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Second Order Infinitesimal Bending of Curves

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Abstract. We investigate a second order infinitesimal bending of curves in a three-dimensional Euclidean space in this paper. We give the necessary and sufficient conditions for the vector fields to be infinitesimal bending fields of the corresponding order, as well as explicit formulas which determine these fields. We examine the first and the second variation of some geometric magnitudes which describe a curve, specially a change of the curvature. Two illustrative examples (a circle and a helix) are studied not only analytically but also by drawing curves using computer program *Mathematica*.

1. Introduction

The problem of infinitesimal bending of curves and surfaces is a special part of the theory of bending which also considers the bending of curves and surfaces as well as the isometric deformations and presents one of the main consisting parts of global differential geometry. The main characteristics of the infinitesimal bending is an appropriate precision. Namely, under an infinitesimal bending arc length is stationary with a given precision, which is described with the condition

$$ds_\epsilon^2 - ds^2 = o(\epsilon^m),\ m \geq 1,\ \epsilon \geq 0,\ \epsilon \to 0.$$

The fundamental tasks at infinitesimal bending problems are: to check the flexibility of surfaces, to find as many surfaces which represent the class of uniquely defined, rigid, surfaces, as well as to find these ones which represent the class of bendable, flexible surfaces. A very important question is to find an application of an infinitesimal bending in different realistic areas and physical situations because it is well known that this theory is in close connection with thin elastic shell theory. The word ”rigidity” has a proper mechanical meaning, although applications can also be found in biology, medicine, etc (see [15], [16]). A nice application in the architecture and roof constructions was given in [17].

H. Liebman [6] had obtained the first results of the infinitesimal bending theory of non-convex surfaces. He proved that the torus and analytic surfaces which contain a convex strip are rigid in the sense of infinitesimal bending. Later, Efimov [2] introduced PDEs as a tool for studying of the infinitesimal bending. Infinitesimal bending theory was also developed thanks to the works of leading mathematicians of the considered area like A. D. Alexandrov, W. Blaschke, S. Conh-Vossen, V. T. Fomenko, I. Kh. Sabitov, I. I. Karatopraklieva, I. N. Vekua, V. A. Alexandrov and many others.

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Infinitesimal bending of curves was widely studied in [2], [7], [11]-[13]. The graphical tool CurveBend, developed in Object Oriented language C++ for graphical presentation of non rigid curves, was presented in [8]. Infinitesimal deformations of curves in the spaces with linear connection were considered in [18]. In [1], the geometry of embedded curves in three dimensional space was described, as well as the effect of a small deformation of the curve on its geometry.

The next step in the theory of infinitesimal bending of curves is to study high order infinitesimal bending and the change of important geometric magnitudes of curves under such infinitesimal bending. Some papers related to the high order infinitesimal bending of surfaces are [4], [9], [14].

This paper is organized as follows: In Section 2, some used notations and preliminaries are introduced. A few properties of an arbitrary order variation are examined. In Section 3, necessary and sufficient conditions for infinitesimal bending fields of the first and the second order are given, as well as their explicit formulas. In Section 4 the behavior of some geometric magnitudes of curves under infinitesimal bending of the second order is described, specially the change of the curvature. Finally, in Section 5, some examples are analytically and graphically studied. It is interesting to see the influence of infinitesimal bending field on flexible curves and their corresponding bent shapes. The computer program *Mathematica* [3] is used for this purpose.

2. Infinitesimal Bending of a Curve in $\mathcal{R}^3$

Let us consider a regular curve

$$C : \mathbf{r} = \mathbf{r}(u), \quad u \in \mathcal{J} \subseteq \mathcal{R} \tag{1}$$

of a class C^α, $\alpha \geq 3$, included in a family of the curves

$$C_\epsilon : \tilde{\mathbf{r}}(u,\epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \overset{(1)}{\mathbf{z}}(u) + \epsilon^2 \overset{(2)}{\mathbf{z}}(u) + \ldots + \epsilon^m \overset{(m)}{\mathbf{z}}(u),\ m \geq 1, \tag{2}$$

where $\epsilon \geq 0$, $\epsilon \to 0$ and we get C for $\epsilon = 0$ ($C = C_0$). The fields $\overset{(j)}{\mathbf{z}}(u) \in C^\alpha$, $\alpha \geq 3$, $j = 1, \ldots m$, are vector functions defined in the points of C.

Definition 2.1. [2] *Family of curves C_ϵ is an* **infinitesimal bending of the order** m **of the curve** C *if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon^m). \tag{3}$$

The field $\overset{(j)}{\mathbf{z}} = \overset{(j)}{\mathbf{z}}(u)$ is the **infinitesimal bending field of the order** j*, $j = 1, \ldots m$, of the curve C.*

The previous condition is equivalent to the system of equations ([2], [5]):

$$d\mathbf{r} \cdot d\overset{(1)}{\mathbf{z}} = 0, \quad 2d\mathbf{r} \cdot d\overset{(j)}{\mathbf{z}} + \sum_{l=1}^{j-1} d\overset{(l)}{\mathbf{z}} \cdot d\overset{(j-l)}{\mathbf{z}} = 0,\ \text{for } j = 2, \ldots, m. \tag{4}$$

where $\cdot$ stands for the scalar product in $\mathcal{R}^3$.

Under an infinitesimal bending, geometric magnitudes of the curve are changed which is described with variations of these geometric magnitudes.

