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Maps Preserving 2-Idempotency of Certain Products of Operators

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Abstract. Let $\mathcal{A}, \mathcal{B}$ be standard operator algebras on complex Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ of dimensions at least 3, respectively. In this paper we give the general form of a surjective (not assumed to be linear or unital) map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\Phi_2 : M_2(\mathbb{C}) \otimes \mathcal{A} \to M_2(\mathbb{C}) \otimes \mathcal{B}$ defined by $\Phi_2((s_{ij})_{2\times 2}) = (\Phi(s_{ij}))_{2\times 2}$ preserves nonzero idempotency of Jordan product of two operators in both directions. We also consider another specific kinds of products of operators, including usual product, Jordan semi-triple product and Jordan triple product. In either of these cases it turns out that Φ is a scalar multiple of either an isomorphism or a conjugate isomorphism.

1. Introduction

The study of (linear or nonlinear) maps leaving invariant certain functions, subsets or relations, is one of the most active research topics in Matrix theory, operator algebras and operator spaces and has attracted the attention of many authors for the past years. One of such problems concerns with the set of idempotents in operator algebras. Completely preserving idempotents between standard operator algebras on real or complex Banach spaces have been considered in [3] and it was shown that such map is either an isomorphism or a conjugate isomorphism.

For a complex Banach space $\mathcal{X}$ with $\dim X \geq 3$, in [2] Fang gave a description of a surjective linear map ϕ on $B(\mathcal{X})$ which preserves nonzero idempotency of Jordan products of operators in one direction. He showed that if X is infinite dimensional, then either there exist a bounded invertible (conjugate) linear operator $A : \mathcal{X} \to \mathcal{X}$ and a constant $\lambda \in \{1, -1\}$ such that

$$\phi(T) = \lambda ATA^{-1} \quad (T \in \mathcal{B}(\mathcal{X})), \tag{1}$$

or (when $\mathcal{X}$ is reflexive) there exist a bounded invertible (conjugate) linear operator $A : \mathcal{X}^* \to \mathcal{X}$ and a constant $\lambda \in \{1, -1\}$ such that

$$\phi(T) = \lambda AT^*A^{-1} \qquad (T \in \mathcal{B}(\mathcal{X})). \tag{2}$$

On the other hand in an earlier paper [7], Wang, Fang and Ji obtained the concrete form of a linear map ϕ on $\mathcal{B}(\mathcal{X})$ which preserves the nonzero idempotency of either product ST or semi-triple Jordan product STS of two operators in one direction. They proved that when X is infinite dimensional, in the case of usual

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product, ϕ has the form (1) and in the case of semi-triple Jordan product, ϕ is either of the form (1) or (2) for a constant $\lambda \in \mathbb{C}$ with $\lambda^3 = 1$.

It may be interesting to find conditions under which the above mentioned results are valid for not necessarily linear maps. As the authors know, the results of [1] and [6] are the only known results of this type. In [1] Fang, Ji and Pang, described the concrete form of unital surjective maps on $\mathcal{B}(\mathcal{X})$ that preserve nonzero idempotency of products of two operators in both directions. In a more general treatment in [6] Petek, among others, characterized unital maps on $\mathcal{B}(\mathcal{X})$ which preserve nonzero idempotency of sequential product (or $\star$-product) of operators $A_1, A_2, ..., A_n \in B(\mathcal{X})$ defined by

$$A_1 \star A_2 \star ... \star A_n = A_{i_1} A_{i_2} ... A_{i_m},$$

where $i_1, i_2, ..., i_m \in \{1, 2, ..., n\}$ are fixed, in both direction. Since the usual product and the semi-triple Jordan product (which is called Jordan triple product in [6]) are sequential, Petek's results provides the description of unital maps on $\mathcal{B}(\mathcal{X})$ preserving nonzero idempotency of product and semi-triple Jordan product of two operators in both directions.

