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Essential Norm of Weighted Composition Operators from the Bloch Space and the Zygmund Space to the Bloch Space

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Abstract. In this paper, we give some estimates for the essential norm of weighted composition operators from the Bloch space and the Zygmund space to the Bloch space.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by $\mathcal{B}$, if

$$\|f\|_\beta = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

$\mathcal{B}$ is a Banach space under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_\beta$. See [29] for the theory of the Bloch space.

The Zygmund space, denoted by $\mathcal{Z}$, is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty.$$

It is easy to see that $\mathcal{Z}$ is a Banach space with the above norm $\|\cdot\|_{\mathcal{Z}}$. See [1, 4, 5, 8, 11, 13, 14, 24, 25] for some results of the Zygmund space and related operators on the Zygmund space.

Let $S(\mathbb{D})$ denote the set of all analytic self-maps of $\mathbb{D}$. Let $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined as follows

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

When $u = 1$, we get the composition operator, denoted by C_φ. When $\varphi(z) = z$, we get the multiplication operator, denoted by M_u.

By Schwarz-Pick lemma, it is easy to see that C_φ is bounded on the Bloch space for any $\varphi \in S(\mathbb{D})$. The compactness of C_φ on $\mathcal{B}$ was studied, for example, in [17, 19, 26–28]. Tjani in [26] proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if

$$\lim_{|a|\to 1}\|C_\varphi\sigma_a\|_{\mathcal{B}} = 0.$$

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Here $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. In [27], Wulan, Zheng and Zhu proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} = 0$. In [28], Zhao obtained the exact value for the essential norm of $C_\varphi : \mathcal{B} \to \mathcal{B}$ as follows.

$$\|C_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} = \Big(\frac{e}{2}\Big) \limsup_{n\to\infty} \|\varphi^n\|_{\mathcal{B}}.$$

Recall that the essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators K mapping X into Y, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \text{ is compact } \},$$

where X, Y are Banach spaces and $\|\cdot\|_{X\to Y}$ is the operator norm.

In [23], Ohno and Zhao studied the boundedness and compactness of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ (see also [22]). In [2], Colonna provided a new characterization of the boundedness and compactness of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ by using $\|u\varphi^n\|_{\mathcal{B}}$. The essential norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ was studied in [7, 18, 20]. In [18], the authors proved that

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \approx \max\Big(\limsup_{|\varphi(z)|\to 1} \frac{|u(z)\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2},\ \limsup_{|\varphi(z)|\to 1} \log\frac{e}{1-|\varphi(z)|^2}|u'(z)|(1-|z|^2)\Big).$$

In [7], the authors obtained a new estimate for the essential norm of $uC_\varphi : \mathcal{B} \to \mathcal{B}$, i.e., they showed that

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \approx \max\Big(\limsup_{j\to\infty} \|I_u(\varphi^j)\|_{\mathcal{B}},\ \limsup_{j\to\infty} \log j \|J_u(\varphi^j)\|_{\mathcal{B}}\Big),$$

where

$$I_u f(z) = \int_0^z f'(\zeta)u(\zeta)d\zeta, \quad J_u f(z) = \int_0^z f(\zeta)u'(\zeta)d\zeta.$$

Various properties of composition operator, as well as weighted composition operators mapping into the Bloch space were studied, for example, in [3, 9, 10, 12–20, 22–28, 30, 31].

In [14], Stević and the second author of this paper studied the boundedness and compactness of the operator $uC_\varphi : \mathcal{Z} \to \mathcal{B}$. Among others, we proved that $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is compact if and only if $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded and

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)|u(z)\varphi'(z)| \log\frac{e}{1-|\varphi(z)|^2} = 0.$$

Motivated by the work of [2, 14, 27], the aim of this article is to give a new estimate for the essential norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ and some estimates for the essential norm of the operator $uC_\varphi : \mathcal{Z} \to \mathcal{B}$. As corollaries, we obtain a new characterization for the compactness of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ and a new characterization for the compactness of the operator $uC_\varphi : \mathcal{Z} \to \mathcal{B}$.

Throughout this paper, we say that $P \lesssim Q$ if there exists a constant C such that $P \le CQ$. The symbol $P \approx Q$ means that $P \lesssim Q \lesssim P$.