Definition 2.2. [10] *Let $\mathcal{A} = \mathcal{A}(u)$ be a magnitude which characterizes a geometric property on the curve C and $\mathcal{A}_\epsilon = \mathcal{A}_\epsilon(u)$ the corresponding magnitude on the curve C_ϵ being infinitesimal bending of the curve C, and let the equation*

$$\Delta\mathcal{A} = \mathcal{A}_\epsilon - \mathcal{A} = \epsilon\, \delta\mathcal{A} + \epsilon^2\, \delta^2\mathcal{A} + \ldots + \epsilon^n\, \delta^n\mathcal{A} + \ldots \tag{5}$$

be a valid one. The coefficients $\delta\mathcal{A}, \delta^2\mathcal{A}, \ldots, \delta^n\mathcal{A}, \ldots$ are **the first, the second, ..., the n-th variation** *of the geometric magnitude $\mathcal{A}$, respectively under the infinitesimal bending C_ϵ of the curve C.*

Let us mark some properties of the variations:

I. For the variations of the product of geometric magnitudes it is effective the equation

$$\delta^n \mathcal{A}\mathcal{B} = \sum_{i=0}^{n} \delta^i \mathcal{A}\, \delta^{n-i}\mathcal{B}, \quad n \geq 0, \quad (\delta^0 \mathcal{A} \stackrel{def}{=} \mathcal{A}). \tag{6}$$

According to Def. (2.2), the variations of geometric magnitudes $\mathcal{A}$ and $\mathcal{B}$, as well as of the product $\mathcal{A}\mathcal{B}$ are:

$$\Delta\mathcal{A} = \mathcal{A}_\epsilon - \mathcal{A} = \epsilon\, \delta\mathcal{A} + \epsilon^2\, \delta^2\mathcal{A} + \ldots + \epsilon^n\, \delta^n\mathcal{A} + \ldots \tag{7}$$

$$\Delta\mathcal{B} = \mathcal{B}_\epsilon - \mathcal{B} = \epsilon\, \delta\mathcal{B} + \epsilon^2\, \delta^2\mathcal{B} + \ldots + \epsilon^n\, \delta^n\mathcal{B} + \ldots \tag{8}$$

$$\Delta\mathcal{A}\mathcal{B} = \mathcal{A}_\epsilon\mathcal{B}_\epsilon - \mathcal{A}\mathcal{B} = \epsilon\, \delta(\mathcal{A}\mathcal{B}) + \epsilon^2\, \delta^2(\mathcal{A}\mathcal{B}) + \ldots + \epsilon^n\, \delta^n(\mathcal{A}\mathcal{B}) + \ldots \tag{9}$$

respectively. On the other hand, the following equalities are satisfied:

$$\Delta\mathcal{A}\mathcal{B} = \mathcal{A}_\epsilon\mathcal{B}_\epsilon - \mathcal{A}\mathcal{B} = \mathcal{A}_\epsilon\mathcal{B}_\epsilon - \mathcal{A}_\epsilon B + \mathcal{A}_\epsilon B - \mathcal{A}\mathcal{B} = \mathcal{A}_\epsilon(\mathcal{B}_\epsilon - B) + (\mathcal{A}_\epsilon - A)B \tag{10}$$

Substituting (7) and (8) into the equation (9) we obtain it is satisfied the result

$$\Delta\mathcal{A}\mathcal{B} = \epsilon(A\delta B + B\delta A) + \epsilon^2(A\delta^2 B + \delta A\, \delta B + B\delta^2 A) + \ldots + \epsilon^n \sum_{i=0}^{n} \delta^i\mathcal{A}\, \delta^{n-i}\mathcal{B} + \ldots \tag{11}$$

As the left sides of the last equation and of the equation (9) are equal, the equation (6) is a valid one.

II. An arbitrary order variation of a derivative is the derivative of the variation, i. e.

$$\delta^n(\frac{d\mathcal{A}}{du}) = \frac{d(\delta^n\mathcal{A})}{du}, \quad n \geq 0. \tag{12}$$

For

$$\frac{d\mathcal{A}}{du} = \mathcal{B}, \tag{13}$$

using (5) we obtain it holds the equation

$$\triangle\frac{d\mathcal{A}}{du} = \triangle\mathcal{B} = \epsilon\, \delta\mathcal{B} + \epsilon^2\, \delta^2\mathcal{B} + \ldots + \epsilon^n\, \delta^n\mathcal{B} + \ldots = \epsilon\, \delta\frac{d\mathcal{A}}{du} + \epsilon^2\, \delta^2\frac{d\mathcal{A}}{du} + \ldots + \epsilon^n\, \delta^n\frac{d\mathcal{A}}{du} + \ldots \tag{14}$$

It is also valid

$$\begin{aligned}\triangle\frac{d\mathcal{A}}{du} &= \triangle\mathcal{B} = \mathcal{B}_\epsilon(u) - \mathcal{B}(u) = \frac{d\mathcal{A}_\epsilon(u)}{du} - \frac{d\mathcal{A}(u)}{du}\\ &= \frac{d}{du}[\mathcal{A}_\epsilon(u) - \mathcal{A}(u)] = \frac{d\triangle\mathcal{A}}{du} = \frac{d(\epsilon\, \delta\mathcal{A} + \epsilon^2\, \delta^2\mathcal{A} + \ldots + \epsilon^n\, \delta^n\mathcal{A} + \ldots)}{du}\\ &= \epsilon\frac{d(\delta\mathcal{A})}{du} + \epsilon^2\frac{d(\delta^2\mathcal{A})}{du} + \ldots + \epsilon^n\frac{d(\delta^n\mathcal{A})}{du} + \ldots\end{aligned} \tag{15}$$

By comparing the equations (14) and (15), we confirm validity of the equation (12). The same case is for the differential, i. e.

