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Existence and Uniqueness Results for an Inverse Problem for Semilinear Parabolic Equations

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Abstract. In the present study, unique solvability of an inverse problem governed by semilinear parabolic equations with an integral overdetermination is investigated. Furthermore, for the approximate solution of this problem a first order of accuracy difference scheme is constructed. Existence and uniqueness results for the solution of this difference scheme are established. Considering a particular example, some numerical results are discussed.

1. Introduction and statement of the problem

There has been a growing interest for investigation of inverse problems due to many models of real life are described by them. In particular, there are many models which consist inverse problems of identifying the unknown source term. Actually, these problems are also called as source identification problems (SIPs) in the literature. The SIPs governed by linear equations are studied from different aspects. Existence and uniqueness of the solutions of such problems are investigated by many researchers (see [1–3] and the references therein). Also several methods and techniques are developed for the solutions of SIPs (for instance, see [4–6]). The numerical solution of these problems is another aspect. The advancements in computer technology allows us to search for more accurate and effective numerical methods. Yet these methods need to be investigated theoretically, as well. Recently, there have been many investigations for the approximate solutions of SIPs [7–11].

In SIPs for identifying the unknown source term an overdetermined condition is given. This condition can be given as local [10, 11], nonlocal [13], or integral condition [14]. These conditions affect the nature of problems and according to them various techniques are applied to study SIPs. It is also possible to classify SIP with respect to source type. While some papers are into SIPs with space-dependent source [8, 9, 15], some papers are into the ones with time-dependent source [10, 11, 16]. In this paper, a time-dependent SIP with an integral overdetermination is considered.

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Guidetti [14] proved the existence and uniqueness of the solution, which is also of maximal regularity type, of a problem of reconstruction of the source term in an abstract parabolic system. Erdogan [16] investigated the stability of a time-dependent SIP with a local overdetermination in a Hölder space and gave a numerical method for the approximate solution of this problem. Yang *et all* [10] developed a numerical method for the approximate solution of a time-dependent SIP for a parabolic equation with integral condition. Borukhov and Zayats [12] applied an approach based on the theory of inverse infinite-dimensional open dynamical systems to solve a nonlinear source identification problem.

On the other hand, the SIPs governed by semilinear equations have not been well-investigated. For this reason, in this paper, we deal with the existence and uniqueness results for the solution of the inverse problem governed by semilinear parabolic equations

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right) = p\left(t\right)q\left(x\right) + f(t,x,u), x \in (0,l), \\ \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right) + b(t,x)u(t,x) = p\left(t\right)q\left(x\right) \\ +g(t,x,u),\ x \in (l,L),\ t \in (0,T), \\ u(0,x) = \varphi(x),\ x \in [0,L], \\ u_x(t,0) = u(t,L) = 0,\ \int_0^L u\left(t,x\right)dx = \psi\left(t\right), \\ u(t,l+) = u(t,l-),\ u_x(t,l+) = u_x(t,l-),\ t \in [0,T]. \end{cases} \tag{1}$$

Here, $u\left(t,x\right)$ and $p\left(t\right)$ are unknown functions, $a(x) \geq a > 0$, $f(t,x)$, $\psi\left(t\right)$ and $\varphi(x)$ are given sufficiently smooth functions, and $q\left(x\right)$ is a sufficiently smooth function assuming $q'\left(0\right) = q\left(L\right) = 0$ and $\int_0^L q\left(x\right)dx \neq 0$. Note that problem (1) is a specific mathematical formulation of two-phase fluid flow in blood vessels [11].

In [17], stability inequalities for this problem governed by linear parabolic equations were presented. Moreover, Rothe and Crank-Nicholson difference schemes for the numerical solution of this linear problem were presented. We may also note that some results of this paper were presented in [18] without proof

The organization of the rest of the paper is as follows: In Section 2, theorems on the unique solvability of the differential problem are established. In Section 3, for the approximate solution of problem (1) a first order of accuracy difference scheme is proposed. Moreover, theorems on unique solvability of this difference scheme are established. In Section 4, some numerical experiments and discussions are given. In the last section, some concluding remarks are mentioned.

2. The Differential Problem

We have the following theorem on the the unique solvability of differential problem (1). Note that throughout the paper, K_i's symbolize the positive constants.

