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Second Hankel Determinant Problem for k-bi-starlike Functions

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Abstract. In this paper we introduce and study some properties of k-bi-starlike functions defined by making use of the Sălăgean derivative operator. Upper bounds on the second Hankel determinant for k-bi-starlike functions are investigated. Relevant connections of the results presented here with various well-known results are briefly indicated.

1. Introduction

As usual, we denote by A the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$.

We also denote by S the subclass of A consisting of functions which are univalent in U. Let $f^{-1}(z)$ be the inverse function of $f(z)$, defined by

$$f^{-1}(f(z)) = z \quad (z \in U) \text{ and } f(f^{-1}(w)) = w \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ...$$

A function $f \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. We denote by σ the class of all functions $f(z)$ which are bi-univalent in U.

Brannan et al. [2] introduced certain subclasses of the bi-univalent function class σ similar to the familiar subclasses $S^*(\beta)$ and $K(\beta)$ of starlike and convex function of order β $(0 \leq \beta < 1)$, respectively (see [9]). For a brief history of functions in the class σ, see the work of Srivastava et al. [19]. In fact, judging by the remarkable flood of papers on the subject ([1], [5], [8], [11], [15]-[18], [20], [21], [23]), the pioneering work

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by Srivastava et al. [19] appears to have revived the study of analytic and bi-univalent functions in recent years. By definition, we have

$$S^*(\beta) = \left\{ f \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta;\ 0 \le \beta < 1,\ z \in U \right\}$$

and

$$K(\beta) = \left\{ f \in S : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta;\ 0 \le \beta < 1,\ z \in U \right\}.$$

The classes $S^*_\sigma(\beta)$ and $K_\sigma(\beta)$ of bi-starlike functions of order β and bi-convex functions of order β, corresponding to the function classes $S^*(\beta)$ and $K(\beta)$, were also considered analogously.

The q^{th} Hankel determinant for $n \ge 0$ and $q \ge 1$ was stated by Noonan et al. ([10]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

We note that $H_2(1) = a_3 - a_2^2$ is well-known as Fekete-Szegö functional (see [4]). For our discussion in the present paper, we examine the Hankel determinant in the case $q = 2$ and $n = 2$, $H_2(2) = a_2a_4 - a_3^2$. We will try to find upper bound for the functional $H_2(2) = a_2a_4 - a_3^2$ for the functions f belonging to the class $S_{\sigma,k}(\beta)$ of k-bi-starlike functions.

For a function $f(z) \in A$, we define

$$\begin{aligned} D^0 f(z) &= f(z); \\ D^1 f(z) &= Df(z) = zf'(z); \\ &\ \vdots \\ D^k f(z) &= D(D^{k-1} f(z)) \ \ (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ where } \mathbb{N} = \{1, 2, 3, ...\}). \end{aligned}$$

The differential operator D^k was considered by Sălăgean [13].

With the help of this differential operator, Sălăgean [13] also defined the class of k-starlike functions of order β $(0 \le \beta < 1)$ defined by

$$S_k(\beta) = \{ f \in A : \Re\left(\frac{D^{k+1} f(z)}{D^k f(z)}\right) > \beta,\ \ z \in U \}.$$

Kanas et al. [7] obtained more general results for $k-$uniformly convex functions by using parameter k. Certain well-known subclasses of S are indeed special cases of $S_k(\beta)$ for suitable choices of parameters k and β. We remark that for $k = 0$, $S_0(\beta) \equiv S(\beta)$ and for $k = 1$, $S_1(\beta) \equiv K(\beta)$ are classes of starlike functions of order β and convex functions of order β, respectively.

Definition 1.1. *A function $f \in \sigma$ is said to be in the class $S_{\sigma,k}(\beta)$, if the following conditions are satisfied:*

$$\Re\left(\frac{D^{k+1} f(z)}{D^k f(z)}\right) > \beta;\ 0 \le \beta < 1,\ z \in U \tag{2}$$

and

$$\Re\left(\frac{D^{k+1} g(w)}{D^k g(w)}\right) > \beta;\ 0 \le \beta < 1,\ w \in U \tag{3}$$

where $g(w) = f^{-1}(w)$.

