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Sturm-Liouville Problem via Coulomb Type in Difference Equations

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Abstract. We present Sturm-Liouville problem via Coulomb type in difference equations. The representation of solutions is found. We proved that these solutions satisfy the equation. Asymptotic formulas of eigenfunctions are set.

1. Introduction

Difference equations have always been an attractive subject because of the discrete analogue of differential equations. The basic theory of linear difference equations was developed by [1, 3, 4, 15, 16, 18–20, 22].

Spectral analysis of difference equations has seen a great interest. Recently, especially Sturm-Liouville difference equations and applications have formed a renewed classical research topic and it has attracted the attention of many scientists, see [1, 3, 16, 23].

Self-adjoint second-order difference equations are considered in [10] and developed by [1]. Spectral properties of second-order difference equations and operators are analyzed in [5, 11, 17, 21, 24, 25]. Eigenvalues of second order difference equations are considered in [13, 14, 26].

The representation of solutions and asymptotic formulas for eigenfunctions for Sturm-Liouville problem in differential equations are obtained in [2]. Sturm-Liouville operator with Coulomb potential in differential equations is studied in [9]. Fractional singular Sturm-Liouville operator for Coulomb potential is studied in [7]. The representation of solutions and asymptotic formulas for eigenfunctions for Sturm-Liouville problem in difference equations are obtained in [8].

We argue that any example of Sturm-Liouville problem via Coulomb type in difference equations isn't known.

Truly, as Sturm-Liouville difference equations are investigated with constant potential function, we present a new approach for the problem via Coulomb type. We obtain the representation of solutions with different initial conditions and asymptotic formulas for eigenfunctions in this study. Besides, it is shown that application of variation of parameters method in difference equations and these results satisfy the equation.

Primarily, we briefly consider Sturm-Liouville operator with Coulomb potential in differential equations. Motion of electrons moving under the Coulomb potential is of significance in quantum theory. This problem enable us to find energy levels for both hydrogen atom and single valence electron atoms. For hydrogen

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atom, the Coulomb potential is given by $U = \frac{-e^2}{r}$, where r is the radius of the nucleus, e is electronic charge. Accordingly, we use the time-dependent Schrödinger equation

$$i\tilde{h}\frac{\partial \omega}{\partial t} = -\frac{\tilde{h}^2}{2m}\frac{\partial^2 \omega}{\partial x^2} + U(x,y,z)\,\omega, \qquad \int\limits_{R^3} |\omega|^2\, dxdydz = 1,$$

where ω is the wave function, $\tilde{h}$ is a Planck's constant and m is the mass of electron. In this equation, if Fourier transform is applied

$$\tilde{\omega} = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-i\lambda t}\omega dt,$$

it will transform the energy equation dependent upon the situation,

$$\frac{\tilde{h}^2}{2m}\nabla^2\tilde{\omega} + \tilde{U}\tilde{\omega} = E\tilde{\omega}.$$

Hence, energy equation in the field with the Coulomb potential transforms

$$\frac{\tilde{h}^2}{2m}\nabla^2\tilde{\omega} + \left(E + \frac{e^2}{r}\right)\tilde{\omega} = 0.$$

If this hydrogen atom is replaced to other potential area, then the energy equation is as follows

$$-\frac{\tilde{h}^2}{2m}\nabla^2\tilde{\omega} + \left(E + \frac{e^2}{r} + q(x,y,z)\right)\tilde{\omega} = 0.$$

As a result of some transformations, we have Sturm-Liouville equation with Coulomb potential in differential equations

$$-y'' + \left[\frac{A}{x} + q(x)\right]y = \lambda y,$$

where λ is a parameter which corresponds to the energy [6].

Our aim is to present similar results to the following studies [2, 7–9]. The following problem (1) – (3)

$$\begin{aligned} -\Delta^2 u(n-1) + \left(\frac{A}{n} + q(n)\right)u(n) &= \lambda u(n),\ n = a, ..., b, && (1)\\ u(a-1) + hu(a) &= 0, && (2)\\ u(b+1) + ku(b) &= 0, && (3) \end{aligned}$$

is called Sturm-Liouville problem via Coulomb type in difference equations.

Basic theorems and definitions are presented in Section 2, representations of solutions with two different initial conditions in Section 3 and asymptotic behaviors of eigenfunctions in Section 3 are presented.