III. $\delta^n(d\mathcal{A}) = d(\delta^n\mathcal{A}), \quad n \geq 0.$

In this paper we will consider the first and the second variation under infinitesimal bending of the second order. For this reason, we can represent the magnitude $\mathcal{A}_\epsilon$ as

$$\mathcal{A}_\epsilon = \mathcal{A} + \epsilon\,\delta\mathcal{A} + \epsilon^2\delta^2\mathcal{A},$$

by neglecting the terms in ϵ^n, $n \geq 3$. The previous considered properties are reduced to

Ia. $\delta\mathcal{A}\mathcal{B} = \mathcal{A}\,\delta\mathcal{B} + \mathcal{B}\,\delta\mathcal{A}, \quad \delta^2\mathcal{A}\mathcal{B} = \mathcal{A}\,\delta^2\mathcal{B} + \mathcal{B}\,\delta^2\mathcal{A} + \delta\mathcal{A}\,\delta\mathcal{B}$

IIa. $\delta(\frac{d\mathcal{A}}{du}) = \frac{d(\delta\mathcal{A})}{du}, \quad \delta^2(\frac{d\mathcal{A}}{du}) = \frac{d(\delta^2\mathcal{A})}{du}$

IIIa. $\delta(d\mathcal{A}) = d(\delta\mathcal{A}), \quad \delta^2(d\mathcal{A}) = d(\delta^2\mathcal{A})$

Let

$$C_\epsilon : \tilde{\mathbf{r}}(u,\epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon\overset{(1)}{\mathbf{z}}(u) + \epsilon^2\overset{(2)}{\mathbf{z}}(u) \tag{16}$$

be a second order infinitesimal bending of a curve C. Then the system (4) is reduced to

$$d\mathbf{r}\cdot d\overset{(1)}{\mathbf{z}} = 0, \quad 2d\mathbf{r}\cdot d\overset{(2)}{\mathbf{z}} + d\overset{(1)}{\mathbf{z}}\cdot d\overset{(1)}{\mathbf{z}} = 0, \tag{17}$$

which is equivalent to the next system of differential equations:

$$\dot{\mathbf{r}}\cdot\overset{(1)}{\dot{\mathbf{z}}} = 0, \quad 2\dot{\mathbf{r}}\cdot\overset{(2)}{\dot{\mathbf{z}}} + \overset{(1)}{\dot{\mathbf{z}}}\cdot\overset{(1)}{\dot{\mathbf{z}}} = 0. \tag{18}$$

Here dot denotes a derivative with respect to u. Based on these equations we obtain it holds the following:

$$\begin{aligned} ds_\epsilon &= \|\dot{\mathbf{r}}_\epsilon(u)\|\,du = \|\dot{\mathbf{r}}(u) + \epsilon\overset{(1)}{\dot{\mathbf{z}}}(u) + \epsilon^2\overset{(2)}{\dot{\mathbf{z}}}(u)\|\,du \\ &= (\|\dot{\mathbf{r}}(u)\|^2 + 2\epsilon^3\overset{(1)}{\dot{\mathbf{z}}}(u)\cdot\overset{(2)}{\dot{\mathbf{z}}}(u) + \epsilon^4\|\overset{(2)}{\dot{\mathbf{z}}}(u)\|^2)^{\frac{1}{2}}\,du \\ &= \|\dot{\mathbf{r}}(u)\|\Big(1 + \frac{2\epsilon^3\overset{(1)}{\dot{\mathbf{z}}}(u)\cdot\overset{(2)}{\dot{\mathbf{z}}}(u) + \epsilon^4\|\overset{(2)}{\dot{\mathbf{z}}}(u)\|^2}{\|\dot{\mathbf{r}}(u)\|^2}\Big)^{\frac{1}{2}}\,du \end{aligned} \tag{19}$$

Further, we get

$$ds_\epsilon = ds\Big(1 + \frac{2\epsilon^3\overset{(1)}{\dot{\mathbf{z}}}(u)\cdot\overset{(2)}{\dot{\mathbf{z}}}(u) + \epsilon^4\|\overset{(2)}{\dot{\mathbf{z}}}(u)\|^2}{2\|\dot{\mathbf{r}}(u)\|^2} - \frac{(2\epsilon^3\overset{(1)}{\dot{\mathbf{z}}}(u)\cdot\overset{(2)}{\dot{\mathbf{z}}}(u) + \epsilon^4\|\overset{(2)}{\dot{\mathbf{z}}}(u)\|^2)^2}{8\|\dot{\mathbf{r}}(u)\|^4} + \dots\Big)$$

i.e.

$$ds_\epsilon = ds + \epsilon^3\frac{\overset{(1)}{\dot{\mathbf{z}}}(u)\cdot\overset{(2)}{\dot{\mathbf{z}}}(u)}{\|\dot{\mathbf{r}}(u)\|^2}\,ds + \epsilon^4\frac{\|\overset{(2)}{\dot{\mathbf{z}}}(u)\|^2}{2\|\dot{\mathbf{r}}(u)\|^2}\,ds - \epsilon^6\dots + \epsilon^7\dots + \dots,$$

wherefrom

$$\delta\,ds = 0, \quad \delta^2\,ds = 0, \quad \delta^3\,ds = \frac{\overset{(1)}{\dot{\mathbf{z}}}(u)\cdot\overset{(2)}{\dot{\mathbf{z}}}(u)}{\|\dot{\mathbf{r}}(u)\|^2}\,ds, \quad \delta^4\,ds = \frac{\|\overset{(2)}{\dot{\mathbf{z}}}(u)\|^2}{2\|\dot{\mathbf{r}}(u)\|^2}\,ds, \dots$$

Therefore, we proved one more property of the variation.

IVa. Under a second order infinitesimal bending of a curve, the first and the second variations of the line element ds are equal to zero.

