Classification of Ricci Semisymmetric Contact Metric Manifolds

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Abstract. The object of the present paper is to study Ricci semisymmetric contact metric manifolds. As a consequence of the main result we deduce some important corollaries. Besides these we study contact metric manifolds satisfying the curvature condition $Q.R = 0$, where Q and R denote the Ricci operator and curvature tensor respectively. Also we study the symmetric properties of a second order parallel tensor in contact metric manifolds. Finally, we give an example to verify the main result.

1. Introduction

Among Riemannian manifolds, the most interesting and most important for applications are the symmetric ones. From the local point of view they were introduced independently by Shirokov [29] and Levy [18] as a Riemannian manifold with covariant constant curvature tensor R, i. e., satisfying

$$\nabla R = 0, \tag{1}$$

where ∇ is the Levi-Civita connection. An extensive theorey of symmetric Riemannian manifolds was worked out by Cartan in 1927.
In 1962, Okumura proved that a Sasakian manifold which is locally symmetric has constant curvature 1 [21]. One way of viewing this result is to say that a Sasakian structure and a locally symmetric metric are incompatible. Since locally symmetry is such a central notion in Riemannian geometry, one would like to have some meaningful notion of symmetry also in contact geometry.
As a generalization of symmetric manifolds Cartan in 1946 introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric if the curvature tensor satisfies

$$R(X, Y).R = 0, \tag{2}$$

where $R(X, Y)$ is considered as a field of linear operators, acting on R. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semisymmetric Riemannian manifolds were first studied by E. Cartan, A. Lichnerowicz, R. S. Couty and

N. S. Sinjukov.

A Riemannian manifold is said to be Ricci semisymmetric if the curvature tensor satisfies

$$R(X, Y).S = 0, \tag{3}$$

where $R(X, Y)$ is considered as a field of linear operators, acting on R and S is the Ricci tensor of type $(0, 2)$. The class of Ricci semisymmetric manifolds includes the set of Ricci symmetric manifolds ($\nabla S = 0$) as a proper subset. Ricci-semisymmetric manifolds were investigated by several authors. Every semisymmetric manifold is Ricci semisymmetric. The converse statement is not true. However, under some additional assumptions (2) and (3) are equivalent.

Semisymmetric manifolds were classified by Szabó, locally in [28]. The classification results of Szabó were presented in the book [12]. A fundamental study on Riemannian semisymmetric manifolds was made by Z. I. Szabó [28], E. Boeckx et al [12] and O. Kowalski [17].

Let (M, g) be an n-dimensional Riemannian manifold with the metric g. A tensor field T of type $(0, q)$ is said to be recurrent [22] if the relation

$$(\nabla_X T)(Y_1, Y_2, .., Y_q)T(Z_1, Z_2, .., Z_q) - T(Y_1, Y_2, .., Y_q)(\nabla_X T)(Z_1, Z_2, .., Z_q) = 0,$$

holds on (M, g). From the definition, it follows that if at a point $x \in M$, $T(x) \neq 0$, then on some nbd. of x, there exists a unique 1-form A satisfying

$$(\nabla_X T)(Y_1, Y_2, .., Y_q) = A(X)T(Y_1, Y_2, .., Y_q).$$

Patterson [20] introduced the notion of Ricci-recurrent manifold. According to Patterson, a Riemannian manifold (M, g) of dimension n, is called Ricci-recurrent if $S \neq 0$ and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z),$$

for some non-zero 1-form A.

Curvature properties of birecurrent semi-Riemannian manifolds were introduced by W. Roter in [23]. A Riemannain manifold is said to be Ricci birecurrent if there exists a non-zero covariant tensor field B such that $(\nabla_X \nabla_W S - \nabla_{\nabla_X W} S)(Y, Z) = B(X, W)S(Y, Z)$. Ricci birecurrent Riemannian manifold have been studied by Chaki et al. [14]. In a recent paper [1] Aikawa and Matsuyama proved that if a tensor field T is birecurrent, then $R(X, Y).T = 0$. Therefore Ricci birecurrent Riemannian manifold implies Ricci semisymmetric.

An example of a curvature condition of semisymmetry type is the following

$$Q.R = 0, \tag{4}$$

where Q is the Ricci operator of type $(1, 1)$ and $S(X, Y) = g(QX, Y)$.

