$, then there exist $(q^*, \lambda^*, \bar{y}^*) \in K(z)$ and $(x^*, \mu^*, v^*) \in H_1(q^*, \lambda^*, \bar{y}^*)$ such that $(q^*, \lambda^*, \bar{y}^*, x^*, \mu^*, v^*)$ is a (D) optimal solution, and the two problems (FP) and (D) have the same optimal values.*

Proof. By Theorem 3.4, there exists $v^* = \frac{f(x^*, y_i^*)}{h(x^*, y_i^*)}$, $i = 1, \cdots, q^*$, satisfying the requirements specified in the theorem, such that $(q^*, \lambda^*, \bar{y}^*, x^*, \mu^*, v^*)$ is a (D) feasible solution, then the optimality of this feasible solution for (D) follows from Theorem 4.1. □

Theorem 4.3 (G-strict converse duality). *Let $\bar{x}$ and $(z, \mu, \nu, q, \lambda, \bar{y})$ be optimal solutions of (FP) and (D), respectively. Suppose that G is both continuously differentiable and strictly increasing on $\mathbb{R}$, G_{g_j} is both continuously differentiable and strictly increasing on $I_{g_j}(X)$ for each $j \in M$. Suppose that $f(\cdot, y_i)$ and $\nu h(\cdot, y_i)$ are (G, β_i)-invex and (G, β_i)-incave at z on E, respectively, for each $i \in Q$. If g_j is (G_j^g, β_j^g)-invex at z on E for each $j \in M$, then $\bar{x} = z$; that is, z is a (FP)-optimal solution and $\sup_{y\in Y} \frac{f(\bar{x}, y)}{h(\bar{x}, y)} = \nu$.*

Proof. Suppose to the contrary that $\bar{x} \neq z$. Using similar arguments as in the proof of Theorem 3.5, we deduce that there exist $\xi_i^f \in \partial_z f(z, y_i)$, $\xi_i^h \in \partial_z h(z, y_i)$ and $\xi_j^g \in \partial g(z)$ for $i \in Q$, $j \in M$ such that

$$
\begin{aligned}
0 &= \left\langle \sum_{i=1}^{q} \lambda_i \left(G'(f(z, y_i))\xi_i^f - \nu G'(\nu h(z, y_i))\xi_i^h \right) + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(z))\xi_j^g, \eta(\bar{x}, z) \right\rangle \\
&< \sum_{i=1}^{q} \lambda_i \frac{G \circ f(\bar{x}, y_i) - G \circ f(z, y_i) - (G \circ (\nu h)(\bar{x}, y_i) - G \circ (\nu h)(z, y_i))}{\beta_i(\bar{x}, z)} + \sum_{j=1}^{m} \mu_j \frac{G_{g_j} \circ g_j(\bar{x}) - G_{g_j} \circ g_j(z)}{\beta_j^g(\bar{x}, z)}
\end{aligned}
$$

and

$$
\sum_{j=1}^{m} \mu_j \frac{G_{g_j} \circ g_j(\bar{x}) - G_{g_j} \circ g_j(z)}{\beta_j^g(\bar{x}, z)} \leq 0.
$$

Therefore,

$$
\sum_{i=1}^{q} \lambda_i \frac{G \circ f(\bar{x}, y_i) - G \circ f(z, y_i) - (G \circ (\nu h)(\bar{x}, y_i) - G \circ (\nu h)(z, y_i))}{\beta_i(\bar{x}, z)} > 0.
$$

From the above inequality, we can conclude that there exists $i_0 \in Q$, such that

$$
G \circ f(\bar{x}, y_{i_0}) - G \circ f(z, y_{i_0}) - (G \circ (\nu h)(\bar{x}, y_{i_0}) - G \circ (\nu h)(z, y_{i_0})) > 0,
$$

or

$$
G \circ f(\bar{x}, y_{i_0}) - G \circ (\nu h)(\bar{x}, y_{i_0}) > G \circ f(z, y_{i_0}) - G \circ (\nu h)(z, y_{i_0}) \geq 0.
$$

Now, by the monotonicity of G, we have

$$
f(\bar{x}, y_{i_0}) > \nu h(\bar{x}, y_{i_0}).
$$

It follows that

$$
\sup_{y\in Y} \frac{f(\bar{x}, y)}{h(\bar{x}, y)} \geq \frac{f(\bar{x}, y_{i_0})}{h(\bar{x}, y_{i_0})} > \nu. \tag{19}
$$