3. Determination of the Infinitesimal Bending Field

Consider a curve

$$C : \mathbf{r} = \mathbf{r}(s) = \mathbf{r}[u(s)], \quad s \in \mathcal{I} \subseteq \mathcal{R}, \tag{20}$$

parameterized by the arc length s. The unit tangent to the curve is $\mathbf{t} = \mathbf{r}'$, where prime denotes a derivative with respect to the arc length s. Clearly, $\mathbf{t}'$ is orthogonal to $\mathbf{t}$, but $\mathbf{t}''$ is not. The classical Frenet equations

$$\begin{aligned} \mathbf{t}' &= k\mathbf{n}_1, \\ \mathbf{n}_1' &= -k\mathbf{t} + \tau\mathbf{n}_2, \\ \mathbf{n}_2' &= -\tau\mathbf{n}_1, \end{aligned} \tag{21}$$

describe the construction of an orthonormal basis $\{\mathbf{t}, \mathbf{n_1}, \mathbf{n_2}\}$ along the curve, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are respectively unit principal normal and binormal vector field of the curve. We choose an orientation with $\mathbf{n}_2 = \mathbf{t} \times \mathbf{n}_1$. k and τ are respectively the curvature and the torsion.

Consider an infinitesimal bending of the second order of the curve (20):

$$C_\epsilon : \tilde{\mathbf{r}}(s, \epsilon) = \mathbf{r}_\epsilon(s) = \mathbf{r}(s) + \epsilon \overset{(1)}{\mathbf{z}}(s) + \epsilon^2 \overset{(2)}{\mathbf{z}}(s). \tag{22}$$

As the vector fields $\overset{(1)}{\mathbf{z}}$ and $\overset{(2)}{\mathbf{z}}$ are defined in the points of the curve (20), they can be presented in the form

$$\overset{(j)}{\mathbf{z}} = \overset{(j)}{z}\mathbf{t} + \overset{(j)}{z}_1\mathbf{n}_1 + \overset{(j)}{z}_2\mathbf{n}_2, \quad j = 1, 2, \tag{23}$$

where $\overset{(j)}{z}\mathbf{t}$ is a tangential and $\overset{(j)}{z}_1\mathbf{n}_1 + \overset{(j)}{z}_2\mathbf{n}_2$ is a normal component, $\overset{(j)}{z}, \overset{(j)}{z}_1, \overset{(j)}{z}_2$ are the functions of s.

In this case the system of differential equations (18) has the form

$$\mathbf{r}' \cdot \overset{(1)}{\mathbf{z}}{}' = 0, \quad 2\mathbf{r}' \cdot \overset{(2)}{\mathbf{z}}{}' + \overset{(1)}{\mathbf{z}}{}' \cdot \overset{(1)}{\mathbf{z}}{}' = 0. \tag{24}$$

Using the equations (24), the expressions (23) for the infinitesimal bending fields of the first and the second order respectively, as well as Frenet equations (21), we proved the next theorem.

Theorem 3.1. *Necessary and sufficient conditions for the fields* $\overset{(j)}{\mathbf{z}}$, $j = 1, 2$, (23) *to be infinitesimal bending fields of the corresponding order of a curve* C (20) *are*

$$\begin{aligned} &\overset{(1)}{z}{}' - k\overset{(1)}{z}_1 = 0, \\ &\overset{(2)}{z}{}' - k\overset{(2)}{z}_1 = -\frac{1}{2}\Big\{[k\overset{(1)}{z} + \overset{(1)}{z}_1{}' - \tau\overset{(1)}{z}_2]^2 + [\tau\overset{(1)}{z}_1 + \overset{(1)}{z}_2{}']^2\Big\} \end{aligned} \tag{25}$$

where k is the curvature and τ is the torsion of C.

The next theorem is related to determination of the infinitesimal bending field of a curve C.

Theorem 3.2. *Infinitesimal bending fields of the first and the second order for the curve* C (20) *are respectively*

$$\overset{(1)}{\mathbf{z}} = \int [p(s)\mathbf{n}_1 + q(s)\mathbf{n}_2]\, ds + C_1, \tag{26}$$

$$\overset{(2)}{\mathbf{z}} = \int [-\frac{p^2(s) + q^2(s)}{2}\mathbf{t} + r(s)\mathbf{n}_1 + g(s)\mathbf{n}_2]\, ds + C_2, \tag{27}$$

where $p(s)$, $q(s)$, $r(s)$, $g(s)$ are arbitrary integrable functions and vectors $\mathbf{t}$, $\mathbf{n}_1$, $\mathbf{n}_2$ *are unit tangent, principal normal and binormal vector fields, respectively, of the curve C. C_1 and C_2 are constants.*

Proof. According to the first equation of the (24), we conclude that $\overset{(1)}{\mathbf{z}}{}'$ lies in the normal plane of the curve C, i. e.

$$\overset{(1)}{\mathbf{z}}{}' = p(s)\mathbf{n}_1 + q(s)\mathbf{n}_2, \tag{28}$$

where $p(s)$ and $q(s)$ are arbitrary integrable functions. Integrating the previous equation we obtain the equation (26). If we put the equation (28) into the second equation in (24) we obtain

$$2\mathbf{r}' \cdot \overset{(2)}{\mathbf{z}}{}' + p^2(s) + q^2(s) = 0,$$

wherefrom we have

$$\mathbf{t} \cdot \overset{(2)}{\mathbf{z}}{}' = -\frac{p^2(s) + q^2(s)}{2}.$$

Therefore,

$$\overset{(2)}{\mathbf{z}}{}' = -\frac{p^2(s) + q^2(s)}{2}\mathbf{t} + r(s)\mathbf{n}_1 + g(s)\mathbf{n}_2, \tag{29}$$

where $r(s)$ and $g(s)$ are arbitrary integrable functions. □

4. Change of Geometric Magnitudes under Second Order Infinitesimal Bending of Curves

Let us describe the behavior of some geometric magnitudes under second order infinitesimal bending of a curve, specially the change of the curvature.