2. Preliminaries

Definition 2.1. *[23] The matrix of Casoratian is given by*

$$w(n) = \begin{pmatrix} u_1(n) & u_2(n) & ... & u_r(n) \\ u_1(n+1) & u_2(n+1) & ... & u_r(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(n+r-1) & u_2(n+r-1) & ... & u_r(n+r-1) \end{pmatrix}$$

where $u_1(n), u_2(n), ..., u_r(n)$ *are given functions. The determinant*

$$W(n) = \det w(n)$$

is called Casoratian. Recall that, it is discrete analogous of Wronskian determinant.

Theorem 2.2. *[1] (Wronskian-Type Identity) Let r and u be solutions of* (1). *Then, for* $a \leq n \leq b$

$$\begin{aligned} W[r,u](n) &= [r(n)\Delta u(n-1) - u(n)\Delta r(n-1)] \\ &= -[r(n)u(n-1) - r(n-1)u(n)] \end{aligned} \tag{4}$$

is a constant (In particular equal to $W[r,u](a)$*).*

Definition 2.3. *[1] Let's express Sturm-Liouville equation* (1) *as follows,*

$$Lu(n) = -\lambda u(n),\ n \in [a,b], \tag{5}$$

with initial conditions

$$\cos\alpha u(a) - \sin\alpha(\nabla u(a)) = 0, \tag{6}$$

where $0 < \alpha$, ∇ *is the backward difference operator,* $\nabla u(n) = u(n) - u(n-1)$, (6) *is equivalent to*

$$u(a-1) + (\cot\alpha - 1)u(a) = 0, \tag{7}$$

in other words,

$$u(a-1) + hu(a) = 0, \tag{8}$$

where L is a self-adjoint Sturm-Liouville operator via Coulomb type and h is a real number. Initial value problem $(5)-(8)$ *is called Sturm-Liouville problem via Coulomb type.*

Theorem 2.4. *[23]* (***Summation by parts***) *If* $m < n$, *then*

$$\sum_{k=m}^{n-1} u(k)\Delta r(k) = [u(k)r(k)]_m^n - \sum_{k=m}^{n-1}\Delta u(k)r(k+1). \tag{9}$$

Theorem 2.5. *[23] If* z_n *is an indefinite sum of* u_n, *then*

$$\sum_{k=m}^{n-1} z(k) = u(n) - u(m). \tag{10}$$

Theorem 2.6. *[23],[12]* (***Annihilator method***) *Suppose that* $u(n)$ *solves the following difference equation, E is the shift operator,* $E^t u(n) = u(n+t)$, *m and t are positive integers,*

$$\left(E^t + p(t-1)E^{t-1} + ... + p(0)\right)u(n) = z(n),$$

and that $z(n)$ *satisfies*

$$\left(E^m + q(m-1)E^{m-1} + ... + q(0)\right)z(n) = 0.$$

Then $u(n)$ *satisfies*

$$\left(E^m + q(m-1)E^{m-1} + ... + q(0)\right)\left(E^t + p(t-1)E^{t-1} + ... + p(0)\right)u(n) = 0.$$

3. Main Results

In this paper, we are interested in the representation of solutions of Sturm-Liouville problem via Coulomb type in difference equations,

$$-\Delta^2 u(n-1)+\left(\frac{A}{n}+q(n)\right)u(n)=\lambda u(n),\ n=a,...,b, \tag{11}$$

with initial conditions,

$$u(a-1)+hu(a)=0, \tag{12}$$

where a, b are nonzero finite integers with $a>0$, $a\leq b$, h, A are real numbers, Δ is the forward difference operator, $\Delta u(n)=u(n+1)-u(n)$, λ is a spectral parameter, $q(n)$ is a real valued potential function for $n\in[a,b]$, n is a finite integer.
A self-adjoint difference operator corresponds to equation (11) shown by,

$$Lu(n)=-\Delta^2 u(n-1)+\left(\frac{A}{n}+q(n)\right)u(n)=\lambda u(n).$$

In $\ell^2(a,b)$, the Hilbert space of sequences of complex numbers $u(a),...,u(b)$ with the inner product,

$$<u(n),r(n)>=\sum_{n=a}^{b}u(n)r(n),$$

for every $u\in D_{L,}$ let's define as follows

$$D_L=\left\{u(n)\in\ell^2(a,b):Lu(n)\in\ell^2(a,b),\ u(1)=-h,\ u(2)=1\right\}.$$

Hence, equation (11) can be written as follows,

$$Lu(n)=\lambda u(n).$$

At this section, we give the representation of solutions of Sturm-Liouville problem via Coulomb type $(11)-(12)$ by the variation of parameters method and show that the solutions satisfy the problem.