It is not well known that if the assumption of being unital, is sufficient to describe the maps satisfying similar properties for the other kinds of products, which are not sequential such as Jordan product. However, considering a similar approach as in [3], in this paper we consider another substitution for the linearity assumption which works for specific kinds of products, including Jordan product without assuming that the map is unital. More precisely, we give a description of a surjective map (not assumed to be linear or unital) $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ between standard operator algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\Phi_2 : M_2(\mathbb{C}) \otimes \mathcal{A} \to M_2(\mathbb{C}) \otimes \mathcal{B}$ defined by $\Phi_2((s_{ij})_{2\times 2}) = (\Phi(s_{ij}))_{2\times 2}$ preserves nonzero idempotency of each of usual products, Jordan products, Jordan semi-triple products and Jordan triple products of operators in both directions.

2. Preliminaries and the Statement of Main Results

Let $\mathcal{X}$ be a complex Banach space with dual space $\mathcal{X}^*$. For each $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, $x \otimes f$ is the rank-one operator defined by $(x \otimes f)y = f(y)x$ on $\mathcal{X}$. We denote the Banach space of all bounded linear operators from $\mathcal{X}$ to a complex Banach space $\mathcal{Y}$ by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and we set $\mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X}, \mathcal{X})$. For $n \in \mathbb{N}$, $M_n(\mathbb{C})$ is the space of $n \times n$ matrices with complex entries.

By a *standard operator algebra* on a Banach space $\mathcal{X}$ we mean a subalgebra of $\mathcal{B}(\mathcal{X})$ which contains the identity and the ideal of all finite rank operators. For each $n \in \mathbb{N}$, let $I_n^*(\mathcal{X})$ be the set of all nonzero idempotents in $M_n(\mathbb{C}) \otimes \mathcal{B}(\mathcal{X})$. For the case where $n = 1$ we use $I^*(\mathcal{X})$ for $I_1^*(\mathcal{X})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Given a map $\Phi : \mathcal{A} \to \mathcal{B}$ and $n \in \mathbb{N}$ let $\Phi_n : M_n(\mathbb{C}) \otimes \mathcal{A} \to M_n(\mathbb{C}) \otimes \mathcal{B}$ be defined by $\Phi_n((s_{ij})_{n\times n}) = (\Phi(s_{ij}))_{n\times n}$.

We recall that Jordan products, Jordan semi-triple products and Jordan triple products of operators are products of the form $ST + TS$, STS and $STU + UTS$, respectively, for $S, T, U \in M_n(\mathbb{C}) \otimes \mathcal{A}$. We note that each $(T_{ij}) \in M_n(\mathbb{C}) \otimes \mathcal{A}$, $n \in \mathbb{N}$, can be considered as an operator in $\mathcal{B}(\mathcal{X}^n)$, where $\mathcal{X}^n$ is equipped with the norm $\|(x_1, ..., x_n)\| = \|x_1\| + ... + \|x_n\|$.

We should note that if S, T are linear operators on a complex vector space $\mathcal{U}$ such that $\operatorname{Ker} T \subseteq \operatorname{Ker} S$ and for all $u \in \mathcal{U}$, Su and Tu are linearly dependent, then by [4, Lemma 1.1 and Remark 1.2] we have $S \in \mathbb{C}T$.

Let $\star$ denote one of the usual product, Jordan product, Jordan semi-triple product or Jordan triple product. The following theorems are the main results of this paper.

Theorem 2.1. *Let $\mathcal{X}, \mathcal{Y}$ be infinite dimensional complex Banach spaces and $\mathcal{A}, \mathcal{B}$ be standard operator algebras on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a surjective map satisfying*

$$S \star T \in I_2^*(\mathcal{X}) \Leftrightarrow \Phi_2(S) \star \Phi_2(T) \in I_2^*(\mathcal{Y})$$

for all $S, T \in M_2(\mathbb{C}) \otimes \mathcal{A}$. Then there exist an invertible bounded linear or conjugate linear operator $A : \mathcal{X} \longrightarrow \mathcal{Y}$ and a scalar λ such that

$$\Phi(T) = \lambda A T A^{-1} \qquad (T \in \mathcal{A}),$$

where in the case of product or Jordan product $\lambda \in \{1, -1\}$ and in the case of Jordan semi-triple product or Jordan triple product $\lambda^3 = 1$.