Theorem 4.1. *Under second order infinitesimal bending of a curve C, the first and the second variation of the unit tangent vector respectively are:*

$$\delta\mathbf{t} = (k\overset{(1)}{z} + \overset{(1)}{z_1}{}' - \tau\overset{(1)}{z_2})\,\mathbf{n}_1 + (\overset{(1)}{z_2}{}' + \tau\overset{(1)}{z_1})\,\mathbf{n}_2, \tag{30}$$

$$\delta^2\mathbf{t} = (\overset{(2)}{z}{}' - k\overset{(2)}{z_1})\,\mathbf{t} + (k\overset{(2)}{z} + \overset{(2)}{z_1}{}' - \tau\overset{(2)}{z_2})\,\mathbf{n}_1 + (\overset{(2)}{z_2}{}' + \tau\overset{(2)}{z_1})\,\mathbf{n}_2, \tag{31}$$

Proof. According to **IIa**, we have it holds $\delta\mathbf{t} = \delta\mathbf{r}' = (\delta\mathbf{r})' = \overset{(1)}{\mathbf{z}}{}'$ and $\delta^2\mathbf{t} = \delta^2\mathbf{r}' = (\delta^2\mathbf{r})' = \overset{(2)}{\mathbf{z}}{}'$. Then, using the equations (23), (25) and the Frenet equations we obtain it is valid the equations (30) and (31). □

Obviously, using the equations (25), (30) and (31), we obtain the connection between the first and the second variation of **t**:

$$\delta^2\mathbf{t} = -\frac{1}{2}\|\delta\mathbf{t}\|^2\,\mathbf{t} + (k\overset{(2)}{z} + \overset{(2)}{z_1}{}' - \tau\overset{(2)}{z_2})\,\mathbf{n}_1 + (\overset{(2)}{z_2}{}' + \tau\overset{(2)}{z_1})\,\mathbf{n}_2. \tag{32}$$

Theorem 4.2. *Under second order infinitesimal bending of a curve C, the first variation of the unit normal and binormal vectors respectively are:*

$$\delta\mathbf{n}_1 = -(k\overset{(1)}{z} + \overset{(1)}{z_1}{}' - \tau\overset{(1)}{z_2})\mathbf{t} + \frac{1}{k}(k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z_1}{}' + \tau'\overset{(1)}{z_1} + \overset{(1)}{z_2}{}'' - \tau^2\overset{(1)}{z_2})\mathbf{n}_2, \tag{33}$$

$$\delta\mathbf{n}_2 = -(\overset{(1)}{z_2}{}' - \tau\overset{(1)}{z_1})\mathbf{t} - \frac{1}{k}(k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z_1}{}' + \tau'\overset{(1)}{z_1} + \overset{(1)}{z_2}{}'' - \tau^2\overset{(1)}{z_2})\mathbf{n}_1, \tag{34}$$

Proof. The unit normal vector remains unit after infinitesimal bending, which means $(\mathbf{n}_1 + \epsilon\delta\mathbf{n}_1 + \epsilon^2\delta^2\mathbf{n}_1) \cdot (\mathbf{n}_1 + \epsilon\delta\mathbf{n}_1 + \epsilon^2\delta^2\mathbf{n}_1) = 1$. This fact gives the following equations:

$$\mathbf{n}_1 \cdot \delta\mathbf{n}_1 = 0, \tag{35}$$

$$\mathbf{n}_1 \cdot \delta^2\mathbf{n}_1 = -\frac{1}{2}\delta\mathbf{n}_1 \cdot \delta\mathbf{n}_1. \tag{36}$$

Also, the unit normal vector remains a perpendicular one to the unit tangent vector, i. e. $(\mathbf{n}_1 + \epsilon\delta\mathbf{n}_1 + \epsilon^2\delta^2\mathbf{n}_1) \cdot (\mathbf{t} + \epsilon\delta\mathbf{t} + \epsilon^2\delta^2\mathbf{t}) = 0$, wherefrom we conclude

$$\mathbf{t} \cdot \delta\mathbf{n}_1 = -\mathbf{n}_1 \cdot \delta\mathbf{t} = -(k\overset{(1)}{z} + \overset{(1)}{z_1}{}' - \tau\overset{(1)}{z_2}), \tag{37}$$

$$\mathbf{t}\cdot\delta^2\mathbf{n}_1 = -\mathbf{n}_1\cdot\delta^2\mathbf{t} - \delta\mathbf{n}_1\cdot\delta\mathbf{t}. \tag{38}$$

Further, we take a first variation of the first equation of (21),

$$\delta\mathbf{t}' = \mathbf{n}_1\delta k + k\delta\mathbf{n}_1. \tag{39}$$

Dotting this equation with $\mathbf{n}_2$ we obtain

$$\mathbf{n}_2\cdot\delta\mathbf{n}_1 = \frac{1}{k}\mathbf{n}_2\cdot\delta\mathbf{t}'.$$

To compute $\delta\mathbf{t}'$ we apply commutativity of the variation and the derivative. For this reason, we obtain it holds $\delta\mathbf{t}' = (\delta\mathbf{t})'$. Based on the equation (30), the Frenet equations and $\overset{(1)}{z}{}' = k\overset{(1)}{z}_1$ (due to the (25)), one obtains it holds

$$\begin{aligned}\delta\mathbf{t}' =& -k(k\overset{(1)}{z} + \overset{(1)}{z}_1{}' - \tau\overset{(1)}{z}_2)\,\mathbf{t} + (k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + (k^2-\tau^2)\overset{(1)}{z}_1 - 2\tau\overset{(1)}{z}_2{}' - \tau'\overset{(1)}{z}_2)\,\mathbf{n}_1\\ &+ (k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z}_1{}' + \tau'\overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2\overset{(1)}{z}_2)\,\mathbf{n}_2.\end{aligned} \tag{40}$$