2. Essential norm of $uC_\varphi : \mathcal{Z} \to \mathcal{B}$

In this section, we give some estimates for the essential norm of the operator $uC_\varphi : \mathcal{Z} \to \mathcal{B}$. For this purpose, we first state some lemmas which will be used in the proofs of the main results in this section.

Lemma 2.1. [26] *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

(1) The point evaluation functionals on Y are continuous.

(2) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(3) $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Lemma 2.2. [11] *If $f \in \mathcal{Z}$, then the following statements hold.*

(i) $|f(z)| \le \|f\|_{\mathcal{Z}}$, *for every* $z \in \mathbb{D}$.

(ii) $|f'(z)| \lesssim \|f\|_{\mathcal{Z}} \log \frac{e}{1-|z|}$, *for every* $z \in \mathbb{D}$.

Lemma 2.3. [5] *Let $\{f_n\}$ be a bounded sequence in $\mathcal{Z}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. Then* $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n(z)| = 0$.

Theorem 2.1. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded. Then*

$$\|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} \approx \limsup_{|a|\to 1} \left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}} \approx E,$$

where

$$E := \limsup_{|\varphi(z)|\to 1} (1-|z|^2)|u(z)\varphi'(z)| \log \frac{e}{1-|\varphi(z)|^2}, \qquad \lambda_a(z) := \left(\log \frac{e}{1-|a|^2}\right)^{-1} \int_0^z \left(\log \frac{e}{1-\bar{a}w}\right)^2 dw.$$

Proof. When $\|\varphi\|_\infty < 1$. It is easy to see that $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is compact by using Lemma 2.1. In this case, the asymptotic relations vacuously holds.

Now we consider the case $\|\varphi\|_\infty = 1$. First we prove that

$$\limsup_{|a|\to 1} \left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}} \lesssim \|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}}.$$

Let $a \in \mathbb{D}$. It is easy to check that $\lambda_a \in \mathcal{Z}$ and $\|\lambda_a\|_{\mathcal{Z}} < \infty$ for all $a \in \mathbb{D}$ and λ_a converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Thus, for any compact operator $K : \mathcal{Z} \to \mathcal{B}$, by Lemma 2.1 we have $\lim_{|a|\to 1} \|K\lambda_a\|_{\mathcal{B}} = 0$. Hence

$$\|uC_\varphi - K\|_{\mathcal{Z}\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)\lambda_a\|_{\mathcal{B}} \ge \|uC_\varphi(\lambda_a)\|_{\mathcal{B}} - \|K(\lambda_a)\|_{\mathcal{B}}.$$

Taking $\limsup_{|a|\to 1}$ to the last inequality on both sides, we obtain

$$\|uC_\varphi - K\|_{\mathcal{Z}\to\mathcal{B}} \gtrsim \limsup_{|a|\to 1} \left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}}.$$

Therefore, from the definition of the essential norm, we get

$$\|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} = \inf_K \|uC_\varphi - K\|_{\mathcal{Z}\to\mathcal{B}} \gtrsim \limsup_{|a|\to 1} \left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}}.$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Define

$$h_j(z) = \left(\log \frac{e}{1-|\varphi(z_j)|^2}\right)^{-1} \int_0^z \left(\log \frac{e}{1-\overline{\varphi(z_j)}w}\right)^2 dw.$$

Similarly to the above proof we see that h_j belongs to $\mathcal{Z}$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Moreover, $h_j'(\varphi(z_j)) = \log \frac{e}{1-|\varphi(z_j)|^2}$. Then for any compact operator $K : \mathcal{Z} \to \mathcal{B}$, we obtain

$$\begin{aligned}
\|uC_\varphi - K\|_{\mathcal{Z}\to\mathcal{B}} &\gtrsim \limsup_{j\to\infty} \|uC_\varphi h_j\|_{\mathcal{B}} - \limsup_{j\to\infty} \|Kh_j\|_{\mathcal{B}} \\
&\ge \limsup_{j\to\infty} (1-|z_j|^2)|u(z_j)||\varphi'(z_j)||h_j'(\varphi(z_j))| - \limsup_{j\to\infty} (1-|z_j|^2)|u'(z_j)||h_j(\varphi(z_j))| \\
&= \limsup_{j\to\infty} (1-|z_j|^2)|u(z_j)||\varphi'(z_j)| \log \frac{e}{1-|\varphi(z_j)|^2} - \limsup_{j\to\infty} (1-|z_j|^2)|u'(z_j)||h_j(\varphi(z_j))|.
\end{aligned}$$