Theorem 2.2. *Let* $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$*,* $n \geq 3$*, be a surjective map satisfying*

$$S \star T \in I_2^*(\mathbb{C}^n) \Leftrightarrow \Phi_2(S) \star \Phi_2(T) \in I_2^*(\mathbb{C}^n)$$

for all $S, T \in M_2(\mathbb{C}) \otimes M_n(\mathbb{C})$*. Then there exist an invertible matrix* $A \in M_n(\mathbb{C})$*, an automorphism* $\tau : \mathbb{C} \to \mathbb{C}$ *and a scalar* $\lambda \in \mathbb{C}$ *such that*

$$\Phi(T) = \lambda A T_\tau A^{-1} \qquad (T \in M_n(\mathbb{C})),$$

where $T_\tau = (\tau(t_{ij}))$ *for* $T = (t_{ij})$*. Moreover, in the case of product or Jordan product* $\lambda \in \{1, -1\}$ *and in the case of Jordan semi-triple product or Jordan triple product* $\lambda^3 = 1$*.*

3. Proofs of Main Results

In this section we assume that $\mathcal{X}$ and $\mathcal{Y}$ are complex Banach spaces with dimensions at least 3, $\mathcal{A}$ and $\mathcal{B}$ are standard operator algebras on $\mathcal{X}$ and $\mathcal{Y}$, respectively and $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective map, not assumed to be linear or unital satisfying

$$S \star T \in I_2^*(\mathcal{X}) \Leftrightarrow \Phi_2(S) \star \Phi_2(T) \in I_2^*(\mathcal{Y}) \quad (S, T \in M_2(\mathbb{C}) \otimes \mathcal{A})$$

where $\star$ denotes one of the usual product, Jordan product, Jordan semi-triple product or Jordan triple product.

Lemma 3.1. $\Phi(0) = 0$*.*

Proof. We first assume that $\star$ is the Jordan product. Assume by the way of contradiction that $\Phi(0) \neq 0$ and set

$$N = \begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix}.$$

We note that $N^2 \neq 0$, since otherwise there exist $y \in \mathcal{Y}$ and $f \in \mathcal{Y}^*$ with $f(\Phi(0)^2(y)) = 0$ and $f(\Phi(0)(y)) = 1$ and hence

$$\begin{aligned} &\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ y \otimes f & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ y \otimes f & 0 \end{pmatrix}\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix} \\ &= \begin{pmatrix} \Phi(0)(y) \otimes f & 0 \\ \Phi(0)(y) \otimes f + y \otimes f\Phi(0) & y \otimes f\Phi(0) \end{pmatrix} \in I_2^*(\mathcal{Y}), \end{aligned}$$

which implies, by the surjectivity of Φ, that $0 \in I_2^*(\mathcal{X})$, a contradiction. Hence $N^2 \neq 0$. We now show that $N^2 = cN$ for some nonzero scalar c. If there exists $y = (y_1, y_2) \in \mathcal{Y}^2$ such that Ny and N^2y are linearly independent, then $\Phi(0)^2(y_1 + y_2)$ and $\Phi(0)(y_1 + y_2)$ are linearly independent and we can choose $f \in \mathcal{Y}^*$ with $f(\Phi(0)^2(y_1 + y_2)) = 0$ and $f(\Phi(0)(y_1 + y_2)) = 1$. Therefore,

$$\begin{aligned} &\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ (y_1 + y_2) \otimes f & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (y_1 + y_2) \otimes f & 0 \end{pmatrix}\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix} \\ &= \begin{pmatrix} \Phi(0)(y_1 + y_2) \otimes f & 0 \\ \Phi(0)(y_1 + y_2) \otimes f + (y_1 + y_2) \otimes f\Phi(0) & (y_1 + y_2) \otimes f\Phi(0) \end{pmatrix} \in I_2^*(\mathcal{Y}), \end{aligned}$$

and so, by the surjectivity of Φ, we have $0 \in I_2^*(\mathcal{X})$ which is impossible. Thus for every $y \in \mathcal{Y}^2$, Ny and N^2y are linearly dependent. Since $\mathrm{Ker\,N} \subseteq \mathrm{Ker\,N}^2$, Lemma 1.1 in [4] implies that $N^2 = cN$ for some $c \in \mathbb{C}$. Being $N^2 \neq 0$ we have $c \neq 0$ and consequently

$$\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix}\begin{pmatrix} \frac{1}{2c}I & 0 \\ 0 & \frac{1}{2c}I \end{pmatrix} + \begin{pmatrix} \frac{1}{2c}I & 0 \\ 0 & \frac{1}{2c}I \end{pmatrix}\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix} = \frac{1}{c}N \in I_2^*(\mathcal{Y}),$$

which leads again to the contradiction $0 \in I_2^*(\mathcal{X})$. Thus we have $\Phi(0) = 0$.