We remark that for $k = 0$ the class $S_{\sigma,0}(\beta) \equiv S^*_\sigma(\beta)$ is the class of bi-starlike functions of order β. When $k = 1$, $S_{\sigma,1}(\beta) \equiv K_\sigma(\beta)$ is the class of bi-convex functions of order β. Our main interest focus on the class $S_{\sigma,k}(\beta)$ of k-bi-starlike functions.

The purpose of this note is to find upper bound for the functional $H_2(2) = a_2a_4 - a_3^2$ for functions f belonging to the class $S_{\sigma,k}(\beta)$.

Now we recall the following lemmas which will be required in our next investigation.

Lemma 1.2. *[12] If $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + ...$ is an analytic function in U with positive real part, then*

$$|p_n| \le 2, \ \ and \ \ |p_2 - \frac{p_1^2}{2}| \le 2 - \frac{|p_2|^2}{2} \ (n \in \mathbb{N}).$$

Lemma 1.3. *[6] If the function $p \in P$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2); \ 4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

2. Main Results

One of our main results is contained in

Theorem 2.1. *Let f given by (1) be in the class $S_{\sigma,k}(\beta)$, $0 \le \beta < 1$. Then, for $k = 1, 2, 3$*

$$|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{2^{2k}}\left[\frac{2^{2k}}{3^{2k}} - \frac{3.2^kM^2}{3^{2k}N}\right]$$

and for $k = 0$ and for every $k \ge 4 (k \in \mathbb{N})$

$$|a_2a_4 - a_3^2| \le \begin{cases} \frac{(1-\beta)^2}{3^{2k+1}2^{5k}}\left[N + 6.2^{2k}M + 3.2^{5k}\right], & \beta \in [0, \beta_1'] \\ \frac{(1-\beta)^2}{2^{2k}}\left[\frac{2^{2k}}{3^{2k}} - \frac{3.2^kM^2}{3^{2k}N}\right], & \beta \in (\beta_1', 1) \end{cases}$$

where

$$\begin{aligned} M &= \left\{6^k + 2.3^{2k} - 2^{3k} - 6^k\beta\right\}, \\ N &= 16.3^{2k}.(3.2^k + 2^{2k} - 3^{k+1})(1-\beta)^2 - 6.3^k.2^{3k}(1-\beta) + 3.2^{5k} - 8.2^{2k}.3^{2k} \end{aligned}$$

and

$$\beta_1' = \frac{3.2^{k+5} + 2^{2k+5} - 2^{3k}3^{1-k} - 32.3^{k+1} - (\frac{2}{3})^k\sqrt{9.2^{4k} + 2^{2k+7}3^{2k} + 2^{k+7}3^{2k+1} - 128.3^{3k+1}}}{2(3.2^{k+4} + 2^{2k+4} - 16.3^{k+1})}.$$

Proof. Let $f \in S_{\sigma,k}(\beta)$. Then

$$\frac{D^{k+1}f(z)}{D^kf(z)} = \beta + (1-\beta)p(z) \tag{4}$$

$$\frac{D^{k+1}g(w)}{D^kg(w)} = \beta + (1-\beta)q(w) \tag{5}$$

where $p, q \in P$ and $g = f^{-1}$. Thus, after some calculations, it follows from (4) and (5) that

$$a_2 = \frac{1-\beta}{2^k}p_1, \tag{6}$$

$$a_3 = \frac{(1-\beta)^2}{2^{2k}}p_1^2 + \frac{1-\beta}{4.3^k}(p_2 - q_2) \tag{7}$$

and

$$a_4 = \frac{(3^{k+1}-2^{2k})(1-\beta)^3}{3.2^{4k}}p_1^3 + \frac{5(1-\beta)^2}{8.6^k}p_1(p_2-q_2) + \frac{(1-\beta)}{6.4^k}(p_3-q_3). \tag{8}$$