Since $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded, applying the operator uC_φ to 1 and z, we easily get that $uC_\varphi(1) = u \in \mathcal{B}$. Using the boundedness of φ, we also get

$$\widetilde{K} := \sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)||u(z)| < \infty.$$

By Lemma 2.3 and the fact that $u \in \mathcal{B}$ we get

$$\limsup_{j\to\infty}(1-|z_j|^2)|u'(z_j)||h_j(\varphi(z_j))| = 0.$$

Thus, by the definition of the essential norm, we obtain

$$\begin{aligned} \|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} &= \inf_K \|uC_\varphi - K\|_{\mathcal{Z}\to\mathcal{B}} \gtrsim \limsup_{j\to\infty}(1-|z_j|^2)|u(z_j)||\varphi'(z_j)|\log\frac{e}{1-|\varphi(z_j)|^2} \\ &= \limsup_{|\varphi(z)|\to1}(1-|z|^2)|u(z)||\varphi'(z)|\log\frac{e}{1-|\varphi(z)|^2} = E. \end{aligned}$$

Next, we prove that

$$\|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} \lesssim \limsup_{|a|\to1}\left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}} \quad \text{and} \quad \|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} \lesssim E.$$

For $r \in [0,1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $(K_rf)(z) = f_r(z) = f(rz)$, $f \in H(\mathbb{D})$. It is obvious that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator K_r is compact on $\mathcal{Z}$ and $\|K_r\|_{\mathcal{Z}\to\mathcal{Z}} \le 1$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for all positive integer j, the operator $uC_\varphi K_{r_j} : \mathcal{Z} \to \mathcal{B}$ is compact. By the definition of the essential norm we have

$$\|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} \le \limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathcal{Z}\to\mathcal{B}}. \tag{1}$$

Thus, we only need to show that

$$\limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathcal{Z}\to\mathcal{B}} \lesssim \limsup_{|a|\to1}\left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}}, \quad \limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathcal{Z}\to\mathcal{B}} \lesssim E.$$

For any $f \in \mathcal{Z}$ such that $\|f\|_{\mathcal{Z}} \le 1$, we consider

$$\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}} = |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| + \|u\cdot(f-f_{r_j})\circ\varphi\|_\beta. \tag{2}$$

It is obvious that

$$\lim_{j\to\infty}|u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0. \tag{3}$$

Now we consider

$$\begin{aligned} &\limsup_{j\to\infty}\|u\cdot(f-f_{r_j})\circ\varphi\|_\beta \\ \le\ &\limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}(1-|z|^2)|(f-f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)| + \limsup_{j\to\infty}\sup_{|\varphi(z)|>r_N}(1-|z|^2)|(f-f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)| \\ &+ \limsup_{j\to\infty}\sup_{z\in\mathbb{D}}(1-|z|^2)|(f-f_{r_j})(\varphi(z))||u'(z)| \\ =\ &T_1 + T_2 + \limsup_{j\to\infty}\sup_{z\in\mathbb{D}}(1-|z|^2)|(f-f_{r_j})(\varphi(z))||u'(z)|, \end{aligned} \tag{4}$$

where $N \in \mathbb{N}$ is large enough such that $r_j \ge \frac{1}{2}$ for all $j \ge N$,

$$T_1 := \limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}(1-|z|^2)|(f-f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)|$$

and

$$T_2 := \limsup_{j\to\infty} \sup_{|\varphi(z)|>r_N} (1-|z|^2)|(f-f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)|.$$

Since $r_j f'_{r_j} \to f'$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, we have

$$T_1 \le \widetilde{K} \limsup_{j\to\infty} \sup_{|w|\le r_N} |f'(w) - r_j f'(r_j w)| = 0. \tag{5}$$