Now let $\star$ be the usual product and $\Phi(0) \neq 0$. Then there exists $y \in \mathcal{Y}$ and $f \in \mathcal{Y}^*$ such that $f(\Phi(0)y) = 1$. Since

$$\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y \otimes f & 0 \end{pmatrix} = \begin{pmatrix} \Phi(0)y \otimes f & 0 \\ \Phi(0)y \otimes f & 0 \end{pmatrix} \in I_2^*(\mathcal{Y}),$$

the hypothesis implies that $0 \in I_2^*(\mathcal{X})$ which is a contradiction.

Assume now that $\star$ denotes Jordan semi-triple product and $\Phi(0) \neq 0$. Choose arbitrary $y \in \mathcal{Y}$ with $\Phi(0)y \neq 0$. If y and $\Phi(0)y$ are linearly independent, then choosing $f_1 \in \mathcal{Y}^*$ with $f_1(y) = f_1(\Phi(0)y) = 1$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & y \otimes f_1 \end{pmatrix} \begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & y \otimes f_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \otimes f_1 \end{pmatrix} \in I_2^*(\mathcal{Y})$$

which leads to $0 \in I_2^*(\mathcal{X})$, a contradiction. Therefore, y and $\Phi(0)y$ are linearly dependent for all $y \in \mathcal{Y}$. Since $\operatorname{Ker} I \subseteq \operatorname{Ker} \Phi(0)$, Lemma 1.1 in [4] implies that $\Phi(0)y = cy$, $y \in \mathcal{Y}$, for some $c \in \mathbb{C}$ which is clearly nonzero. Choosing $\lambda \in \mathbb{C}$ with $\lambda^2 = c$ and $f_2 \in \mathcal{Y}^*$ with $f_2(y) = \frac{1}{\lambda}$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & y \otimes f_2 \end{pmatrix} \begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & y \otimes f_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda y \otimes f_2 \end{pmatrix} \in I_2^*(\mathcal{Y}),$$

which again leads to the contradiction $0 \in I_2^*(\mathcal{X})$. Thus $\Phi(0) = 0$.

The case where $\star$ denotes Jordan triple product has a similar argument. $\square$

Remark. Using the above lemma we can show that Φ preserves multiplicatively nonzero idempotents, that is

$$A \star B \in I^*(\mathcal{X}) \Leftrightarrow \Phi(A) \star \Phi(B) \in I^*(\mathcal{Y})$$

for all $A, B \in \mathcal{A}$. For example in the case where $\star$ denotes the Jordan product, since $\Phi(0) = 0$, for each $A, B \in \mathcal{A}$ we have $AB + BA \in I^*(X)$ if and only if

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in I_2^*(\mathcal{X})$$

and this is equivalent to

$$\begin{pmatrix} \Phi(A) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi(B) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Phi(B) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi(A) & 0 \\ 0 & 0 \end{pmatrix} \in I_2^*(\mathcal{Y})$$

or equivalently

$$\Phi(A)\Phi(B) + \Phi(B)\Phi(A) \in I^*(\mathcal{Y}).$$

The same conclusion holds for the other kinds of products of operators.