A natural extension of such curvature conditions form curvature conditions of pseudosymmetry type. The curvature condition $Q.R = 0$ have been studied by Verstraelen et al. in [31]. The main result of [31] is the following:

Theorem 1.1. *[31] Let M be a hypersurface in a Euclidean space E^{n+1}, $n \geq 3$. Then* (4) *holds on M if and only if M is a hypercylinder.*

Recently, Sharma and Koufogiorgos [24] studied locally symmetric and Ricci symmetric contact metric manifolds of dimension greater than 3, by assuming certain conditions on the curvature and Ricci curvature along the characteristic vector field of the contact structure. Contact metric manifolds have also been studied by several authors such as ([2], [6], [7], [8], [9], [10], [11], [13], [16]) and many others. Motivated by these studies we study in this paper Ricci semisymmetric contact metric manifolds such that $\nabla_\xi h = 0$ and the sectional curvature of any plane section containing ξ equals to a constant c. Besides these we study contact metric manifolds satisfying the curvature condition $Q.R = 0$ under the same conditions, where Q and R denote the Ricci operator and curvature tensor respectively.

In 1925 Levy [18] proved that a second order symmetric parallel non-singular tensor on a space of constant

curvature is a constant multiple of the metric tensor. In recent papers Sharma ([25], [26], [27]) generalized Levy's result and also studied a second order parallel tensor on Kaehler space of constant holomorphic sectional curvature as well as on contact manifolds . In 1996 De [15] studied second order parallel tensors on $P-$Sasakian manifolds. Recently, Mondal, De and Özgür [19] studied Second order parallel tensors on (k, μ)-contact metric manifolds. Motivated by the above studies we study contact metric manifolds admitting a second order symmetric parallel tensor under certain conditions.

The present paper is organized as follows:

After preliminaries in section 3, we consider Ricci semisymmetric contact metric manifolds such that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals to a constant c and prove that under such a condition either $c = 0$ or it is an Einstein manifold. As a consequence of the result we deduce some important corollaries. Section 4 deals with contact metric manifolds satisfying the curvature condition $Q.R = 0$ under the same conditions. Section 5 is devoted to study contact metric manifolds admitting a second order symmetric parallel tensor under the some conditions mentioned in the previous section. Finally, we give an example to verify the main result.

2. Contact Metric Manifolds

In this section, we collect the basic formulas and results which we need about contact metric manifolds. Full details can be found in ([3], [4]). All manifolds are assume to be connected and smooth.

A contact manifold is by definition an odd dimensional manifold M^{2n+1} equipped with a global 1-form satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere. It is well-known that there exists a unique vector field ξ, the characteristic vector field for which $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. Further, one can find an associated Riemannian metric g and a vector field ϕ of type $(1, 1)$ such that

$$\eta(X) = g(X, \xi), d\eta(X, Y) = g(X, \phi Y), \phi^2 X = -X + \eta(X)\xi \tag{5}$$

where X and Y are vector fields on M. From (5) it follows that

$$\phi\xi = 0, \eta \circ \phi = 0, g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{6}$$

The manifold M^{2n+1} together with the structure tensor (η, ξ, ϕ, g) is called a contact metric manifold.

Given the contact metric manifold (M, η, ξ, ϕ, g), we define a symmetric $(1,1)$-tensor field h as $h = \frac{1}{2}L_\xi\phi$, where $L_\xi\phi$ denotes Lie differentiation in the direction of ξ. We have the following identities:

$$h\xi = 0, \quad h\phi + \phi h = 0, \tag{7}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{8}$$

$$\nabla_\xi \phi = 0, \tag{9}$$

$$R(\xi, X)\xi - \phi R(\xi, \phi X)\xi = 2(h^2 + \phi^2)X, \tag{10}$$

$$(\nabla_\xi h)X = \phi X - h^2\phi X + \phi R(\xi, X)\xi, \tag{11}$$

$$S(\xi, \xi) = 2n - trh^2, \tag{12}$$

$$R(X, Y)\xi = -(\nabla_X \phi)Y + (\nabla_Y \phi)X - (\nabla_X \phi h)Y + (\nabla_Y \phi h)X. \tag{13}$$

Here, ∇ is the Levi-Civita connection and R the Riemannian curvature tensor of (M, g) with the sign convention

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for vector fields X, Y, Z on M. The tensor $l = R(., \xi)\xi$ is the Jacobi operator with respect to the characteristic field ξ.

If the characteristic vector field ξ is a Killing vector field , the contact metric space (M, η, ξ, ϕ, g) is called K-contact manifold. This is the case if and only if $h = 0$. Finally, if the Riemann curvature tensor satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

or, equivalently, if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

holds, then the manifol is Sasakian. We note that a Sasakian manifold is always K-contact, but the converse only holds in dimension three.