Similarly, from the fact that $u \in \mathcal{B}$, $f_{r_j} \to f$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$ and by Lemma 2.3, we have

$$\limsup_{j\to\infty} \sup_{z\in\mathbb{D}} (1-|z|^2)|(f-f_{r_j})(\varphi(z))||u'(z)| \le \|u\|_{\mathcal{B}} \limsup_{j\to\infty} \sup_{w\in\mathbb{D}} |f(w) - f(r_j w)| = 0. \tag{6}$$

We consider T_2. We have $T_2 \le \limsup_{j\to\infty}(J_1+J_2)$, where

$$J_1 := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f'(\varphi(z))||\varphi'(z)||u(z)|, \qquad J_2 := \sup_{|\varphi(z)|>r_N} (1-|z|^2) r_j |f'(r_j\varphi(z))||\varphi'(z)||u(z)|.$$

First we estimate J_1. Using the fact that $|f'(z)| \lesssim \|f\|_{\mathcal{Z}} \log \frac{e}{1-|z|^2}$ (by Lemma 2.2) and $\|f\|_{\mathcal{Z}} \le 1$, we have

$$\begin{aligned} J_1 &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f'(\varphi(z))||\varphi'(z)||u(z)| \\ &\lesssim \sup_{|\varphi(z)|>r_N} (1-|z|^2)\|f\|_{\mathcal{Z}} \log \frac{e}{1-|\varphi(z)|^2} |\varphi'(z)||u(z)| \\ &\lesssim \sup_{|\varphi(z)|>r_N} (1-|z|^2)|\varphi'(z)||u(z)| \log \frac{e}{1-|\varphi(z)|^2}. \end{aligned}$$

It is easy to check that $\lambda_a \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a|\to 1$. From Lemma 2.3 we know that $\lim_{|a|\to1} |\lambda_a(a)| \le \lim_{|a|\to1} \sup_{z\in\mathbb{D}} |\lambda_a(z)| = 0$. Then by the fact that $u\in\mathcal{B}$ and letting $N\to\infty$, we obtain

$$\limsup_{|\varphi(z)|\to1} (1-|z|^2)|u'(z)||\lambda_{\varphi(z)}(\varphi(z))| = 0.$$

Since

$$\begin{aligned} \sup_{|a|>r_N} \|uC_\varphi \lambda_a\|_\beta &\ge \sup_{|\varphi(z)|>r_N} (1-|z|^2)|u'(z)(\lambda_{\varphi(z)}(\varphi(z))) + u(z)\varphi'(z) \log \frac{e}{1-|\varphi(z)|^2}| \\ &\ge \sup_{|\varphi(z)|>r_N} (1-|z|^2)|u(z)\varphi'(z)| \log \frac{e}{1-|\varphi(z)|^2} - \sup_{|\varphi(z)|>r_N} (1-|z|^2)|u'(z)(\lambda_{\varphi(z)}(\varphi(z)))|, \end{aligned}$$

we get

$$\begin{aligned} \limsup_{j\to\infty} J_1 &\lesssim \limsup_{|\varphi(z)|\to1} (1-|z|^2)|\varphi'(z)||u(z)| \log \frac{e}{1-|\varphi(z)|^2} = E \\ &\lesssim \limsup_{|a|\to1} \|uC_\varphi(\lambda_a)\|_\beta + \limsup_{|\varphi(z)|\to1} (1-|z|^2)|u'(z)||\lambda_{\varphi(z)}(\varphi(z))| \\ &\lesssim \limsup_{|a|\to1} \left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}}. \end{aligned}$$

Similarly, we have

$$\limsup_{j\to\infty} J_2 \lesssim \limsup_{|\varphi(z)|\to1} (1-|z|^2)|\varphi'(z)||u(z)| \log \frac{e}{1-|\varphi(z)|^2} = E \lesssim \limsup_{|a|\to1} \left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}},$$

i.e., we get

$$T_2 \lesssim E \lesssim \limsup_{|a|\to1} \left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}}. \tag{7}$$

Hence, by (2)-(7) we get

$$\begin{aligned}\limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathcal{Z}\to\mathcal{B}} &= \limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{Z}}\le 1}\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}} \\ &= \limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{Z}}\le 1}\|u\cdot(f-f_{r_j})\circ\varphi\|_{\beta} \\ &\lesssim E \lesssim \limsup_{|a|\to 1}\left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}}. \qquad (8)\end{aligned}$$

Therefore, by (1) and (8) we obtain

$$\|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} \lesssim E \quad \text{and} \quad \|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} \lesssim \limsup_{|a|\to 1}\left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}}.$$

The proof of this theorem is complete. □

From Theorem 2.1, we immediately get the following new characterization of the compactness of the operator $uC_\varphi : \mathcal{Z}\to\mathcal{B}$.