Theorem 2.1. *Suppose that* $f \in C\left([0,T], L_2[0,l]\right)$, $g \in C\left([0,T], L_2[l,L]\right)$, *and there are constants* $K_1, K_2 > 0$ *such that* f *and* g *satisfy the Lipschitz conditions*

$$\left\|f\left(t,\cdot,u\right) - f\left(t,\cdot,v\right)\right\|_{L_2[0,l]} \leq K_1 \left\|u-v\right\|_{L_2[0,l]}, \tag{2}$$

$$\left\|g\left(t,\cdot,u\right) - g\left(t,\cdot,v\right)\right\|_{L_2[l,L]} \leq K_2 \left\|u-v\right\|_{L_2[l,L]} \tag{3}$$

for all $t \in [0,T]$, $u, v \in L_2[0,L]$. *Then problem (1) has a unique solution in* $C\left([0,T], L_2[0,L]\right)$. *Here,* $C\left([0,T], H\right)$ *is the space of all continuous functions* $\phi\left(t\right)$ *defined on* $[0,T]$ *with values in the Hilbert space* H, *and is equipped with the norm*

$$\left\|\phi\right\|_{C([0,T],H)} = \max_{0\leq t\leq T}\left\|\phi\left(t\right)\right\|_H.$$

Proof. For the solution of problem (1), we consider substitution

$$u\left(t,x\right) = \eta\left(t\right)q\left(x\right) + w\left(t,x\right), \eta\left(t\right) = \int_0^t p\left(s\right)ds, \eta\left(0\right) = 0, \tag{4}$$

and $w(t,x)$ is the solution of problem

$$\begin{cases} \frac{\partial w(t,x)}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial w(t,x)}{\partial x}\right)=f(t,x,w(t,x)+\eta(t)q(x)) \\ +\frac{\partial}{\partial x}\left(a(x)\eta(t)\frac{\partial q(x)}{\partial x}\right),\ x\in(0,l), \\ \\ \frac{\partial w(t,x)}{\partial t}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial w(t,x)}{\partial x}\right)+b(t,x)w(t,x) \\ =g(t,x,w(t,x)+\eta(t)q(x))+\frac{\partial}{\partial x}\left(a(x)\eta(t)\frac{\partial q(x)}{\partial x}\right) \\ -b(t,x)\eta(t)q(x),\ x\in(l,L),\ t\in(0,T), \\ \\ w(0,x)=\varphi(x),\ x\in[0,L], \\ w_x(t,0)=w(t,L)=0, \\ w(t,l+)=w(t,l-),\ w_x(t,l+)=w_x(t,l-),\ t\in[0,T]. \end{cases} \tag{5}$$

Hence, the proof of Theorem 2.1 is based on substitution (4) and the following theorem on the uniqueness of solution of problem (5). □

Theorem 2.2. *Assume that the conditions of Theorem 2.1 are satisfied, then problem (5) has a unique solution in* $C([0,T], L_2[0,L])$.

Proof. Let us introduce the positive definite self-adjoint operator A defined by the formula

$$Aw=-\frac{\partial}{\partial x}\left(a(x)\frac{\partial w(x)}{\partial x}\right) \tag{6}$$

with the domain

$$D(A)=\{w: w,w''\in L_2[0,L],\ w'(0)=w(L)=0,\ w(l-)=w(l+),\ w'(l-)=w'(l+)\},$$

and also the bounded operator $B(t)$ defined by the formula

$$B(t)w=\begin{cases} 0, & x\in(0,l), \\ b(t,x)w(t,x), & x\in(l,L) \end{cases} \tag{7}$$

with the domain $D(B(t))=C([0,T],L_2[0,L])$. Then, with the help of these operators problem (5) can be written in the abstract form as

$$\begin{cases} \frac{dw(t)}{dt}+Aw(t)=F(t,w(t)+\eta(t)q)-B(t)w(t) \\ \\ -\eta(t)(A+B(t))q,\ 0<t<T,\ w(0)=\varphi, \end{cases} \tag{8}$$

where

$$F(t,\cdot,w(t)+\eta(t)q)=\left\{\begin{array}{l} f(t,x,w(t,x)+\eta(t)q(x)),\ 0<x<l, \\ \\ g(t,x,w(t,x)+\eta(t)q(x)),\ l<x<L \end{array}\right\}.$$