Now we have

$$\mathbf{n}_2\cdot\delta\mathbf{n}_1 = \frac{1}{k}(k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z}_1{}' + \tau'\overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2\overset{(1)}{z}_2) \tag{41}$$

Comparing the equations (35), (37) and (41) we obtain the equation (33) is a valid one. Similarly, from the conditions $\mathbf{n}_{2\epsilon}\cdot\mathbf{n}_{2\epsilon} = 1$, $\mathbf{n}_{2\epsilon}\cdot\mathbf{t}_\epsilon = 0$ and $\mathbf{n}_{2\epsilon}\cdot\mathbf{n}_{1\epsilon} = 0$ we confirm the validity of the equation (34). □

Theorem 4.3. *Under a second order infinitesimal bending of a curve C, a unit vector of the orthonormal basis and its first variation are orthogonal, i. e.*

$$\mathbf{t}\cdot\delta\mathbf{t} = 0, \quad \mathbf{n}_1\cdot\delta\mathbf{n}_1, \quad \mathbf{n}_2\cdot\delta\mathbf{n}_2 = 0. \tag{42}$$

A unit vector of the orthonormal basis and its second variation are not orthogonal, i. e.

$$\mathbf{t}\cdot\delta^2\mathbf{t} = -\frac{1}{2}\|\delta\mathbf{t}\|^2 = \overset{(2)}{z}{}' - k\overset{(2)}{z}_1, \tag{43}$$

$$\mathbf{n}_1\cdot\delta^2\mathbf{n}_1 = -\frac{1}{2}[(k\overset{(1)}{z} + \overset{(1)}{z}_1{}' - \tau\overset{(1)}{z}_2)^2 + \frac{1}{k^2}(k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z}_1{}' + \tau'\overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2\overset{(1)}{z}_2)^2], \tag{44}$$

$$\mathbf{n}_2\cdot\delta^2\mathbf{n}_2 = -\frac{1}{2}[(\overset{(1)}{z}_2{}' - \tau\overset{(1)}{z}_1)^2 + \frac{1}{k^2}(k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z}_1{}' + \tau'\overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2\overset{(1)}{z}_2)^2] \tag{45}$$

Proof. The previous result comes directly from the equations (30), (33), (34), (36) and the same for $\mathbf{n}_2$ as (36). □

Theorem 4.4. *Under a second order infinitesimal bending of a curve C, the first and the second variation of the curvature are respectively*

$$\delta k = k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + (k^2-\tau^2)\overset{(1)}{z}_1 - 2\tau\overset{(1)}{z}_2{}' - \tau'\overset{(1)}{z}_2, \tag{46}$$

$$\delta^2 k = -\frac{k}{2}(\tau\overset{(1)}{z}_1 + \overset{(1)}{z}_2{}')^2 + \frac{1}{2k}(k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z}_1{}' + \tau'\overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2\overset{(1)}{z}_2)^2 + (k\overset{(2)}{z} + \overset{(2)}{z}_1{}' - \tau\overset{(2)}{z}_2)' - \tau(\overset{(2)}{z}_2{}' + \tau\overset{(2)}{z}_1). \tag{47}$$

Proof. Dotting Eq. (39) with $\mathbf{n}_1$ and using Theorem 4.3, we obtain $\delta k = \mathbf{n}_1 \cdot \delta \mathbf{t}'$. This leads to Eq. (46) after using (40).

Let us take a second variation of the first Frenet equation. We obtain it holds $\delta^2 \mathbf{t}' = \mathbf{n}_1 \delta^2 k + k\delta^2 \mathbf{n}_1 + \delta k \delta \mathbf{n}_1$. After scalar product with $\mathbf{n}_1$ we obtain

$$\delta^2 k = \mathbf{n}_1 \cdot \delta^2 \mathbf{t}' - k\mathbf{n}_1 \cdot \delta^2 \mathbf{n}_1 - \delta k \mathbf{n}_1 \cdot \delta \mathbf{n}_1 \tag{48}$$

As it is $\delta^2 \mathbf{t}' = (\delta^2 \mathbf{t})'$, applying the equation (31), Frenet equations and dotting with $\mathbf{n}_1$ we get

$$\mathbf{n}_1 \cdot \delta^2 \mathbf{t}' = k(\overset{(2)}{z}{}' - k\overset{(2)}{z}_1) + (k\overset{(2)}{z} + \overset{(2)}{z}_1{}' - \tau\overset{(2)}{z}_2)' - \tau(\overset{(2)}{z}_2{}' + \tau\overset{(2)}{z}_1). \tag{49}$$

Using the previous equation, Theorem 4.3 and the second equation in Eq. (25) we complete the proof of this theorem . □

Directly from Theorem 4.4, in the case of a plane curve ($\tau = 0$), we obtain the next corollary.

Corollary 4.5. *Under a second order infinitesimal bending of a plane curve, the first and the second variation of the curvature are respectively*

$$\delta k = k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + k^2\overset{(1)}{z}_1, \tag{50}$$

$$\delta^2 k = -\frac{k}{2}(\overset{(1)}{z}_2{}')^2 + \frac{1}{2k}(\overset{(1)}{z}_2{}'')^2 + (k\overset{(2)}{z} + \overset{(2)}{z}_1{}')'. \tag{51}$$

5. Examples

Example 5.1 Let us examine a second order infinitesimal bending of a circle

$$\mathbf{r} = (\cos u, \sin u, 0), \quad (\text{or } x^2 + y^2 = 1). \tag{52}$$

Here is $R = 1$ and $u = s$, i. e. the curve can be parameterized by the arc length and we have

$$\mathbf{t} = \mathbf{r}' = (-\sin s, \cos s, 0), \quad \mathbf{n}_1 = \frac{\mathbf{r}''}{\|\mathbf{r}''\|} = (-\cos s, -\sin s, 0), \quad \mathbf{n}_2 = \mathbf{t} \times \mathbf{n}_1 = (0, 0, 1).$$