Theorem 3.1. *Let's define Sturm-Liouville problem via Coulomb type in difference equations as follows*

$$L\varphi(n)=\lambda\varphi(n), \tag{13}$$

$$\varphi(1)=-h,\ \varphi(2)=1, \tag{14}$$

then the problem $(13)-(14)$ *has a unique solution for* $\varphi(n)$ *as follows,*

$$\begin{aligned}\varphi(n,\lambda) &= [-2h\cos\theta-1+h(A+q(1))]\cos n\theta+\left(\frac{[1-h(A+q(1))]\cos\theta-h\cos 2\theta}{\sin\theta}\right)\sin n\theta\\ &\quad+\frac{1}{\sin\theta}\sum_{i=1}^{n}\left(\frac{A}{i}+q(i)\right)\varphi(i)\sin(i-n)\theta,\end{aligned}$$

where, $|\lambda-2|<2$, $\theta=\arctan\left(\frac{\sqrt{\lambda(4-\lambda)}}{2-\lambda}\right)$, $\sum_{i=1}^{0}.=0.$

Proof. It is proved at two parts. At the first part, it is proved that how solution is found and at the second part, it is proved that the result satisfies the equation.

Firstly, if $\varphi_1(n)$ and $\varphi_2(n)$ are linearly independent solutions for homogeneous part of (13), then it is found by characteristic polynomial and since $|\lambda-2|<2$, characteristic roots are complex pair [23],

$$\varphi_h(n)=v_1\varphi_1(n)+v_2\varphi_2(n), \tag{15}$$

$$\varphi_h(n)=v_1\cos n\theta+v_2\sin n\theta. \tag{16}$$

By the variation of parameters method [23],[3], we take

$$\varphi_p(n)=v_1(n)\varphi_1(n)+v_2(n)\varphi_2(n), \tag{17}$$

$$\begin{aligned}\Delta\varphi_p(n-1) &= \Delta v_1(n-1)\varphi_1(n)+\Delta v_2(n-1)\varphi_2(n)\\ &\quad +v_1(n-1)\Delta\varphi_1(n-1)+v_2(n-1)\Delta\varphi_2(n-1).\end{aligned}$$

Let's take the first two terms are zero, so that

$$\Delta v_1(n-1)\varphi_1(n)+\Delta v_2(n-1)\varphi_2(n)=0, \tag{18}$$

we obtain,

$$\Delta\varphi_p(n-1)=v_1(n-1)\Delta\varphi_1(n-1)+v_2(n-1)\Delta\varphi_2(n-1), \tag{19}$$

and if we apply Δ operator to the equation (18) and substitute obtained datas into (13) and collect terms involving $v_1(n)$ and $v_2(n)$, we find

$$\begin{aligned}&v_1(n-1)\Delta^2\varphi_1(n-1)+\Delta\varphi_1(n)\Delta v_1(n-1)+v_2(n-1)\Delta^2\varphi_2(n-1)\\ &+\Delta\varphi_2(n)\Delta v_2(n-1)+\lambda\left[v_1(n)\varphi_1(n)+v_2(n)\varphi_2(n)\right] = \left(\frac{A}{n}+q(n)\right)\varphi(n).\end{aligned} \tag{20}$$

We can write $v_1(n-1)\varphi_1(n)+v_2(n-1)\varphi_2(n)$ in place of the expression in bracket from (18) and collect terms involving $v_1(n-1)$ and $v_2(n-1)$, so

$$\begin{aligned}&v_1(n-1)\left[\Delta^2\varphi_1(n-1)+\lambda\varphi_1(n)\right]+v_2(n-1)\left[\Delta^2\varphi_2(n-1)+\lambda\varphi_2(n)\right]\\ &+\Delta\varphi_1(n)\Delta v_1(n-1)+\Delta\varphi_2(n)\Delta v_2(n-1) = \left(\frac{A}{n}+q(n)\right)\varphi(n),\end{aligned} \tag{21}$$

and since $\varphi_1(n)$ and $\varphi_2(n)$ hold for homogeneous part of (13), first two terms are zero,

$$\Delta v_1(n-1)\Delta\varphi_1(n)+\Delta v_2(n-1)\Delta\varphi_2(n)=\left(\frac{A}{n}+q(n)\right)\varphi(n). \tag{22}$$