It is clear that problem (8) can be written in equivalent operator form

$$w(t)=Fw(t),$$

where

$$Fw(t)=e^{-tA}\varphi+\int_0^t e^{-(t-s)A}\{F(s,w(s)+\eta(s)q)-B(s)w(s)-\eta(s)(A+B(s))q\}ds.$$

Obviously, $F : C([0,T], L_2[0,L]) \to C([0,T], L_2[0,L])$ is a continuous mapping. Now, we will prove that F is a contraction mapping in $C([0,T], L_2[0,L])$. Note that in $C([0,T], H)$, the norms

$$\left\|\varphi\right\|_{C^*([0,T],H)} = \max_{0\le t\le T} e^{-\mu t}\left\|\varphi(t)\right\|_H$$

and

$$\left\|\varphi\right\|_{C([0,T],H)} = \max_{0\le t\le T}\left\|\varphi(t)\right\|_H$$

are equivalent. Applying the triangle inequality, Lipschitz conditions (2) and (3), we reach

$$e^{-\mu t}\left\|Fw - Fv\right\|_{L_2[0,L]} \le K_4\frac{1-e^{-\mu T}}{\mu}\left\|w-v\right\|_{C^*([0,T],L_2[0,L])}$$

for all $t \in [0,T]$ and $w(t), v(t) \in C^*([0,T], L_2[0,L])$. Here, $K_4 = \|B(t)\|_{L_2[0,L]\to L_2[0,L]} + \max\{K_1, K_2\} + K_3$. Therefore,

$$\left\|Fw - Fv\right\|_{C^*([0,T],L_2[0,L])} \le \alpha_\mu\left\|w-v\right\|_{C^*([0,T],L_2[0,L])},$$

where

$$\alpha_\mu = K_4\frac{1-e^{-\mu T}}{\mu}.$$

It is easy to see that as $\mu\to\infty$, $\alpha_\mu\to 0$. That means F is a contraction mapping, so by Banach fixed-point theorem, problem (5) has a unique solution in $C([0,T], L_2[0,L])$. □

3. The Difference Problem

In this section, for the approximate solution of problem (1) we study the first order of accuracy Rothe difference scheme. We consider the set of mesh points (t_k, x_n), where

$$t_k = k\tau,\ 0\le k\le N,\ N\tau = T$$

and

$$x_n = \begin{cases} nh_0,\ 0\le n\le M_1,\ M_1h_0 = l, \\ l + (n-M_1)h,\ M_1 < n\le M,\ (M-M_1)h = L-l. \end{cases}$$

3.1. *Rothe Difference Scheme*

The Rothe difference scheme for the numerical solution of problem (1) is

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{1}{h_0}\left(a(x_{n+1})\frac{u_{n+1}^k-u_n^k}{h_0} - a(x_n)\frac{u_n^k-u_{n-1}^k}{h_0}\right) = p_k q(x_n) + f\left(t_k, x_n, u_n^k\right), \\ 1\le k\le N,\ 1\le n\le M_1-1, \\ \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{1}{h}\left(a(x_{n+1})\frac{u_{n+1}^k-u_n^k}{h} - a(x_n)\frac{u_n^k-u_{n-1}^k}{h}\right) + b(t_k, x_n)u_n^k = p_k q(x_n) \\ +g\left(t_k, x_n, u_n^k\right),\ 1\le k\le N,\ M_1+1\le n\le M, \\ u_n^0 = \varphi(x_n),\ 0\le n\le M, \\ u_1^k - u_0^k = u_M^k = 0,\ 0\le k\le N, \\ \sum\limits_{j=1}^{M} u_j^k h^* = \psi(t_k),\ 0\le k\le N, \\ h_0\left(u_{M_1+1}^k - u_{M_1}^k\right) = h\left(u_{M_1}^k - u_{M_1-1}^k\right),\ 0\le k\le N. \end{cases} \tag{9}$$

Here, it is assumed that $q_1 - q_0 = q_M = 0$, and $\sum_{j=1}^M q_j h^* \ne 0$. Here and throughout the paper,

$$h^* = \begin{cases} h_0, 1\le j\le M_1, \\ h, M_1+1\le j\le M. \end{cases}$$

Theorem 3.1. *Assume that f and g satisfy Lipschitz conditions (2) and (3), respectively. Then difference scheme (9) has a unique solution $U_h^\tau = \left\{ \left\{ u_n^k \right\}_{n=0}^{M} \right\}_{k=1}^{N}$ in $C_\tau (L_{2h})$, where $C_\tau (L_{2h})$ is the space of grid functions v_h^τ with the norm*

$$\left\| v_h^\tau \right\|_{C_\tau(L_{2h})} = \max_{1\le k\le N} \left(\sum_{j=0}^{M} \left| v_j^k \right|^2 h^* \right)^{1/2}.$$