Lemma 3.2. *Φ is injective.*

Proof. We prove the lemma for the case where $\star$ is the Jordan semi-triple product, the other cases are proved similarly. For this let $A, B \in \mathcal{A}$ such that $\Phi(A) = \Phi(B)$. Then

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & I \\ A - A^2 & I - A \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \in I_2^*(\mathcal{X}).$$

Therefore, by the hypothesis,

$$\begin{pmatrix} \Phi(I) & 0 \\ 0 & \Phi(I) \end{pmatrix} \begin{pmatrix} \Phi(A) & \Phi(I) \\ \Phi(A - A^2) & \Phi(I - A) \end{pmatrix} \begin{pmatrix} \Phi(I) & 0 \\ 0 & \Phi(I) \end{pmatrix} \in I_2^*(\mathcal{Y}).$$

Since by the assumption $\Phi(A) = \Phi(B)$, it follows that

$$\begin{pmatrix} \Phi(I) & 0 \\ 0 & \Phi(I) \end{pmatrix}\begin{pmatrix} \Phi(B) & \Phi(I) \\ \Phi(A-A^2) & \Phi(I-A) \end{pmatrix}\begin{pmatrix} \Phi(I) & 0 \\ 0 & \Phi(I) \end{pmatrix} \in I_2^*(\mathcal{Y})$$

which is equivalent to

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} B & I \\ A-A^2 & I-A \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \in I_2^*(\mathcal{X}).$$

Thus

$$M = \begin{pmatrix} B & I \\ A-A^2 & I-A \end{pmatrix} \in I_2^*(\mathcal{X}),$$

that is, $M^2 = M$ which implies that $A = B$, as desired. □

Lemma 3.3. *If $\star$ denotes the Jordan product, then Φ preserves zero Jordan products of operators in both directions, that is for $A, B \in \mathcal{A}$,*

$$AB + BA = 0 \Leftrightarrow \Phi(A)\Phi(B) + \Phi(B)\Phi(A) = 0.$$

In particular, Φ is square zero preserving, that is $A^2 = 0$ holds if and only if $\Phi(A)^2 = 0$.

Proof. Let $A, B \in \mathcal{A}$ such that $AB + BA = 0$. Choosing $U, V \in \mathcal{A}$ with $UV + VU \in I^*(\mathcal{X})$ we have

$$\begin{pmatrix} UV+VU & 0 \\ 0 & 0 \end{pmatrix} \in I_2^*(\mathcal{X})$$

and since $AB + BA = 0$ we have

$$\begin{pmatrix} U & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} V & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} UV+VU & 0 \\ 0 & 0 \end{pmatrix} \in I_2^*(\mathcal{X}).$$

Now the hypothesis implies that

$$\begin{pmatrix} \Phi(U) & 0 \\ 0 & \Phi(A) \end{pmatrix}\begin{pmatrix} \Phi(V) & 0 \\ 0 & \Phi(B) \end{pmatrix} + \begin{pmatrix} \Phi(V) & 0 \\ 0 & \Phi(B) \end{pmatrix}\begin{pmatrix} \Phi(U) & 0 \\ 0 & \Phi(A) \end{pmatrix} =$$
$$\begin{pmatrix} \Phi(U)\Phi(V)+\Phi(V)\Phi(U) & 0 \\ 0 & \Phi(A)\Phi(B)+\Phi(B)\Phi(A) \end{pmatrix} \in I_2^*(\mathcal{Y}).$$

So there are two possible cases, either

$$\Phi(A)\Phi(B) + \Phi(B)\Phi(A) \in I^*(\mathcal{Y})$$

or

$$\Phi(A)\Phi(B) + \Phi(B)\Phi(A) = 0.$$

The first case concludes that $0 = AB + BA \in I^*(\mathcal{X})$, a contradiction. Therefore, $\Phi(A)\Phi(B) + \Phi(B)\Phi(A) = 0$, as desired. The converse statement is similar. Clearly this implies that for $A \in \mathcal{A}$, $A^2 = 0$ iff $\Phi(A)^2 = 0$. □

Remark. One can easily check that similar results, as in the above lemma, hold for the case where $\star$ denotes one of usual product, Jordan semi-triple product or Jordan triple product. The last statement for Jordan semi-triple product and Jordan triple product is as follows: $A^3 = 0$ iff $\Phi(A)^3 = 0$ for $A \in \mathcal{A}$.