On the other hand, we know from Theorem 4.1 that

$$
\sup_{y\in Y} \frac{f(\bar{x}, y)}{h(\bar{x}, y)} = \nu.
$$

This contradicts to (19). □

5. Conclusion

In this paper, we have discussed the applications of (G, β)-invexity for a class of nonsmooth minimax fractional programming Problem (FP). We not only established G-optimality conditions but also constructed a dual model (D) and derived G-duality results between Problems (FP) and (D). More exactly, we construct an important auxiliary minimax programming problem to deal with the nonsmooth minimax fractional programming Problem (FP) addressed in this paper and obtain G-necessary optimality conditions for this Problem (FP). Under the nondifferential (G, β)-invexity assumptions, we have also derived the sufficiency of the G-necessary optimality conditions for the same problem. Further, we have constructed a dual model (D) and derived G-duality results between Problems (FP) and (D). Note that many researchers are interesting in dealing with the minimax programming under generalized invexity assumptions; see [1, 10, 11, 14–17, 19]. However, we have not found results for minimax fractional programming problems under the assumptions of G-invexity and its extension. Hence, this work extends the applications of G-invexity to the generalized minimax fractional programming as well as to the nonsmooth case.

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References

[1] Antczak, T.: Generalized fractional minimax programming with B-(p, r)-invexity, Comput. Math. Appl., **56**(6), 1505-1525 (2008).
[2] Hanson, M. A.: On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl., **80**, 545-550 (1981).
[3] Ben-Israel, A., Mond, B.: What is invexity? J. Aust. Math. Soc. Series B, **28**, 1-9 (1986).
[4] Antczak, T.: New optimality conditions and duality results of G-type in differentiable mathematical programming, Nonlinear Anal., **66**, 1617-1632 (2007).
[5] Antczak, T.: On G-invex multiobjective programming. Part I. Optimality, J. Global Optim., **43(1)**, 97-109 (2009).
[6] Antczak, T.: On G-invex multiobjective programming. Part II. Duality, J. Global Optim., **43(1)**, 111-140 (2009).
[7] Antczak, T.: (p, r)-invex sets and functions, J. Math. Anal. Appl. **80**, 545-550 (2001)
[8] Preda, V.: On Sufficiency and Duality for Multiobjective Programs, J. Math. Anal. Appl., **166**, 365-377 (1992)
[9] Liang, Z.A., Huang, H.X., Pardalos, P.M.: Optimality Conditions and Duality for a Class of Nonlinear Fractional Programming Problems, J. Optim. Theory Appl., **110**, 611-619 (2001)
[10] Yuan, D.H., Liu, X.L, Chinchuluun, A., Pardalos, P.M.: Nondifferentiable minimax fractional programming problems with (C, α, ρ, d)-convexity, J. Optim. Theory Appl., **129(1)**, 185-199 (2006)
[11] Stefanescu, M. V., Stefanescu, A.: Minimax programming under new invexity assumptions, Revue Roumaine de Mathematiques Pures et Appliquees, 52(3), 367-376 (2007).
[12] Antczak, T.: The notion of V-r-invexity in differentiable multiobjective programming, J. Appl. Anal., **11**(1), 63-79 (2005).
[13] Lai, H.C., Lee, J.C.: On duality theorems for a nondifferentiable minimax fractional programming, J. Comput. Appl. Math., **146**(1), 115-126 (2002).
[14] Antczak, T.: Minimax programming under (p, r)-invexity, European J. Oper. Res., **158**(1), 1-19 (2004).
[15] Ahmad, I., Gupta, S., Kailey, N., Agarwal, R.P.: Duality in nondifferentiable minimax fractional programming with B-(p, r)-invexity, J. Inequal. Appl., 2011(75), 1-14 (2011).
[16] Antczak, T.: Nonsmooth minimax programming under locally Lipschitz (ϕ, ρ)-invexity, Appl. Math. Comput., **217**(23), 9606-9624 (2011).
[17] Mishra, S. K., Shukla K.: Nonsmooth minimax programming problems with V-r-invex functions, Optimization 59(1), 95-103 (2010).
[18] Yuan, D. H., Liu, X. L., Yang, S. Y., Lai,G.M.: Nondifferential mathematical programming involving (G, β)-invexity, J. Inequal. Appl., 2012(256), 1-17 (2012).
[19] Yuan, D. H., Liu, X. L.: Generalized minimax programming with nondifferentiable (G, β)-invexity, J. Appl. Math., ID 854125, 1-7 (2013).
[20] I. Ahmad, Second order nondifferentiable minimax fractional programming with square root terms, Filomat, 27, 135-142 (2013).
[21] A. Jayswal, A. K. Prasad, K. Kummari, On nondifferentiable minimax fractional programming involving higher order generalized convexity, Filomat, 27, 1497- 1504 (2013).
[22] Clarke, F.: Optimization and nonsmooth analysis. John Wiley, New York, 1983.
[23] Noor, M.: On generalized preinvex functions and monotonicities, J. Inequal. Pure Appl. Math., 5(4), 1-9 (2004).
[24] Studniarski, M., Taha, A.W.A.: A characterization of strict local minimizers of order one for nonsmooth static minmax problems, J. Math. Anal. Appl., **259**(2), 368-376 (2001).