Proof. For the solution of difference scheme (9) we consider the substitution

$$u_n^k = \eta^k q_n + w_n^k, \tag{10}$$

where

$$q_n = q(x_n),\ \eta^k = \sum_{j=1}^{k} p^j \tau, 1 \le k \le N, \eta^0 = 0,$$

and w_n^k is the solution of the problem

$$\begin{cases} \frac{w_n^k - w_n^{k-1}}{\tau} - \frac{1}{h_0}\left[a(x_{n+1}) \left(\frac{w_{n+1}^k - w_n^k}{h_0} + \eta^k \left(\frac{q_{n+1}-q_n}{h_0} \right) \right) \right. \\ \left. -a(x_n) \left(\frac{w_n^k - w_{n-1}^k}{h_0} + \eta^k \left(\frac{q_n - q_{n-1}}{h_0} \right) \right) \right] = f\left(t_k, x_n, w_n^k + \eta^k q_n \right), \\ 1 \le k \le N,\ 1 \le n \le M_1 - 1, \\ \frac{w_n^k - w_n^{k-1}}{\tau} - \frac{1}{h}\left[a(x_{n+1}) \left(\frac{w_{n+1}^k - w_n^k}{h} + \eta^k \left(\frac{q_{n+1}-q_n}{h} \right) \right) \right. \\ \left. -a(x_n) \left(\frac{w_n^k - w_{n-1}^k}{h} + \eta^k \left(\frac{q_n - q_{n-1}}{h} \right) \right) \right] + b(t_k, x_n) \left(w_n^k + \eta^k q_n \right) \\ = g\left(t_k, x_n, w_n^k + \eta^k q_n \right), 1 \le k \le N,\ M_1 + 1 \le n \le M, \\ w_n^0 = \varphi(x_n),\ 0 \le n \le M, \\ w_1^k - w_0^k = w_M^k = 0,\ 0 \le k \le N, \\ h_0 \left(w_{M_1+1}^k - w_{M_1}^k \right) = h \left(w_{M_1}^k - w_{M_1-1}^k \right),\ 0 \le k \le N. \end{cases} \tag{11}$$

Thus, the proof of Theorem 3.1 is based on substitution (10) and the following theorem on the uniqueness of solution of difference scheme (11). □

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are satisfied, then difference scheme (11) has a unique solution in $C_\tau (L_{2h})$.*

Proof. Firstly, to the differential operators A and $B(t)$ defined by (6) and (7), we introduce the difference operator A_h defined by the formula

$$A_h w_k^h = \left\{ -\frac{1}{h^*} \left[a(x_{n+1}) \frac{w_{n+1}^k - w_n^k}{h^*} - a(x_n) \frac{w_n^k - w_{n-1}^k}{h^*} \right] \right\}_{n=1}^{M-1}$$

acting on the grid functions $\left\{ w_k^h \right\}_{k=0}^{N}$ satisfying the conditions

$$w_1^k - w_0^k = w_M^k = 0,\ h_0 \left(w_{M_1+1}^k - w_{M_1}^k \right) = h \left(w_{M_1}^k - w_{M_1-1}^k \right),\ 0 \le k \le N,$$

and the bounded operator $B_h(t_k)$ defined by the formula

$$B_h(t_k)w_k^h = \begin{cases} 0, & 1 \leq n \leq M_1, \\ b(t_k, x_n)\, w_n^k, & M_1 + 1 \leq n \leq M - 1 \end{cases}$$

acting on the grid functions $\left\{w_k^h\right\}_{k=0}^N$. Then, with the help of these operators difference scheme (11) can be written in the abstract form as

$$\begin{cases} \frac{w_k^h - w_{k-1}^h}{\tau} + A_h w_k^h = F\left(t_k, w_k^h + \eta^k q^h\right) - B_h(t_k) w_k^h \\ -\eta^k \left(A_h + B_h(t_k)\right) q^h, \ 1 \leq k \leq N\ , \ w_0^h = \varphi, \end{cases} \tag{12}$$

where

$$F\left(t_k, \cdot, w_k^h + \eta^k q^h\right) = \left\{ \begin{array}{c} f\left(t_k, x_n, w_n^k + \eta^k q_n\right), 1 \leq n \leq M_1 - 1, \\ 0, \\ g\left(t_k, x_n, w_n^k + \eta^k q_n\right), M_1 + 1 \leq n \leq M \end{array} \right\}.$$