Corollary 2.1. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : \mathcal{Z}\to\mathcal{B}$ is bounded. Then $uC_\varphi : \mathcal{Z}\to\mathcal{B}$ is compact if and only if $\limsup_{|a|\to 1}\left\|uC_\varphi(\lambda_a)\right\|_{\mathcal{B}} = 0$.*

3. A new characterization of $uC_\varphi : \mathcal{Z}\to\mathcal{B}$

In this section, we give another new characterization for the boundedness, compactness and essential norm of the operator $uC_\varphi : \mathcal{Z}\to\mathcal{B}$. For this purpose, we state some definitions and some lemmas which will be used.

Let $v : \mathbb{D}\to R_+$ be a continuous, strictly positive and bounded function. The weighted space, denoted by H_v^∞, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

H_v^∞ is a Banach space under the norm $\|\cdot\|_v$. If $v(z) = v(|z|)$ for all $z \in \mathbb{D}$, the weighted v is called radial. The associated weight $\widetilde{v}$ of v is defined by

$$\widetilde{v} = (\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \le 1\})^{-1}, z \in \mathbb{D}.$$

When $v = v_{\log}(z) = (\log\frac{e}{1-|z|^2})^{-1}$, then $\widetilde{v}_{\log}(z) = v_{\log}(z)$ (see [7]). When $v = v_\alpha(z) = (1-|z|^2)^\alpha (0 < \alpha < \infty)$, it is easy to check that $\widetilde{v}_\alpha(z) = v_\alpha(z)$. In this case, we denote H_v^∞ by $H_{v_\alpha}^\infty$, where,

$$H_{v_\alpha}^\infty = \{f \in H(\mathbb{D}) : \|f\|_{v_\alpha} = \sup_{z\in\mathbb{D}}|f(z)|(1-|z|^2)^\alpha < \infty\}.$$

Lemma 3.1. [7] *For $\alpha > 0$, we have*

$$\lim_{k\to\infty} k^\alpha\|z^{k-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha \quad \text{and} \quad \lim_{k\to\infty}(\log k)\|z^k\|_{v_{\log}} = 1.$$

Lemma 3.2. [21] *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.*

(a) The weighted composition operator $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if $\sup_{z\in\mathbb{D}}\frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)| < \infty$. Moreover, the following holds

$$\|uC_\varphi\|_{H_v^\infty\to H_w^\infty} = \sup_{z\in\mathbb{D}}\frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|.$$

(b) Suppose $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded. Then

$$\|uC_\varphi\|_{e,H_v^\infty\to H_w^\infty} = \lim_{s\to 1^-} \sup_{|\varphi(z)|>s} \frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|.$$

Lemma 3.3. [6] *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.*

(a) $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if $\sup_{k\geq 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty$, with the norm comparable to the above supermum.

(b) Suppose $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded. Then

$$\|uC_\varphi\|_{e,H_v^\infty\to H_w^\infty} = \limsup_{k\to\infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

Theorem 3.1. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the operator $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded if and only if $u \in \mathcal{B}, \sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)||\varphi'(z)| < \infty$ and*

$$\sup_{j\geq 2} \frac{\log(j-1)}{j} \|I_u(\varphi^j)\|_{\mathcal{B}} < \infty. \tag{9}$$

Proof. By Theorem 1 of [14], $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded if and only if $u \in \mathcal{B}$ and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)||\varphi'(z)| \log \frac{e}{1-|\varphi(z)|^2} < \infty. \tag{10}$$

By Lemma 3.2, (10) is equivalent to the weighted composition operator $u\varphi' C_\varphi : H_{v_{\log}}^\infty \to H_{v_1}^\infty$ is bounded. By Lemma 3.3, this is equivalent to

$$\sup_{j\geq 1} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{\log}}} < \infty.$$