Then, we can establish that

$$\begin{aligned} |a_2a_4 - a_3^2| &= |-\frac{(1-\beta)^4(3^{k+1}-2^{2k}-3.2^k)}{3.2^{5k}}p_1^4 \\ &+\frac{(1-\beta)^3}{8.12^k}p_1^2(p_2-q_2) + \frac{(1-\beta)^2}{6.2^{3k}}p_1(p_3-q_3) - \frac{(1-\beta)^2}{16.9^k}(p_2-q_2)^2|. \end{aligned} \tag{9}$$

Making use of Lemma 1.3, we have

$$p_2 - q_2 = \frac{4-p_1^2}{2}(x-y) \tag{10}$$

and

$$p_3 - q_3 = \frac{p_1^3}{2} + \frac{(4-p_1^2)p_1}{2}(x+y) - \frac{(4-p_1^2)p_1}{4}(x^2+y^2) + \frac{4-p_1^2}{2}\left[(1-|x|^2)z - (1-|y|^2)w\right]. \tag{11}$$

Then, by using equations (10) and (11) in (9) we may set

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq (\frac{(1-\beta)^4(3.2^k+2^{2k}-3^{k+1})}{3.2^{5k}} + \frac{(1-\beta)^2}{12.3^k})p_1^4 + \frac{(1-\beta)^2}{6.2^{3k}}p_1(4-p_1^2) \\ &+\left[\frac{(1-\beta)^2}{6.2^{3k}}p_1^2\frac{(4-p_1^2)}{2} + \frac{(1-\beta)^3}{8.12^k}p_1^2\frac{(4-p_1^2)}{2}\right](|x|+|y|) \\ &+\left[\frac{(1-\beta)^2}{6.2^{3k}}p_1^2\frac{(4-p_1^2)}{4} - \frac{(1-\beta)^2}{6.2^{3k}}p_1\frac{(4-p_1^2)}{2}\right](|x|^2+|y|^2) \\ &+\frac{(1-\beta)^2}{16.9^k}\frac{(4-p_1^2)^2}{4}(|x|+|y|)^2. \end{aligned} \tag{12}$$

Since $p \in P$, so $|p_1| \leq 2$. Letting $|p_1| = p$, we may assume without restriction that $p \in [0,2]$. For $\eta = |x| \leq 1$ and $\mu = |y| \leq 1$, we get

$$|a_2a_4 - a_3^2| \leq T_1 + (\eta+\mu)T_2 + (\eta^2+\mu^2)T_3 + (\eta+\mu)^2T_4 = G(\eta,\mu)$$

where

$$\begin{aligned} T_1 &= T_1(p) = \frac{(1-\beta)^2}{3.2^{3k}}\left[\left((1-\beta)^2\frac{(3.2^k+2^{2k}-3^{k+1})}{2^{2k}} + \frac{1}{4}\right)p^4 - \frac{p^3}{2} + 2p\right] \geq 0 \\ T_2 &= T_2(p) = \frac{(1-\beta)^2p^2(4-p^2)}{2^{2k+2}}\left[\frac{1}{3.2^k} + \frac{(1-\beta)}{4.3^k}\right] \geq 0 \\ T_3 &= T_3(p) = \frac{(1-\beta)^2p(4-p^2)(p-2)}{24.2^{3k}} \leq 0 \\ T_4 &= T_4(p) = \frac{(1-\beta)^2}{16.9^k}.\frac{(4-p^2)^2}{4} \geq 0. \end{aligned}$$

We now need to maximize the function $G(\eta,\mu)$ on the closed region $[0,1]\times[0,1]$.

Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $p \in [0,2)$, we conclude that $G_{\eta\eta}G_{\mu\mu} - (G_{\eta\mu})^2 < 0$.

Thus the function G can't have a local maximum in the interior of the region. Now, we investigate the maximum value of G on the boundary of the region.