Lemma 3.4. $\Phi(I) = \lambda I$ *for some scalar $\lambda \in \mathbb{C}$. Moreover, in the cases that $\star$ is the usual product or Jordan product we have $\lambda \in \{1, -1\}$ and in the other two cases $\lambda^3 = 1$.*

Proof. Let $\star$ be the Jordan product. To prove the lemma for this case we first show that $\Phi(I)$ is not a divisor of zero with respect to the Jordan product and $\Phi(I)^3 = \Phi(I)$. We note that since

$$\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\in I_2^*(\mathcal{X}),$$

it follows from the hypothesis that

$$\begin{pmatrix} 0 & \Phi(I) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \Phi(I) & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ \Phi(I) & 0 \end{pmatrix}\begin{pmatrix} 0 & \Phi(I) \\ 0 & 0 \end{pmatrix}\in I_2^*(\mathcal{Y}),$$

and hence

$$\begin{pmatrix} \Phi(I)^2 & 0 \\ 0 & \Phi(I)^2 \end{pmatrix}\in I_2^*(\mathcal{Y}).$$

This clearly implies that $\Phi(I)^2 \in I^*(\mathcal{Y})$. Assume now that $\Phi(I)A + A\Phi(I) = 0$ for some nonzero $A \in \mathcal{B}$. Since Φ is surjective $A = \Phi(A_0)$ for some $A_0 \in \mathcal{A}$. Therefore, $\Phi(I)\Phi(A_0) + \Phi(A_0)\Phi(I) = 0$ and Lemma 3.3 implies that $A_0 = 0$ which is impossible, since $\Phi(0) = 0$. Hence $\Phi(I)$ is not a divisor of zero with respect to the Jordan product. Now set $T = \Phi(I)^3$. Then by the above argument, $\Phi(I)^4 = T\Phi(I) = \Phi(I)^2$, that is $(T - \Phi(I))\Phi(I) = \Phi(I)(T - \Phi(I)) = 0$. This implies that $T = \Phi(I)$, by the first part, that is $\Phi(I)^3 = \Phi(I)$.

We now show that $\Phi(I)$ is a scalar operator. We note that since $\Phi(I)^2$ is an idempotent operator, if $\Phi(I) = \lambda I$ for some scalar λ, then we have necessarily $\lambda \in \{1, -1\}$. So assume by the way of contradiction that $A = \Phi(I)$ is not a scalar operator. Then since $A^3 = A$, applying Molnar's comment [5] at the end of p.297, A can be written as $A = P - Q$ for some orthogonal idempotents P and Q in $\mathcal{B}(\mathcal{Y})$. Obviously, $P, Q \in \mathcal{B}$, since $A^2 = P + Q \in \mathcal{B}$. Now we claim that $A^2 \neq I$. Assume on the contrary that $A^2 = I$. Since A is assumed to be a non-scalar operator and $\operatorname{Ker} I \subseteq A$, it follows that there exists $y_0 \in \mathcal{Y}$ such that y_0 and Ay_0 are linearly independent. Hence there exists $f \in \mathcal{Y}^*$ such that $f(y_0) = 0$ and $f(Ay_0) = 1$. Therefore, $(y_0 \otimes f)^2 = 0 = (y_0 \otimes fA)(Ay_0 \otimes f)$ and consequently

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}\begin{pmatrix} 0 & 0 \\ y_0\otimes f & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ y_0\otimes f & 0 \end{pmatrix}\begin{pmatrix} A & A \\ A & A \end{pmatrix}\in I_2^*(\mathcal{Y}).$$

Hence, choosing $T_0 \in \mathcal{A}$ with $\Phi(T_0) = y_0 \otimes f$ we conclude that

$$\begin{pmatrix} I & I \\ I & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ T_0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ T_0 & 0 \end{pmatrix}\begin{pmatrix} I & I \\ I & I \end{pmatrix}\in I_2^*(\mathcal{X}).$$

This implies that $T_0^2 = T_0$ and since $\Phi(T_0)^2 = (y_0 \otimes f)^2 = 0$ it follows from Lemma 3.3 that $T_0 = T_0^2 = 0$, which is impossible. This argument proves our claim, that is $A^2 \neq I$. Hence $I - P - Q \neq 0$. Choosing a nonzero $y_0 \in (I - P - Q)(\mathcal{Y})$, we can find $g \in \mathcal{Y}^*$ with $g(y_0) = 1$ and $g = 0$ on $P(\mathcal{Y}) \cup Q(\mathcal{Y})$. Clearly $g(Ay) = 0$ for all $y \in \mathcal{Y}$, and since $P(I - P - Q) = 0 = Q(I - P - Q)$ it follows that $Ay_0 = 0$. Therefore, $(y_0 \otimes g)A + A(y_0 \otimes g) = 0$ which is a contradiction, since $A = \Phi(I)$ is not a divisor of zero with respect to the Jordan product. This completes the proof for the case where $\star$ denotes the Jordan product.