Thus, the proof of this theorem follows from the discrete analogue of the proof of Theorem 2.2 using the mapping F^h defined by formula

$$F^h w_k^h = R^k \varphi + \sum_{i=1}^{k-1} R^{k-i+1} \left[F\left(t_i, w_i^h + \eta^i q^h\right) - B_h(t_i) w_i^h - \eta^i \left(A_h^x + B_h(t_i)\right) q^h \right] \tau, \ 1 \leq k \leq N,$$

where $R = (I + \tau A_h)^{-1}$. $\square$

In order to obtain the approximate solution of difference scheme (9), we consider the auxiliary difference scheme (11) with the following formulas. Using the integral overdetermination, one can easily show that

$$\eta^k = \frac{\psi(t_k) - \sum\limits_{j=1}^{M} w_j^k h^*}{\sum\limits_{j=1}^{M} q_j h^*}, \ p^k = \frac{\left(\psi(t_k) - \sum\limits_{j=1}^{M} w_j^k h^*\right) - \left(\psi(t_{k-1}) - \sum\limits_{j=1}^{M} w_j^{k-1} h^*\right)}{\tau \sum\limits_{j=1}^{M} q_j h^*}. \tag{13}$$

Thus, the solution of difference scheme (11) is obtained by the iterative difference scheme

$$\begin{cases} \frac{w_n^{k,m} - w_n^{k-1,m}}{\tau} - \frac{1}{h_0}\left[a(x_{n+1})\left(\frac{w_{n+1}^{k,m} - w_n^{k,m}}{h_0} + \eta^k\left(\frac{q_{n+1} - q_n}{h_0}\right)\right)\right. \\ \left. -a(x_n)\left(\frac{w_n^{k,m} - w_{n-1}^{k,m}}{h_0} + \eta^k\left(\frac{q_n - q_{n-1}}{h_0}\right)\right)\right] = f\left(t_k, x_n, w_n^{k,m-1} + \eta^k q_n\right), \\ 1 \leq k \leq N, \ 1 \leq n \leq M_1 - 1, \ m = 1, 2, \cdots, \\ \frac{w_n^{k,m} - w_n^{k-1,m}}{\tau} - \frac{1}{h}\left[a(x_{n+1})\left(\frac{w_{n+1}^{k,m} - w_n^{k,m}}{h} + \eta^k\left(\frac{q_{n+1} - q_n}{h}\right)\right)\right. \\ \left. -a(x_n)\left(\frac{w_n^{k,m} - w_{n-1}^{k,m}}{h_0} + \eta^k\left(\frac{q_n - q_{n-1}}{h_0}\right)\right)\right] + b(t_k, x_n)\left(w_n^{k,m} + \eta^k q_n\right) \\ = g\left(t_k, x_n, w_n^{k,m-1} + \eta^k q_n\right), 1 \leq k \leq N, \ M_1 + 1 \leq n \leq M, \ m = 1, 2, \cdots, \\ w_n^{k,0} \text{ is given}, \ w_n^{0,m} = \varphi(x_n), \ 0 \leq n \leq M, \ m = 1, 2, \cdots, \\ w_1^{k,m} - w_0^{k,m} = w_M^{k,m} = 0, \ 0 \leq k \leq N, \ m = 1, 2, \cdots, \\ h_0\left(w_{M_1+1}^{k,m} - w_{M_1}^{k,m}\right) = h\left(w_{M_1}^{k,m} - w_{M_1-1}^{k,m}\right), \ 0 \leq k \leq N, \ m = 1, 2, \cdots. \end{cases} \tag{14}$$

When the the maximum difference at grid points of two successive results gets less than the termination criteria ϵ the iterative computations are stopped. Once we obtain the results, using formula (10), we get the numerical solution for difference scheme (9).