For $\eta = 0$ and $0 \leq \mu \leq 1$ (similarly $\mu = 0$ and $0 \leq \eta \leq 1$), we obtain $G(0, \mu) = H(\mu) = (T_3 + T_4)\mu^2 + T_2\mu + T_{1.}$

Case 1: $T_3 + T_4 \geq 0$: In this case for $0 \leq \mu \leq 1$ and any fixed p with $0 \leq p < 2$, it's clear that $H'(\mu) = 2(T_3 + T_4)\mu + T_2 > 0$, that is, $H(\mu)$ is increasing function. Hence, for fixed $p \in [0, 2)$, the maximum of $H(\mu)$ occurs at $\mu = 1$, and $\max H(\mu) = H(1) = T_1 + T_2 + T_3 + T_4$.

Case 2: $T_3 + T_4 < 0$: Since $T_2 + 2(T_3 + T_4) \geq 0$ for $0 < \mu < 1$ and any fixed p with $0 \leq p < 2$,it is clear that $T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\mu + T_2 < T_2$ and so $H'(\mu) > 0$. Hence for fixed $p \in [0, 2)$, the maximum of $H(\mu)$ occurs at $\mu = 1$.

Also for $p = 2$ we obtain

$$G(\eta, \mu) = \frac{(1-\beta)^2}{3.2^{3k}} \left[\frac{(1-\beta)^2 (3.2^k + 2^{2k} - 3^{k+1})}{2^{2k-4}} + 4 \right]. \tag{13}$$

Taking into consideration the value (13), and the cases 1 and 2, for $0 \leq \mu \leq 1$ and any fixed p with $0 \leq p \leq 2$, $\max H(\mu) = H(1) = T_1 + T_2 + T_3 + T_4$.

For $\eta = 1$ and $0 \leq \mu \leq 1$ (similarly $\mu = 1$ and $0 \leq \eta \leq 1$), we have $G(1, \mu) = F(\mu) = (T_3 + T_4)\mu^2 + (T_2 + 2T_4)\mu + T_1 + T_2 + T_3 + T_4$.

Similarly to the above cases of $T_3 + T_4$, we get that $\max F(\mu) = F(1) = T_1 + 2T_2 + 2T_3 + 4T_{4.}$

Since $H(1) \leq F(1)$ for $p \in [0, 2]$, $\max G(\eta, \mu) = G(1, 1)$ on the boundary of the region. Thus, the maximum value of G occurs at $\eta = 1$ and $\mu = 1$ in the closed region.

Let $K : [0, 2] \to R$

$$K(p) = \max G(\eta, \mu) = G(1, 1) = T_1 + 2T_2 + 2T_3 + 4T_{4.} \tag{14}$$

Substituting the values of T_1, T_2, T_3 and T_4 in the function K defined by (14), yields

$$K(p) = \frac{(1-\beta)^2}{2^{2k}} \left\{ \frac{N}{48.2^{3k}3^{2k}} p^4 + \frac{M}{3^{2k}2^{k+1}} p^2 + \frac{2^{2k}}{3^{2k}} \right\}.$$

Assume that $K(p)$ has a maximum value in an interior of $p \in [0, 2]$, by elementary calculations, we arrive at

$$K'(p) = \frac{(1-\beta)^2}{2^{2k}} \left\{ \frac{N}{12.2^{3k}3^{2k}} p^3 + \frac{M}{2^k 3^{2k}} p \right\}.$$

Setting $K'(p) = 0$, we have the real critical points $p_{0_1} = 0$ and $p_{0_2} = \sqrt{\frac{-12.2^{2k}M}{N}}$.

It can be showed easily that M is a positive real number for every $\beta \in [0, 1)$ and for every $k \in \mathbb{N}$. That is, $M > 0$.

Besides, by using Mathematica Program we can obtain that one of roots of equation $N = 0$ is

$$\begin{aligned} \beta_1 \quad = \quad & \frac{3.2^{k+5} + 2^{2k+5} - 2^{3k+1}3^{1-k} - 32.3^{k+1}}{2(3.2^{k+4} + 2^{2k+4} - 16.3^{k+1})} \\ & - \frac{2.3^{-2k}\sqrt{2^{4k+7}3^{4k} - 2^{7k+4}3^{2k+1} - 5.2^{6k}3^{2k+3} + 2^{5k+4}3^{3k+2} + 2^{3k+7}3^{4k+1} - 2^{2k+7}3^{5k+1}}}{2(3.2^{k+4} + 2^{2k+4} - 16.3^{k+1})}. \end{aligned}$$

As a result of some calculations we can deduce that N is a negative real number for every $\beta \in [0, 1)$ and for $k = 1, 2, 3, 4, 5$ (see Figure 1) but N is not always a negative real number for $k \geq 6 (k \in \mathbb{N})$ and for some values of $\beta \in [0, 1)$. Also, if below Figure 1 is scrutinized, we can conclude that N is a negative real number for every $\beta \in [0, 1)$ and for $k = 1, 2, 3, 4, 5$.