Thus, we obtain equation system from (18) and (22), and its solution is

$$\begin{aligned}v_1(n-1) &= \sum_{i=1}^{n-1}\frac{\left(\frac{A}{i}+q(i)\right)\varphi(i)\varphi_2(i)}{W(\varphi_1(i),\varphi_2(i))},\\ v_2(n-1) &= -\sum_{i=1}^{n-1}\frac{\left(\frac{A}{i}+q(i)\right)\varphi(i)\varphi_1(i)}{W(\varphi_1(i),\varphi_2(i))}.\end{aligned}$$

Finally, we find general solution

$$\varphi(n,\lambda)=v_1\cos n\theta+v_2\sin n\theta+\sum_{i=1}^{n}\frac{\left(\frac{A}{i}+q(i)\right)\varphi(i)}{W(\varphi_1(i),\varphi_2(i))}\sin(i-n)\theta.$$

Where, Casoratian determinant W is a constant by Theorem 2.2,

$$W(\varphi_1(i), \varphi_2(i)) = \sin\theta.$$

If we use the initial conditions (14), then we obtain the representation of the solution of Sturm-Liouville problem via Coulomb type in difference equations as follows,

$$\begin{aligned}\varphi(n,\lambda) &= \left[-2h\cos\theta - 1 + h(A+q(1))\right]\cos n\theta + \left(\frac{\left[1-h\left(A+q(1)\right)\right]\cos\theta - h\cos 2\theta}{\sin\theta}\right)\sin n\theta \\ &\quad + \frac{1}{\sin\theta}\sum_{i=1}^{n}\left(\frac{A}{i}+q(i)\right)\varphi(i)\sin(i-n)\theta,\end{aligned} \tag{23}$$

Secondly, let's prove that this result satisfies the problem. Since $\varphi(n,\lambda)$ satisfies problem $(13)-(14)$, we have

$$\left(\frac{A}{n}+q(n)\right)\varphi(n) = \Delta^2\varphi(n-1) + \lambda\varphi(n),$$

$$\sum_{i=1}^{n}\frac{\left[\Delta^2\varphi(i-1)+\lambda\varphi(i)\right]}{\sin\theta}\sin(i-n)\theta = \sum_{i=1}^{n}\frac{\Delta^2\varphi(i-1)}{\sin\theta}\sin(i-n)\theta + \sum_{i=1}^{n}\frac{\lambda\varphi(i)}{\sin\theta}\sin(i-n)\theta. \tag{24}$$

By using twice summation by parts method to the first term at the right hand side of equation (24) by Theorem 2.4 and substituting initial conditions (14), we have (23).
The proof completes. □

Theorem 3.2. *Let's define Sturm-Liouville problem via Coulomb type in difference equations as follows;*

$$L\psi(n) = \lambda\psi(n), \tag{25}$$

$$\psi(1) = 1,\ \psi(2) = 0, \tag{26}$$

then the problem $(25)-(26)$ *has a unique solution for* $\psi(n)$ *as follows,*

$$\begin{aligned}\psi(n,\lambda) &= \left[2\cos\theta - A - q(1)\right]\cos n\theta - \left(\frac{\cos\theta(A+q(1)) - \cos 2\theta}{\sin\theta}\right)\sin n\theta \\ &\quad + \frac{1}{\sin\theta}\sum_{i=1}^{n}\left(\frac{A}{i}+q(i)\right)\psi(i)\sin(i-n)\theta.\end{aligned} \tag{27}$$

Proof. This is proved similarly to the proof of Theorem 3.1. □

4. Asymptotic Formulas for Sturm-Liouville Problems via Coulomb Type in Difference Equations

At this section, we give the asymptotic formulas for the solution of Sturm-Liouville problem via Coulomb type.