Since $I_u(\varphi^j)(0) = 0$, $\left(I_u(\varphi^j)(z)\right)' = ju(z)\varphi'(z)\varphi^{j-1}(z)$, by Lemma 3.1, we see that $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded if and only if $u \in \mathcal{B}$ and

$$\begin{aligned} \infty &> \sup_{j\geq 1} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{\log}}} = \sup_{j\geq 1} \frac{j^{-1}\|I_u(\varphi^j)\|_{\mathcal{B}}}{\|z^{j-1}\|_{v_{\log}}} \\ &\approx \max\Big\{\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)||\varphi'(z)|,\ \sup_{j\geq 2} \frac{\log(j-1)}{j}\|I_u(\varphi^j)\|_{\mathcal{B}}\Big\}. \end{aligned}$$

The proof is complete. □

Theorem 3.2. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that the operator $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded. Then*

$$\|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} \approx \limsup_{j\to\infty} \frac{\log(j-1)}{j}\|I_u(\varphi^j)\|_{\mathcal{B}}.$$

Proof. From Theorem 2.1, Lemmas 3.1 and 3.2, we have

$$\begin{aligned} \|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}} \approx E &= \|u\varphi' C_\varphi\|_{e,H_{v_{\log}}^\infty\to H_{v_1}^\infty} = \limsup_{j\to\infty} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{\log}}} \\ &\approx \limsup_{j\to\infty} \log(j-1)\|u\varphi'\varphi^{j-1}\|_{v_1} = \limsup_{j\to\infty} \frac{\log(j-1)}{j}\|I_u(\varphi^j)\|_{\mathcal{B}}, \end{aligned}$$

as desired. The proof is complete. □

From Theorem 3.2, we immediately get the following new characterization of the compactness of the operator $uC_\varphi : \mathcal{Z} \to \mathcal{B}$.

Corollary 3.1. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that the operator $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded. Then $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is compact if and only if*

$$\limsup_{j\to\infty} \frac{\log(j-1)}{j} \|I_u(\varphi^j)\|_{\mathcal{B}} = 0.$$

4. Essential norm of $uC_\varphi : \mathcal{B} \to \mathcal{B}$

In this section, we give a new estimate of the essential norm for the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$.

Theorem 4.1. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded. Then*

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \approx \max\Big\{\limsup_{|a|\to1} \big\|uC_\varphi(x_a)\big\|_{\mathcal{B}}, \limsup_{|a|\to1} \big\|uC_\varphi(y_a)\big\|_{\mathcal{B}}\Big\},$$

where

$$x_a(z) := \frac{\left(\log \frac{e}{1-\bar{a}z}\right)^2}{\log \frac{e}{1-|a|^2}}, \qquad y_a(z) := \frac{\left(\log \frac{e}{1-\bar{a}z}\right)^3}{\left(\log \frac{e}{1-|a|^2}\right)^2}.$$

Proof. When $\|\varphi\|_\infty < 1$. It is easy to see that $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is compact by using Lemma 2.1. In this case, the asymptotic relations vacuously holds.

Now we consider the case $\|\varphi\|_\infty = 1$. For the simplicity of the proof, we denote

$$A := \limsup_{|a|\to1} \big\|uC_\varphi(x_a)\big\|_{\mathcal{B}}, \quad B := \limsup_{|a|\to1} \big\|uC_\varphi(y_a)\big\|_{\mathcal{B}}.$$

Let $a \in \mathbb{D}$.

$$\begin{aligned}
\|x_a\|_{\mathcal{B}} &= \sup_{z\in\mathbb{D}} (1-|z|^2)\frac{2|a|}{e}\left|\frac{1}{1-\bar{a}z}\frac{\log \frac{e}{1-\bar{a}z}}{\log \frac{e}{1-|a|^2}}\right| \\
&\lesssim \sup_{z\in\mathbb{D}} \left|\frac{\log \frac{e}{1-\bar{a}z}}{\log \frac{e}{1-|a|^2}}\right| = \left(\log \frac{e}{1-|a|^2}\right)^{-1} \sup_{z\in\mathbb{D}} \left|\log \frac{e}{1-\bar{a}z}\right|.
\end{aligned}$$