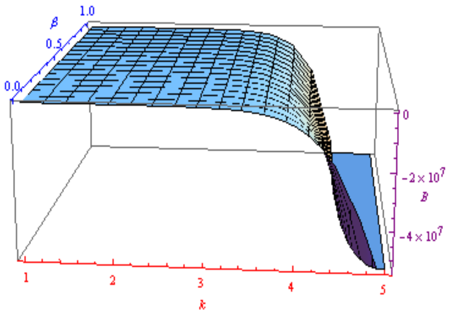

Figure 1: We can see that N is a negative real number for k=1,2,3,4,5.

We can do the following examine in consequence of above explanations:

First all, let $k = 1, 2, 3$. In this case $M > 0$ and $N < 0$ for every $\beta \in [0, 1)$. Since $p_{0_2} < 2$ $(k = 1, 2, 3)$ for every $\beta \in [0, 1)$ and so $K''(p_{0_2}) < 0$, the maximum value of $K(p)$ corresponds to $p = p_{0_2}$, that is,

$$\max_{0 \le p \le 2} K(p) = K(p_{0_2}) = \frac{(1-\beta)^2}{2^{2k}} \left[\frac{2^{2k}}{3^{2k}} - \frac{3.2^k M^2}{3^{2k} N} \right].$$

Consequently, since $K(0) < K(2) \le K(p_{0_2})$ we obtain $\max K(p) = K(p_{0_2})$.

Now, let $k = 4, 5$. In this case, we can deduce that for some values of $\beta \in [0, 1)$ is $p_{0_2} \ge 2$ (see Figure 2). If Figure 2 is analyzed , we conclude that for $k > 3$ $(k \in \mathbb{N})$ and for some value of $\beta \in [0, 1)$ is $p_{0_2} \ge 2$ or $p_{0_2} < 2$.

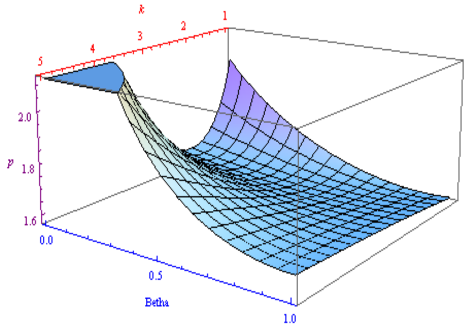

Figure 2: We can observe that for $k > 3$ and some values of β are $p_{0_2} \ge 2$ or $p_{0_2} < 2$.

Case 1: If $\beta \in [0, \beta_1']$ then $p_{0_2} \ge 2$, that is, p_{0_2} is out of the interval $(0, 2)$. Therefore, the maximum value of $K(p)$ occurs at $p = p_{0_1}$ or $p = p_{0_2}$ which contradicts our assumption of having the maximum value at the interior point of $p \in [0, 2]$. Since K is an increasing function in the interval $[0, 2]$, maximum point of K must be on the boundary of $p \in [0, 2]$, that is, $p = 2$. Thus, we have

$$\max_{0 \le p \le 2} K(p) = K(2) = \frac{(1-\beta)^2}{3^{2k+1} 2^{5k}} \left[N + 6.2^{2k} M + 3.2^{5k} \right].$$

Case 2: When $\beta \in (\beta_1', 1)$ we observe that $p_{0_2} \le 2$, that is, p_{0_2} is interior of the interval $[0, 2]$. Since $K''(p_{0_2}) < 0$, the maximum value of $K(p)$ occurs at $p = p_{0_2}$. Thus, we have

$$\max_{0 \le p \le 2} K(p) = K(p_{0_2}) = \frac{(1-\beta)^2}{2^{2k}} \left[\frac{2^{2k}}{3^{2k}} - \frac{3.2^k M^2}{3^{2k} N} \right].$$

Finally, we examined the cases of $k \geq 6$ (and $k = 0$) in below, in this case, we see that N is negative real number for $\beta \in (\beta_1, 1)$ (see Figure 3).Thus, p_{0_2} is a real number.