Theorem 4.1. *Sturm-Liouville problem via Coulomb type* $(13)-(14)$ *has the estimates*

$$\varphi(n) = O(|h|),$$

$$\varphi(n) = \left[-2h\cos\theta - 1 - h(A+q(1))\right]\cos n\theta + O(|h|).$$

Proof. If we apply triangle inequality to the equation (23) , we have

$$\left|\varphi(n)\right| \leq \frac{2\left[|h|+1+|h|\left(|A|+\left|q(1)\right|\right)\right]}{|\sin\theta|} + \frac{1}{|\sin\theta|}\sum_{j=1}^{n}\left(\frac{|A|}{|j|}+\left|q(j)\right|\right)\left|\varphi(j)\right|. \tag{28}$$

By the equation (24) and Theorem 2.5,

$$\left|\varphi(n)\right| \leq \tfrac{2\left[|h|+1+|h|\left(|A|+\left|q(1)\right|\right)\right]}{|\sin\theta|} + \tfrac{1}{|\sin\theta|}\left(\left|\Delta\varphi(n)\right| + 2|h| + 1 + |h|\left(|A|+\left|q(1)\right|\right) + \lambda\sum_{j=1}^{n}|\varphi(j)|\right). \tag{29}$$

Applying Δ operator both side of the equation (29) by Theorem 2.5,

$$\begin{aligned}\left(E^2+E\left(|\sin\theta|+2+\lambda\right)-1-|\sin\theta|\right)\left|\varphi(n)\right| &\leq \lambda|h|,\\ R(E)\varphi(n) &\leq \lambda|h|,\\ \varphi(n) &\leq \frac{1}{R(E)}\lambda|h|,\\ \varphi(n) &\leq \frac{1}{R(1+\Delta)}\lambda|h|,\end{aligned}$$

By means of Theorem 2.6

$$\frac{|\varphi(n)|}{|h|} \leq 24,$$

$$\varphi(n) = O(|h|),$$

where $R(E) = E^2 + E(|\sin\theta| + 2 + \lambda) - 1 - |\sin\theta|$. For obtaining the second formula, similar operations are applied.
The proof completes. □

Theorem 4.2. *Sturm-Liouville problem via Coulomb type* (25) – (26) *has the estimates*

$$\psi(n) = O(1),$$

$$\psi(n) = [2\cos\theta - A - q(1)]\cos n\theta + O(1).$$

Proof. Proof is similar to the proof of Theorem 4.1. □

5. Applications

Application 1. Obtain eigenpairs for Sturm-Liouville problem

$$\Delta^2 u(n-1) + (\lambda+6)u(n) = 0, \tag{30}$$

$$u(0) = 0,\ u(5) = 0.$$

The characteristic equation for equation (30) is

$$m^2 + (\lambda+4)m + 1 = 0,$$

hence

$$m_{1,2} = \frac{(-4-\lambda) \pm \sqrt{(\lambda+4)^2-4}}{2}.$$

It is straightforward there are no eigenvalues, when $|\lambda+4|\geq 2$. Take that $|\lambda+4|<2$ and so

$$\lambda+4=-2\cos\theta.$$

Then

$$m_{1,2}=e^{\pm i\theta},$$

thus,

$$u(n)=c_1\cos\theta n+c_2\sin\theta n.$$

Due to the boundary conditions,

$$\begin{aligned} u(0) &= c_1=0, \\ u(5) &= c_2\sin 5\theta=0. \end{aligned}$$

Assuming

$$\theta=\frac{k\pi}{5},\ (k=1,2,...,4);$$

then

$$\lambda_k=-4-2\cos\frac{k\pi}{5},\ (k=1,2,...,4),$$

so

$$u_k(n)=\sin\frac{k\pi n}{5},\ (k=1,2,...,4),\ (n=0,1,...,5),$$

are eigenpairs for this problem.

Application 2. Obtain eigenpairs for Sturm-Liouville problem

$$\Delta^2u(n-1)+(\lambda+10)u(n)=0, \tag{31}$$

$$x(0)=0,\ x(7)=0.$$

The characteristic equation for equation (31) is

$$m^2+(\lambda+8)m+1=0,$$

hence

$$m_{1,2}=\frac{(-8-\lambda)\pm\sqrt{(\lambda+8)^2-4}}{2}.$$

Suppose that $|\lambda+8|<2$ and so

$$\lambda+8=-2\cos\theta.$$

Then

$$m_{1,2}=e^{\pm i\theta},$$

thus,

$$u(n)=c_1\cos\theta n+c_2\sin\theta n.$$

Due to the boundary conditions

$$\begin{aligned} u(0) &= c_1 = 0, \\ u(7) &= c_2 \sin 7\theta = 0. \end{aligned}$$

Assuming

$$\theta = \frac{k\pi}{7},\ (k = 1, 2, ..., 6);$$

then

$$\lambda_k = -8 - 2\cos\frac{k\pi}{7},\ (k = 1, 2, ..., 7),$$

so

$$u_k(n) = \sin\frac{k\pi n}{7},\ (k = 1, 2, ..., 6),\ (n = 0, 1, ..., 7),$$

are eigenpairs for this problem.