We need the following Lemma which was obtained by Blair in [5]

Lemma 2.1. *[5] If a contact metric manifold is locally symmetric, then* $\nabla_\xi h = 0$

It follows from Lemma 2.1 and formula (2.7) that

$$R(\xi, X)\xi = (h^2 + \phi^2)X. \tag{14}$$

Now, if the sectional curvature $K(\xi, X)$ is equal to a constant c, then it is not hard to see that $h^2 = (1-c)(I - \eta \otimes \xi)$ and

$$R(\xi, X)\xi = c\{-X + \eta(X)\xi\} \tag{15}$$

3. Ricci Semisymmetry on Contact Metric Spaces

This section deals with the study of Ricci semisymmetric contact metric manifolds. Let us assume that $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals to a constant c. Therefore we have

$$R(\xi, X)\xi = c\{-X + \eta(X)\xi\}. \tag{16}$$

Now,

$$(R(X, Y).S)(U, V) = 0, \tag{17}$$

implies

$$S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \tag{18}$$

Putting $Y = U = \xi$ in (18) implies

$$S(R(X, \xi)\xi, V) + S(\xi, R(X, \xi)V) = 0. \tag{19}$$

Using (16) in (19) yields

$$c\{-S(X, V) + \eta(X)S(\xi, V)\} + S(\xi, R(X, \xi)V) = 0. \tag{20}$$

Again putting $V = \xi$ in (20) yields

$$c\{-S(X, \xi) + \eta(X)S(\xi, \xi)\} + S(\xi, R(X, \xi)\xi) = 0. \tag{21}$$

In view of (12), (16) and (21) we have

$$c\{S(X, \xi) - \eta(X)(2n - trh^2)\} = 0. \tag{22}$$

Therefore either $c = 0$ or, $S(X, \xi) = \eta(X)(2n - trh^2)$.
For $c \neq 0$, using $S(X, \xi) = \eta(X)(2n - trh^2)$ in (20) we have

$$S(X, V) = (2n - trh^2)g(X, V). \tag{23}$$

Thus in view of the above result we can state the following:

Theorem 3.1. *Let a contact metric manifold M^{2n+1} ($n > 1$) satisfy $\nabla_\xi h = 0$ and the sectional curvature of any plane section containing ξ equals to a non-zero constant c. If the manifold is Ricci semisymmetric, then it is an Einstein manifold.*

Again for an Einstein manifold, $R.S = 0$ holds. That is, Einstein manifold implies Ricci semisymmetric manifold. Thus we can state the following:

Corollary 3.2. *Let a contact metric manifold M^{2n+1} ($n > 1$) satisfy $\nabla_\xi h = 0$ and the sectional curvature of any plane section containing ξ equals to a non-zero constant c. Then the manifold is Ricci semisymmetric if and only if it is an Einstein manifold.*

Moreover if $\nabla S = 0$, then $R.S = 0$, that is, Ricci parallelity implies Ricci semisymmetry . Therefore we can state the following:

Corollary 3.3. *Let a contact metric manifold M^{2n+1} ($n > 1$) satisfy $\nabla_\xi h = 0$ and the sectional curvature of any plane section containing ξ equals to non-zero constant c. If the manifold is Ricci parallel, then it is an Einstein manifold.*

Also $R.R = 0$ implies $R.S = 0$, that is, semisymmetry implies Ricci semisymmetry. Therefore under the same conditions mentioned in the Corollary 3.2 we can state the following:

Corollary 3.4. *A semisymmetric contact metric manifold is an Einstein manifold.*

On the other hand, it is known that Ricci birecurrent Riemannian manifold is always Ricci semisymmetric ([1]). Thus we can state the following:

Corollary 3.5. *Let a contact metric manifold M^{2n+1} ($n > 1$) satisfy $\nabla_\xi h = 0$ and the sectional curvature of any plane section containing ξ equals to non-zero constant c. If the manifold is Ricci birecurrent, then it is an Einstein manifold.*