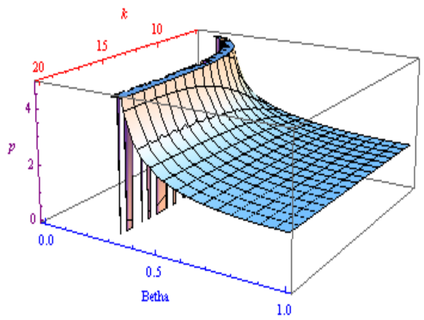

Figure 3: It can be showed both p and β for values of $k \geq 6$.

Therefore, there are two cases;

Case 1: For $N \geq 0$, that is, $\beta \in [0, \beta_1)$.Therefore, $K'(p) > 0$ for $p \in (0, 2)$. Since K is an increasing function in the interval $(0, 2)$, maximum point of K must be on the boundary of $p \in [0, 2]$, that is, $p = 2$. Thus, we have

$$\max_{0 \leq p \leq 2} K(p) = K(2) = \frac{(1-\beta)^2}{3^{2k+1} 2^{5k}} \left[N + 6.2^{2k} M + 3.2^{5k} \right].$$

Case 2: When $\beta \in [\beta_1, \beta_1']$, we observe that $p_{0_2} \geq 2$, that is, p_{0_2} is out of the interval $(0, 2)$. Therefore, the maximum value of $K(p)$ occurs at $p_{0_1} = 0$ or $p = p_{0_2}$ which contradicts our assumption of having the maximum value at the interior point of $p \in [0, 2]$. Since K is an increasing function in the interval $[0, 2]$, maximum point of K must be on the boundary of $p \in [0, 2]$, that is, $p = 2$. Thus, we have

$$\max_{0 \leq p \leq 2} K(p) = K(2) = \frac{(1-\beta)^2}{3^{2k+1} 2^{5k}} \left[N + 6.2^{2k} M + 3.2^{5k} \right].$$

When $\beta \in (\beta_1', 1)$, we observe that $p_{0_2} \leq 2$, that is, p_{0_2} is interior of the interval $[0, 2]$. Since $K''(p_{0_2}) < 0$, the maximum value of $K(p)$ occurs at $p = p_{0_2}$. Thus, we have

$$\max_{0 \leq p \leq 2} K(p) = K(p_{0_2}) = \frac{(1-\beta)^2}{2^{2k}} \left[\frac{2^{2k}}{3^{2k}} - \frac{3.2^k M^2}{3^{2k} N} \right].$$

We thus have completed our proof of Theorem 2.1. □

Corollary 2.2. *[3]Let f given by (1) be in the class* $S^*_\sigma(\beta)$ *and* $0 \leq \beta < 1$. *Then*

$$|a_2 a_4 - a_3^2| \leq \begin{cases} \frac{4(1-\beta)^2}{3}(4\beta^2 - 8\beta + 5), & \beta \in [0, \frac{29-\sqrt{137}}{32}) \\ (1-\beta)^2(\frac{13\beta^2 - 14\beta - 7}{16\beta^2 - 26\beta + 5}), & \beta \in (\frac{29-\sqrt{137}}{32}, 1) \end{cases}$$

Corollary 2.3. *[3]Let f given by (1) be in the class* $K_\sigma(\beta)$ *and* $0 \leq \beta < 1$. *Then*

$$|a_2 a_4 - a_3^2| \leq \frac{(1-\beta)^2}{24} \left(\frac{5\beta^2 + 8\beta - 32}{3\beta^2 - 3\beta - 4} \right).$$

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