6. Conclusion

The article aims to contribute to the spectral theory of Sturm-Liouville problem via Coulomb type in difference equations by obtaining the representation of solution and asymptotic formulas.

References

[1] A. Jirari, Second Order Sturm-Liouville Difference Equations.and Orthogonal Polynomials, Memoirs of the American Mathematical Society, Providence Rhode Island, 1995.
[2] B. M. Levitan, I. S. Sargsjan, Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators, American Mathematical Society, Providence Rhode Island, p.5–8, 1975.
[3] C. M. Bender, S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory, Springer-Verlag, Newyork, 1999.
[4] C. Ahlbrandt, J. Hooker, Disconjugacy criteria for second order linear difference equations, Qualitative Properties of Differential Equations, Proceedings of the 1984 Edmonton Conference, University of Alberta, Edmonton, (1987) 15–26.
[5] D. B. Hinton, R. T. Lewis, Spectral Theory of Second-Order Difference Equations, Journal of Mathematical Analysis and Applications 63 (1978) 421–438.
[6] D. I. Blohincev, Foundations of Quantum Mechanics. GITTL, Moscow, 1949.
[7] E. Bas, F. Metin, Fractional singular Sturm-Liouville operator for Coulomb potential, Advances in Difference Equations 300 (2013).
[8] E. Bas, R. Ozarslan, New Estimations for Sturm-Liouville Problems in Difference Equations, arXiv:1505.02181v1 [math.CA].
[9] E. S. Panakhov, M. Sat, Reconstruction of potential function for Sturm-Liouville operator with Coulomb potential, Boundary Value Problems 49 (2013).
[10] F. V. Atkinson, Discrete and Continuous Boundary Value Problems, Academic Press, Newyork, 1964.
[11] G. Shi, H. Wu, Spectral Theory of Sturm-Liouville Difference Operators, Linear Algebra and Its Applications 430 (2009) 830–846.
[12] H. Bereketoğlu, V., Kutay, Fark Denklemleri, Gazi Kitabevi, Ankara, 2012.
[13] H. Sun, Y. Shi, Eigenvalues of second-order difference equations with coupled boundary conditions, Linear Algebra and its Applications 414 (2006) 361–372.
[14] J. Ji, B. Yang, Eigenvalue comparisons for second order difference equations with Neumann boundary conditions,Linear Algebra and its Applications 425 (2007) 171–183.
[15] P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, Complete monocity, Trans. Amer. Math. Soc. 246 (1978) 1–30.
[16] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, Newyork, 2000.
[17] R. S. Hilscher, Spectral and oscillation theory for general second order Sturm-Liouville difference equations, Advances in Difference Equations 82 (2012).
[18] R. Mickens, Difference Equations, Van Nostrand, Newyork, 1990.
[19] S. Goldberg, Introduction to Difference equations with Illustrative examples from Economics, Psychology and Sociology, Dover, Newyork, 1986.
[20] S. Elaydi, An introduction to difference equations, Springer Science+Business Media, Newyork, 2005.

[21] S. L. Clark, A Spectral Analysis for Self-Adjoint Operators Generated by a Class of Second Order Difference Equations, Journal of Mathematical Analysis and Applications 197 (1978) 267–285.
[22] J. C. Butcher, Theory of Difference Equations: Numerical Methods and Applications, V. Lakshmikantham and D. Trigiante 31 (2012) 689–690.
[23] W. G. Kelley and A. C. Peterson, Difference Equations: An Introduction with Applications, Academic Press, San Diego, 2001.
[24] Y. Shi and S. Chen, Spectral Theory of Second-Order Vector Difference Equations, Journal of Mathematical Analysis and Applications 239 (1999) 195–212.
[25] Y. Shi, H. Sun, Self-adjoint extensions for second-order symmetric linear difference equations, Linear Algebra and its Applications 434 (2011) 903–930.
[26] Y. Wang, Y. Shi, Eigenvalues of second-order difference equations with periodic and antiperiodic boundary conditions, Journal of Mathematical Analysis and Applications 309 (2005) 56–69.