4. Numerical Experiments and Discussion

For the discussion of the numerical results we consider the problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(\cos x \frac{\partial u(t,x)}{\partial x}\right) = p(t)\cos\frac{x\pi}{2} \\ +e^{-2t}\left(2x^2 - 2x^3 + \left(3x^2 - 2x\right)\sin x - (6x-2)\cos x\right) \\ -e^{-t}\cos\frac{x\pi}{2} - \sin\left(e^{-2t}\left(x^3 - x^2\right)\right) + \sin\left(u\left(t,x\right)\right),\ 0 < t \le 1,\ 0 < x < 3/4, \\ \frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(\cos x \frac{\partial u(t,x)}{\partial x}\right) + (t+x)\,u(t,x) = p(t)\cos\frac{x\pi}{2} \\ +e^{-2t}\left(2x^2 - 2x^3 + \left(3x^2 - 2x\right)\sin x - (6x-2)\cos x\right) \\ +e^{-2t}\left(t+x\right)\left(x^3 - x^2\right) - e^{-t}\cos\frac{x\pi}{2} - \sin\left(e^{-2t}\left(x^3 - x^2\right)\right) + \sin\left(u\left(t,x\right)\right), \\ 0 < t \le 1,\ 3/4 < x < 1, \\ u(0,x) = x^3 - x^2,\ 0 \le x \le 1, \\ u_x(t,0) = u(t,1) = 0,\ 0 \le t \le 1, \\ u(t,\frac{3}{4}+) = u(t,\frac{3}{4}-),\ u_x(t,\frac{3}{4}+) = u_x(t,\frac{3}{4}-),\ 0 \le t \le 1, \\ \int_0^1 u(t,x)dx = \frac{-e^{-2t}}{12},\ 0 \le t \le 1. \end{cases} \tag{15}$$

The exact solution pair of this problem is $(u,p) = \left(e^{-t}(x^3 - x^2), -e^{-t}\right)$.

4.1. Rothe difference scheme

In this section, we implement the difference scheme constructed in the previous section. In the computations the difference schemes start with the identical zero grid function $w_n^{k,0} = 0$ and the iterations are terminated after m-th iteration when the error becomes less than 10^{-4}.

For the approximate solution of problem (15) applying iterative difference scheme (14) we obtain a system of linear equations in $(M+1)\,(N+1)$ unknowns, whose matrix representation is

$$\begin{cases} A_k W^{k,m} + B W^{k-1,m} = \varphi^{k,m-1}, 1 \le k \le N, \\ \\ m = 1, 2, \cdots,\ W^0 = \{\cos 2x_n\}_{n=0}^{M}. \end{cases} \tag{16}$$

Here A_k and B are $(M+1)\times(M+1)$ square matrices, $W^{k,m}$ and $\varphi^{k,m}$ are $(M+1)\times 1$ column matrices. Note that as k changes, so does the coefficient matrix A_k. From (16) it follows that

$$W^{k,m} = (A_k)^{-1}\left(\varphi^{k,m-1} - BW^{k-1,m}\right),\ k = 1,\ldots,N,\ m = 1,2,\cdots. \tag{17}$$

For the computation we use MATLAB software. Finally using formulas (10), (13) and values of W^k obtained in the last iteration, the values of u_n^k are obtained.

4.2. Error Analysis

In this subsection, we present the errors between exact and approximate solutions of problem (15). Table 1 gives the error analysis between the exact solution and the solutions derived by difference schemes for different values of N and M. The following table presents the errors between exact solution $u\left(t_k, x_n\right)$ and the numerical solution of the difference scheme u_n^k. For comparison of the result, the error is computed by

$$E_m = \max_{1\le k\le N}\left(\sum_{n=0}^{M}\left|u_n^k - u(t_k,x_n)\right|^2 h^*\right)^{1/2}.$$

Table 1: Error analysis for the exact solution $u(t,x)$.

Method	$N=M=5$	$N=M=10$	$N=M=20$
1^{st} order of accuracy d.s.	0.0415	0.0244	0.0123

5. Concluding Remarks

In the present study, a time-dependent source identification problem governed by semilinear parabolic equations subject to an integral overdetermination is investigated. Moreover, for the numerical solution of this problem the first order of accuracy Rothe difference scheme is constructed. The existence and uniqueness results for these differential problem and difference scheme are established under the Lipschitz condition. For showing the validity of proposed numerical method, it is tested on an example. The numerical results confirm the theoretical results of this paper.

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