Since

$$\left|\log \frac{e}{1-\bar{a}z}\right| \lesssim \log\left|\frac{e}{1-\bar{a}z}\right| = \log\left|\frac{\frac{e}{1-\bar{a}z}}{\frac{e}{1-|a|^2}} \cdot \frac{e}{1-|a|^2}\right| \le \log\left|2 \cdot \frac{e}{1-|a|^2}\right| = \log 2 + \log \frac{e}{1-|a|^2},$$

we get $\|x_a\|_{\mathcal{B}} < \infty$ for all $a \in \mathbb{D}$. Similarly we have $\|y_a\|_{\mathcal{B}} < \infty$ for all $a \in \mathbb{D}$. Clearly x_a, y_a converge to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Thus, for any compact operator $K : \mathcal{B} \to \mathcal{B}$, we have

$$\lim_{|a|\to1} \|K(x_a)\|_{\mathcal{B}} = 0, \quad \lim_{|a|\to1} \|K(y_a)\|_{\mathcal{B}} = 0.$$

Similarly to the proof of Theorem 2.1, we get

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} = \inf_K \|uC_\varphi - K\|_{\mathcal{B}\to\mathcal{B}} \gtrsim \max\big\{A, B\big\}.$$

Next, we prove that

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \lesssim \max\big\{A, B\big\}.$$

For $r \in [0,1)$, it is easy to check that the operator K_r is also compact on $\mathcal{B}$ and $\|K_r\|_{\mathcal{B}\to\mathcal{B}} \leq 1$ (see also [18]). Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for all positive integer j, the operator $uC_\varphi K_{r_j} : \mathcal{B} \to \mathcal{B}$ is compact. By the definition of the essential norm, we get

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \leq \limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathcal{B}\to\mathcal{B}}. \tag{11}$$

Therefore, we only need to prove that

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathcal{B}\to\mathcal{B}} \lesssim \max\Big\{A, B\Big\}.$$

For any $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{B}} \leq 1$, we consider

$$\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}} = |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| + \|u\cdot(f - f_{r_j})\circ\varphi\|_\beta. \tag{12}$$

It is clear that

$$\lim_{j\to\infty} |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0. \tag{13}$$

Now we estimate

$$\limsup_{j\to\infty} \|u\cdot(f - f_{r_j})\circ\varphi\|_\beta \leq P_1 + P_2 + P_3 + P_4, \tag{14}$$

where

$$P_1 := \limsup_{j\to\infty} \sup_{|\varphi(z)|\leq r_N} (1-|z|^2)|(f - f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)|,$$

$$P_2 := \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|(f - f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)|,$$

$$P_3 := \limsup_{j\to\infty} \sup_{|\varphi(z)|\leq r_N} (1-|z|^2)|(f - f_{r_j})(\varphi(z))||u'(z)|,$$

$$P_4 := \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|(f - f_{r_j})(\varphi(z))||u'(z)|$$

and $N \in \mathbb{N}$ is large enough such that $r_j \geq \frac{1}{2}$ for all $j \geq N$. Similarly to the proof of Theorem 2.1 we have

$$P_1 \leq \widetilde{K} \limsup_{j\to\infty} \sup_{|w|\leq r_N} |f'(w) - r_j f'(r_j w)| = 0 \tag{15}$$

and

$$P_3 \leq \|u\|_{\mathcal{B}} \limsup_{j\to\infty} \sup_{|w|\leq r_N} |f(w) - f(r_j w)| = 0. \tag{16}$$

We consider P_2. We have $P_2 \leq \limsup_{j\to\infty}(I_1 + I_2)$, where

$$I_1 := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f'(\varphi(z))||\varphi'(z)||u(z)|, \qquad I_2 := \sup_{|\varphi(z)|>r_N} (1-|z|^2) r_j |f'(r_j\varphi(z))||\varphi'(z)||u(z)|.$$