Patterson [20] introduced the notion of Ricci-recurrent manifolds. According to Patterson, a manifold (M, g) of dimension n, is called Ricci-recurrent if the Ricci tensor is non-zero and satisfies the condition

$$\nabla S = A \otimes S, \tag{24}$$

for some non-zero 1-form A. Now we prove that (24) implies $R.S = 0$.
For this, we now define a function f on M^{2n+1} by

$$f^2 = g(Q, Q),$$

where $g(QX, Y) = S(X, Y)$ and the Riemannian metric g is extended to the inner product between the tensor fields in the standard fashion. Then we obtain

$$f(Yf) = f^2 \alpha(Y).$$

So from this we have

$$Yf = f\alpha(Y). \tag{25}$$

From (25) we have

$$X(Yf) - Y(Xf) = [X\alpha(Y) - Y\alpha(X)]f. \tag{26}$$

Therefore, we get

$$[\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}]f = [X\alpha(Y) - Y\alpha(X) - \alpha[X, Y]]f. \tag{27}$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M^{2n+1} by our assumption, we obtain

$$d\alpha(X, Y) = 0, \tag{28}$$

that is, the 1-form α is closed. Now from $(\nabla_Y S)(U, V) = \alpha(Y)S(U, V)$, we get

$$(\nabla_X \nabla_Y S)(U, V) = [X\alpha(Y) + \alpha(X)\alpha(Y)]S(U, V).$$

Hence from (28) we get

$$(R(X, Y) \cdot S)(U, V) = 2d\alpha(X, Y)S(U, V).$$

That is, our manifold satisfies $R(X, Y) \cdot S = 0$.
Thus a Ricci-recurrent manifold is Ricci semi-symmetric.
Hence from Corollary 3.4, we can state the following:

Corollary 3.6. *Let a contact metric manifold M^{2n+1} ($n > 1$) satisfy $\nabla_\xi h = 0$ and the sectional curvature of any plane section containing ξ equals to a non-zero constant c. If the manifold is Ricci recurrent, then it is an Einstein manifold.*

4. Contact metric manifolds satifying $Q.R = 0$

This section is devoted to study contact metric manifolds satisfying $Q.R = 0$. Therefore

$$(Q.R)(X, Y)Z = 0. \tag{29}$$

This implies

$$Q(R(X, Y)Z) - R(QX, Y)Z - R(X, QY)Z - R(X, Y)QZ = 0, \tag{30}$$

where X, Y, Z are vector fields on M.
Putting $X = Z = \xi$ in (30) we have

$$Q(R(\xi, Y)\xi) - R(Q\xi, Y)\xi - R(\xi, QY)\xi - R(\xi, Y)Q\xi = 0. \tag{31}$$

Let X be the eigen vector of Q corresponding to the eigen value λ and $X \perp \xi$. Therefore

$$QX = \lambda X, \quad X \neq 0. \tag{32}$$

It follows that

$$Q\xi = \lambda\xi. \tag{33}$$

Using $R(\xi, X)\xi = c\{-X + \eta(X)\xi\}$ and (32), (33) in (31) we have

$$\lambda c[-Y + \eta(Y)\xi] - 3\lambda c[-Y + \eta(Y)\xi] = 0. \tag{34}$$

Again since Y is orthogonal to ξ, that is, $\eta(Y) = 0$, from (34) we have

$$2\lambda cY = 0. \tag{35}$$

Therefore either $c = 0$ or $\lambda = 0$. Assume that $c \neq 0$, then from (32), $QX = 0$ and hence $S(X, Y) = 0$.
Thus in view of the above result we can state the following:

Theorem 4.1. *Let a contact metric manifold M^{2n+1} ($n > 1$) satisfy $\nabla_\xi h = 0$ and the sectional curvature of any plane section containing ξ equals to a non-zero constant c. If such a contact metric manifold satisfies $Q.R = 0$ having X is an eigen vector of Q corresponding to eigen value λ, then the manifold is Ricci flat.*

5. Second Order Symmetric Parallel Tensor in Contact Metric Manifolds

Definition 5.1. *A tensor α of second order is said to be a parallel tensor if $\nabla\alpha = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.*

Let α be a $(0,2)$-symmetric tensor field on a contact metric manifold M such that $\nabla\alpha = 0$. Then it follows that

$$\alpha(R(W,X)Y,Z) + \alpha(Y,R(W,X)Z) = 0, \tag{36}$$

for all vector fields W, X, Y, Z.