Now suppose that $\star$ is the usual product. By the remark after Lemma 3.1, $\Phi(I)^2$ is a nonzero idempotent operator. So it suffices to show that $\Phi(I)$ is a scalar operator. Let $A = \Phi(I)$ and assume on the contrary that A is a non-scalar operator. Then since $\operatorname{Ker} I \subseteq \operatorname{Ker} A$, there exists a vector $y_4 \in \mathcal{Y}$ such that Ay_4 and y_4 are linearly independent. Choosing $f_4 \in \mathcal{Y}^*$ with $f_4(Ay_4) = 1$ and $f_4(y_4) = 0$, we have $Ay_4 \otimes f_4 \in I^*(\mathcal{Y})$. By the remark after Lemma 3.1, $\Phi^{-1}(y_4 \otimes f_4) = \Phi^{-1}(A)\Phi^{-1}(y_4 \otimes f_4) \in I^*(\mathcal{X})$ which is impossible, since $y_4 \otimes f_4$ is square zero (see remark after Lemma 3.3).

Assume now that $\star$ denotes the Jordan semi-triple product. Then clearly $\Phi(I)^3$ is a nonzero idempotent operator. So, as before, it suffices to show that $A = \Phi(I)$ is a scalar operator. If there exists a vector $y_5 \in \mathcal{Y}$ such that A^2y_5 and y_5 are linearly independent, then choosing $f_5 \in \mathcal{Y}^*$ with $f_5(y_5) = 0$ and $f_5(A^2y_5) = 1$, we have $Ay_5 \otimes f_5A \in I^*(\mathcal{Y})$. Therefore, $\Phi^{-1}(y_5 \otimes f_5) \in I^*(\mathcal{X})$ which is impossible, by the remark after Lemma 3.3, since $(y_5 \otimes f_5)^3 = 0$. Thus for every $y \in \mathcal{Y}$, A^2y and y are linearly dependent. Since $\operatorname{Ker} I \subseteq \operatorname{Ker} A^2$, it follows that A^2 is a scalar operator, that is $A^2 = \lambda_0 I$ for some $\lambda_0 \in \mathbb{C}$. Clearly $\lambda_0 \neq 0$, since $A^3 \neq 0$. On the other hand, since $A^3 = \lambda_0 A$ and A^3 is an idempotent operator, we get ${\lambda_0}^3 I = A^6 = A^3 = \lambda_0 A$. Thus $A = {\lambda_0}^2 I$, as desired.

Finally consider the case where $\star$ is the Jordan triple product. As

$$\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}+\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in I_2^*(\mathcal{X}),$$

we have

$$\begin{pmatrix} \Phi(I)^3 & 0 \\ 0 & \Phi(I)^3 \end{pmatrix} \in I_2^*(\mathcal{Y}),$$

which implies that $\Phi(I)^3$ is a nonzero idempotent operator. So we need to show that $A = \Phi(I)$ is a scalar operator. If there exists a vector $y_6 \in \mathcal{Y}$ such that $A^2 y_6$ and y_6 are linearly independent, then choosing $f_6 \in \mathcal{Y}^*$ with $f_6(y_6) = 0$ and $f_6(A^2 y_6) = 1/2$, we get $Ay_6 \otimes f_6 A + Ay_6 \otimes f_6 A \in I^*(\mathcal{Y})$. Hence $2\Phi^{-1}(y_6 \otimes f_6) \in I^*(\mathcal{X})$ which is impossible by the remark after Lemma 3.3, since $(y_6 \otimes f_6)^3 = 0$. Thus for every $y \in \mathcal{Y}$, $A^2 y$ and y are linearly dependent and, as before, we have $A^2 = \lambda_1 I$ for some nonzero $\lambda_1 \in \mathbb{C}$. In particular, we have $A^3 = \lambda_1 A$, and since A^3 is an idempotent operator we get ${\lambda_1}^3 I = A^6 = A^3 = \lambda_1 A$. Therefore, $A = {\lambda_1}^2 I$, as desired. □