According to (26) we have

$$\overset{(1)}{\mathbf{z}} = \int [p(s)(-\cos s\,\mathbf{i} - \sin s\,\mathbf{j}) + q(s)\,\mathbf{k}]\,ds + C_1,$$

where $p(s)$ and $q(s)$ are arbitrary functions. For instance, for $p(s) = C_1 = 0$, $q(s) = 1$, an infinitesimal bending field of the first order is

$$\overset{(1)}{\mathbf{z}} = (0, 0, s) = s\,\mathbf{k} = s\,\mathbf{n}_2. \tag{53}$$

Now according to the equation (27) for $r(s) = C_2 = 0$ and $g(s) = 1$, we obtain it holds:

$$\overset{(2)}{\mathbf{z}} = \int (\frac{1}{2}\sin s\,\mathbf{i} - \frac{1}{2}\cos s\,\mathbf{j} + \mathbf{k})\,ds,$$

$$\overset{(2)}{\mathbf{z}} = (-\frac{1}{2}\cos s, -\frac{1}{2}\sin s, s) = -\frac{1}{2}\cos s\,\mathbf{i} - \frac{1}{2}\sin s\,\mathbf{j} + s\,\mathbf{k},$$

or after determination of this vector field via the vectors of the Frenet frame $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$,

$$\overset{(2)}{\mathbf{z}} = \frac{1}{2}\mathbf{n}_1 + s\mathbf{n}_2. \tag{54}$$

Thus the curves we got under infinitesimal bending of the second order of a circle with these infinitesimal bending fields present a family of helices

$$\mathbf{r}_\epsilon = (\cos s - \frac{1}{2}\cos s\,\epsilon^2, \sin s - \frac{1}{2}\sin s\,\epsilon^2, s\epsilon + s\epsilon^2) \tag{55}$$

that are not on the cylinder $x^2 + y^2 = 1$. Let us examine if the relation (3) is a valid one. According to $u = s$ we have $ds^2 = du^2$ and also

$$ds_\epsilon^2 = d\mathbf{r}_\epsilon \cdot d\mathbf{r}_\epsilon = du^2 + 2\epsilon^3 du^2 + \frac{5}{4}\epsilon^4 du^2,$$

wherefrom we get

$$ds_\epsilon^2 - ds^2 = 2\epsilon^3 du^2 + \frac{5}{4}\epsilon^4 du^2 = o(\epsilon^2),$$

i. e. the fields (53) and (54) are corresponding fields of a second order infinitesimal bending of the circle (52). The resulting deformation is shown in Figure 1. The influence of $\epsilon \in \{0, 0.01, 0.02, 0.03, 0.05, 0.07, 0.1, 0.2\}$ to the circle is presented. On the last picture we have the family of deformed curves C_ϵ for given values of ϵ.

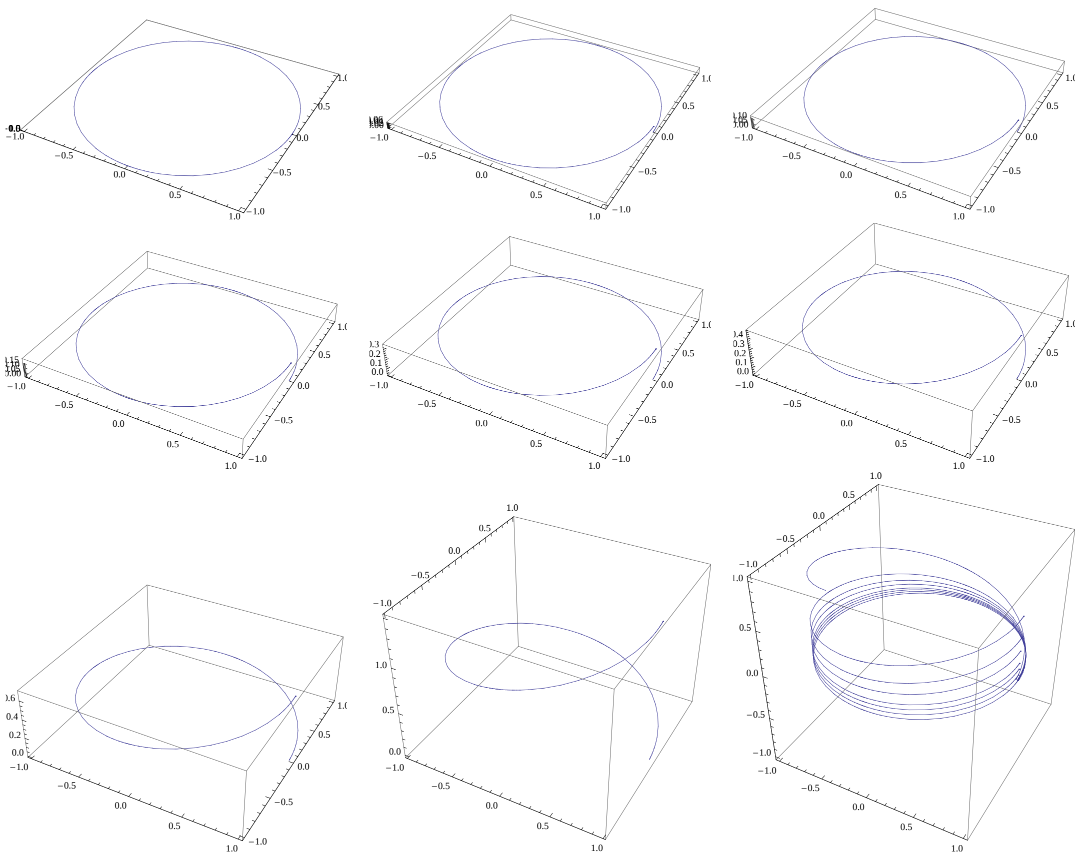

Figure 1: *Infinitesimal bending of the second order of a circle to a helix for* $\epsilon \in \{0, 0.01, 0.02, 0.03, 0.05, 0.07, 0.1, 0.2\}$

Let us compute a variation of the curvature. Obviously, $k = \frac{1}{R} = \|\mathbf{r}''\| = 1$, $\tau = 0$ (plane curve).