Substitution of $W = Y = Z = \xi$ in (36) which gives us

$$\alpha(R(\xi,X)\xi,\xi) = 0, \tag{37}$$

since α is symmetric. Using (15) in (37) we have

$$c\{g(X,\xi)\alpha(\xi,\xi) - \alpha(\xi,X)\} = 0. \tag{38}$$

Therefore either $c = 0$, or

$$\{g(X,\xi)\alpha(\xi,\xi) - \alpha(\xi,X)\} = 0. \tag{39}$$

Differentiating (39) covariantly along Y, we get

$$g(\nabla_Y X,\xi)\alpha(\xi,\xi) + g(X,\nabla_Y\xi)\alpha(\xi,\xi) + 2g(X,\xi)\alpha(\nabla_Y\xi,\xi) - \alpha(\nabla_Y X,\xi) - \alpha(X,\nabla_Y\xi) = 0. \tag{40}$$

Changing X by $\nabla_Y X$ in (40) yields

$$g(\nabla_Y X,\xi)\alpha(\xi,\xi) - \alpha(\nabla_Y X,\xi) = 0. \tag{41}$$

From (40) and (41) it follows that

$$g(X,\nabla_Y\xi)\alpha(\xi,\xi) + 2g(X,\xi)\alpha(\nabla_Y\xi,\xi) - \alpha(X,\nabla_Y\xi) = 0. \tag{42}$$

Using $\nabla_Y\xi = -\phi X - \phi hX$ in (42) we have

$$-g(X,\phi Y)\alpha(\xi,\xi) - g(X,\phi hY)\alpha(\xi,\xi) - 2g(X,\xi)\alpha(\phi Y,\xi)$$
$$-2g(X,\xi)\alpha(\phi hY,\xi) + \alpha(X,\phi Y) + \alpha(X,\phi hY) = 0. \tag{43}$$

Substituting $Y = \phi Y$ in (39) we obtain

$$\alpha(\xi,\phi Y) = 0. \tag{44}$$

Thus from (43) and (44) we have

$$- \quad g(X,\phi Y)\alpha(\xi,\xi) - g(X,\phi hY)\alpha(\xi,\xi) + \alpha(X,\phi Y) + \alpha(X,\phi hY) = 0. \tag{45}$$

Using $Y = \phi Y$ in (45) we get

$$\{g(X,hY) - g(\phi X,\phi Y)\}\alpha(\xi,\xi) \quad = \quad -\alpha(X,Y) + \eta(Y)\alpha(X,\xi) + \alpha(X,hY). \tag{46}$$

Putting $Y = hY$ in (46) we have

$$\{-g(X,hY) + g(X,h^2 Y)\}\alpha(\xi,\xi) = -\alpha(X,hY) + \alpha(X,h^2 Y). \tag{47}$$

In view of $h^2 = (1-c)(I - \eta\otimes\xi)$, (47) gives us

$$\{g(X,hY) - g(\phi X,\phi Y)\}\alpha(\xi,\xi) \quad = \quad -cg(\phi X,\phi Y)\alpha(\xi,\xi) + \alpha(X,hY)$$
$$-(1-c)\alpha(X,Y) + (1-c)\eta(Y)\alpha(X,\xi). \tag{48}$$

Thus from (46) and (48) we have

$$c\{g(\phi X, \phi Y)\alpha(\xi, \xi) - \alpha(X, Y) + \eta(Y)\alpha(X, \xi)\} = 0. \tag{49}$$

Since $c \neq 0$, from (49) it follows that

$$\alpha(X, Y) - \alpha(\xi, \xi)g(X, Y) - \{\alpha(X, \xi) - \alpha(\xi, \xi)g(X, \xi)\}\eta(Y) = 0. \tag{50}$$

Again substituting $Y = \phi Y$ in (44), we obtain

$$\alpha(Y, \xi) = \eta(Y)\alpha(\xi, \xi). \tag{51}$$

Thus from (50) and (51) we have

$$\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y). \tag{52}$$

In view of the above result we can state the following:

Theorem 5.2. *Let a contact metric manifold M^{2n+1} ($n > 1$) have $\nabla_\xi h = 0$ and the sectional curvature of the plane section containing ξ equals to a non-zero constant c. If the contact metric manifold admits a second order symmetric parallel tensor then the second order symmetric parallel tensor is a constant multiple of the associated metric tensor.*

   Application: We consider the Ricci parallel contact metric manifold. Then $\nabla S = 0$. Hence from Theorem 5.2, we obtain the Corollary 3.3.

6. Example

   In a K-contact manifold $h = 0$ and hence $\nabla_\xi h = 0$ holds. Also in a K-contact manifold the sectional curvature of the plane section containing ξ equals to 1 [3]. That is, the sectional curvature of the plane section containing ξ is a non-zero constant. Therefore from Corollary 3.3 we conclude that a K-contact manifold is Ricci semisymmetric if and only if the manifold is an Einstein manifold.
The above result was proved by S. Tanno [30] in another way.

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