First we estimate I_1. Using the fact that $\|f\|_{\mathcal{B}} \leq 1$, we have

$$\begin{aligned}
I_1 &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f'(\varphi(z))||\varphi'(z)||u(z)| \\
&\lesssim \frac{1}{r_N}\|f\|_{\mathcal{B}} \sup_{|\varphi(z)|>r_N} (1-|z|^2)|\varphi'(z)||u(z)| \frac{|\varphi(z)|}{1-|\varphi(z)|^2} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \frac{(1-|z|^2)|\varphi'(z)||u(z)||\varphi(z)|}{1-|\varphi(z)|^2} \\
&\lesssim \sup_{|a|>r_N} \left\|uC_\varphi(x_a - y_a)\right\|_{\mathcal{B}} \lesssim \sup_{|a|>r_N} \left\|uC_\varphi(x_a)\right\|_{\mathcal{B}} + \sup_{|a|>r_N} \left\|uC_\varphi(y_a)\right\|_{\mathcal{B}}.
\end{aligned}$$

Taking the limit as $N \to \infty$ we obtain

$$\limsup_{j\to\infty} I_1 \lesssim \limsup_{|a|\to 1} \left\|uC_\varphi(x_a)\right\|_{\mathcal{B}} + \limsup_{|a|\to 1} \left\|uC_\varphi(y_a)\right\|_{\mathcal{B}} = A + B.$$

Similarly, we have $\limsup_{j\to\infty} I_2 \lesssim A + B$, *i.e.*, we get that

$$P_2 \lesssim A + B \lesssim \max\left\{A, B\right\}. \tag{17}$$

We have $P_4 \le \limsup_{j\to\infty}(I_3 + I_4)$, where

$$I_3 := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f(\varphi(z))||u'(z)|, \qquad I_4 := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f(r_j\varphi(z))||u'(z)|.$$

Since $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \le 1$, we know that

$$|f(z)| \le \|f\|_{\mathcal{B}} \log\frac{e}{1-|z|^2} \lesssim \log\frac{e}{1-|z|^2}.$$

After a calculation, we have

$$\begin{aligned}
I_3 &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f(\varphi(z))||u'(z)| \\
&\lesssim \sup_{|\varphi(z)|>r_N} (1-|z|^2)|u'(z)|\|f\|_{\mathcal{B}} \log\frac{e}{1-|\varphi(z)|^2} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \frac{1}{3}(1-|z|^2)|u'(z)| \log\frac{e}{1-|\varphi(z)|} \\
&\lesssim \sup_{|a|>r_N} \left\|uC_\varphi(x_a - \frac{2}{3}y_a)\right\|_{\mathcal{B}} \lesssim \sup_{|a|>r_N} \left\|uC_\varphi(x_a)\right\|_{\mathcal{B}} + \frac{2}{3}\sup_{|a|>r_N} \left\|uC_\varphi(y_a)\right\|_{\mathcal{B}} \\
&\le \sup_{|a|>r_N} \left\|uC_\varphi(x_a)\right\|_{\mathcal{B}} + \sup_{|a|>r_N} \left\|uC_\varphi(y_a)\right\|_{\mathcal{B}}.
\end{aligned}$$

Taking limit as $N \to \infty$ we obtain

$$\limsup_{j\to\infty} I_3 \lesssim \limsup_{|a|\to 1} \left\|uC_\varphi(x_a)\right\|_{\mathcal{B}} + \limsup_{|a|\to 1} \left\|uC_\varphi(y_a)\right\|_{\mathcal{B}} = A + B.$$

Similarly, we have $\limsup_{j\to\infty} I_4 \lesssim A + B$, *i.e.*, we get that

$$P_4 \lesssim A + B \lesssim \max\left\{A, B\right\}. \tag{18}$$

Hence, by (12)-(18) we get

$$\begin{aligned}
\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathcal{B}\to\mathcal{B}} &= \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}}\le 1} \|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}} = \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}}\le 1} \|u\cdot(f - f_{r_j})\circ\varphi\|_{\beta} \\
&\lesssim \max\left\{A, B\right\}.
\end{aligned} \tag{19}$$

Therefore, by (11) and (19), we obtain

$$\|uC_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} \lesssim \max\left\{A, B\right\}.$$

The proof is complete. □

From Theorem 4.1, we immediately get the following new characterization of the compactness of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$.

Corollary 4.1. *Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded. Then $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if*

$$\limsup_{|a|\to 1} \left\|uC_\varphi(x_a)\right\|_{\mathcal{B}} = \limsup_{|a|\to 1} \left\|uC_\varphi(y_a)\right\|_{\mathcal{B}} = 0.$$

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