As it is

$$\overset{(1)}{z} = \overset{(1)}{z}_1 = \overset{(2)}{z} = 0, \quad \overset{(1)}{z}_2 = \overset{(2)}{z}_2 = s, \quad \overset{(2)}{z}_1 = \frac{1}{2},$$

according to (50) and (51) we obtain $\delta k = 0$ and $\delta^2 k = -\frac{1}{2}$.

Example 5.2 Let us observe a helix

$$\mathbf{r} = (a\cos u, a\sin u, bu), \quad u \in \mathcal{R}, \quad a \neq 0, \quad b \neq 0,$$

which lies on a cylinder $x^2 + y^2 = a^2$. If we measure s from the point $\mathbf{r}(u) = (a, 0, 0)$, we get $s = \int_0^u \sqrt{a^2\sin^2 u + a^2\cos^2 u + b^2}\,du = \sqrt{a^2+b^2}\,u$, i.e. $u = u(s) = \frac{s}{\sqrt{a^2+b^2}}$. Substituting this result into the starting equation gives the parametrization of the helix by the arc length

$$\mathbf{r} = \Big(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\Big).$$

Specially, for $a = b = \frac{\sqrt{2}}{2}$, we have a helix

$$\mathbf{r} = (\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}s). \tag{56}$$

The vectors of the Frenet frame $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ are

$$\mathbf{t} = (-\frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}), \quad \mathbf{n}_1 = (-\cos s, -\sin s, 0), \quad \mathbf{n}_2 = (\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}).$$

For $p(s) = 1$, $q(s) = r(s) = 0$, $g(s) = \frac{1}{2}$, $C_1 = C_2 = 0$, we get infinitesimal bending fields of the first and the second order respectively

$$\overset{(1)}{\mathbf{z}} = (-\sin s, \cos s, 0) = -\sin s\,\mathbf{i} + \cos s\,\mathbf{j}, \tag{57}$$

$$\overset{(2)}{\mathbf{z}} = (-\frac{\sqrt{2}}{2}\cos s, -\frac{\sqrt{2}}{2}\sin s, 0) = -\frac{\sqrt{2}}{2}\cos s\,\mathbf{i} - \frac{\sqrt{2}}{2}\sin s\,\mathbf{j}, \tag{58}$$

i. e. in the basis $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$:

$$\overset{(1)}{\mathbf{z}} = \frac{\sqrt{2}}{2}\mathbf{t} - \frac{\sqrt{2}}{2}\mathbf{n}_2, \quad \overset{(2)}{\mathbf{z}} = \frac{\sqrt{2}}{2}\mathbf{n}_1. \tag{59}$$

The curve obtained under this infinitesimal bending is

$$\mathbf{r}_\epsilon = (\frac{\sqrt{2}}{2}\cos s - \epsilon\sin s - \epsilon^2\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s + \epsilon\cos s - \epsilon^2\frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}s). \tag{60}$$

Graphical presentation of the family of curves C_ϵ is given in Figure 2. For the sake of illustration, we have chosen large parameters for ϵ. More realistically, one has to choose a much smaller ϵ since we assume $o(\epsilon^2)$ in the definition of the infinitesimal bending.

It is easy to check the equations (24), so we conclude that the vector fields (57) and (58) are infinitesimal bending fields of the first and the second order respectively of a given helix. Moreover, it is valid $ds_\epsilon^2 - ds^2 = \frac{\epsilon^4}{2}du^2 = o(\epsilon^3)$, which means that this infinitesimal bending is the bending of the third order (for $\overset{(3)}{\mathbf{z}} = \mathbf{0}$).

For the helix (56) we have $k = \tau = \frac{\sqrt{2}}{2}$. According to the equation (59), the relations

$$\overset{(1)}{z}_1 = \overset{(2)}{z} = \overset{(2)}{z}_2 = 0, \quad \overset{(1)}{z} = \overset{(2)}{z}_1 = -\overset{(1)}{z}_2 = \frac{\sqrt{2}}{2}$$

are valid. Using Theorem 4.4 we obtain it holds $\delta k = \delta^2 k = 0$.

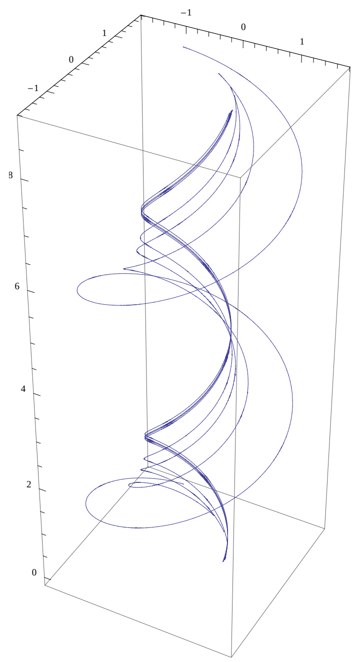

Figure 2: *Infinitesimal bending of the second order of a helix for* $\epsilon \in \{0, 0.05, 0.07, 0.1, 0.5, 0.7, 1, 1.5\}$

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