Lemma 3.5. *If $\phi(I) = I$, then Φ preserves nonzero 2-idempotents in both directions, that is for each $(T_{ij}) \in M_2(\mathbb{C}) \otimes \mathcal{A}$*

$$(T_{ij}) \in I_2^*(\mathcal{X}) \Leftrightarrow (\Phi(T_{ij})) \in I_2^*(\mathcal{Y}).$$

Proof. We first show that in cases where $\star$ denotes either the Jordan product or Jordan triple product we have $\Phi(\frac{1}{2}I) = \frac{1}{2}I$. If $\star$ is the Jordan product, then since

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}I & I \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} \frac{1}{2}I & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in I_2^*(\mathcal{X}),$$

it follows that

$$\begin{pmatrix} 2\Phi(\frac{1}{2}I) & I \\ 0 & 0 \end{pmatrix} \in I_2^*(\mathcal{Y}),$$

which easily implies that $\Phi(\frac{1}{2}I) = \frac{1}{2}I$. Clearly the case where $\star$ denotes the Jordan triple product follows immediately from the preceding case, since by assumption $\Phi(I) = I$.

Now let $(T_{ij}) \in M_2(\mathbb{C}) \otimes \mathcal{A}$ be given. If $\star$ denotes the Jordan product, then since $\Phi(\frac{1}{2}I) = \frac{1}{2}I$ and

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} \frac{1}{2}I & 0 \\ 0 & \frac{1}{2}I \end{pmatrix}+\begin{pmatrix} \frac{1}{2}I & 0 \\ 0 & \frac{1}{2}I \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

it follows that

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in I_2^*(\mathcal{X}) \Leftrightarrow \begin{pmatrix} \Phi(T_{11}) & \Phi(T_{12}) \\ \Phi(T_{21}) & \Phi(T_{22}) \end{pmatrix} \in I_2^*(\mathcal{Y}).$$

In the case where $\star$ denotes the Jordan triple product, since $\Phi(I) = I$ it follows that Φ satisfies the same condition for Jordan product and the result follows from the previous case.

The case where $\star$ is the usual product is obvious. Indeed,

$$\begin{aligned}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in I_2^*(\mathcal{X}) &\Leftrightarrow \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \in I_2^*(\mathcal{X}) \\ &\Leftrightarrow \begin{pmatrix} \Phi(T_{11}) & \Phi(T_{12}) \\ \Phi(T_{21}) & \Phi(T_{22}) \end{pmatrix}\begin{pmatrix} \Phi(I) & 0 \\ 0 & \Phi(I) \end{pmatrix} \in I_2^*(\mathcal{X}) \\ &\Leftrightarrow \begin{pmatrix} \Phi(T_{11}) & \Phi(T_{12}) \\ \Phi(T_{21}) & \Phi(T_{22}) \end{pmatrix} \in I_2^*(\mathcal{Y}).\end{aligned}$$

Finally the case where $\star$ is the Jordan semitriple product, is a consequence of the preceding case. □

We now state the proofs of the main theorems.

Proof. [Proof of Theorem 2.1] In either of cases, by Lemma 3.4, there exists a scalar $\lambda \in \mathbb{C}$ such that $\Phi(I) = \lambda I$, where $\lambda \in \{1, -1\}$ whenever $\star$ denotes either the usual product or Jordan product and $\lambda^3 = 1$ whenever $\star$ is either the Jordan semi-triple product or Jordan triple product. Now by Lemma 3.5, $\lambda^{-1}\Phi$ preserves nonzero 2-idempotents in both directions, thus by [3, Theorem 2.1], there exists a bounded invertible linear or conjugate linear operator $A : \mathcal{X} \longrightarrow \mathcal{Y}$ such that $\lambda^{-1}\Phi(T) = ATA^{-1}$, for every $T \in \mathcal{A}$. □

Proof. [Proof of Theorem 2.2] Proof in either of cases follows from Lemmas 3.4 and 3.5 and [3, Theorem 2.2]. □

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