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A New Expression for the Transform with Respect to the Gaussian Process

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Abstract. In this paper we first introduce the concept of the $*_w$-product. We then proceed to show that the transform with respect to the Gaussian process for the functional F can be expressed in terms of the $*_w$-product without the concept of the transform. We then proceed to obtain several relationships involving the $*_w$-product.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m_w denote Wiener measure. $(C_0[0,T], \mathcal{M}, m_w)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm_w(x).$$

A subset B of $C_0[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{M}$-measurable for all $\rho > 0$, and a scale-invariant measurable set $\mathcal{N}$ is said to be a scale-invariant null set provided $m_w(\rho\mathcal{N}) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.) [12].

In this paper, we first introduce the concept of the $*_w$-product. We then proceed to show that the transform with respect to the Gaussian process for the functional F can be expressed in terms of the $*_w$-product without using the concept of the transform. That is to say, the $*_w$-product is a very useful tool to obtain the transform with respect to the Gaussian process without using the concept of the transform. Finally we obtain several relationships involving the $*_w$-product.

2. Definitions and Preliminaries

In this section we first state several definitions and then we introduce various notations which are used throughout this paper[7, 8, 10].

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For $h \in L^2[0,T]$, we define the Gaussian process Z_h by

$$Z_h(x,t) = \int_0^t h(s)\tilde{d}x(s) \tag{1}$$

where $\int_0^t h(s)\tilde{d}x(s)$ denotes the Paley-Wiener-Zygmund stochastic integral. For each $v \in L^2[0,T]$, let $\langle v,x\rangle = \int_0^T v(t)\tilde{d}x(t)$. From [6], we note that $\langle v, Z_h(x,\cdot)\rangle = \langle vh, x\rangle$ for $h \in L^\infty[0,T]$ and s-a.e. $x \in C_0[0,T]$. Thus, throughout this paper, we require h to be in $L^\infty[0,T]$ rather than simply in $L^2[0,T]$.

Let $K_0[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t = 0$ and whose real and imaginary parts are elements of $C_0[0,T]$.

Now, we state the definitions of the transform with respect to the Gaussian process and the first variation, [8, 10, 11].

Definition 2.1. *Let F and G be functionals on $K_0[0,T]$ and let γ and β be non-zero complex numbers. Then the transform with respect to the Gaussian process and the first variation are defined by the formulas*

$$T^{h_1,h_2}_{\gamma,\beta}(F)(y) = \int_{C_0[0,T]} F(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot))dm_w(x) \tag{2}$$

and

$$\delta F(Z_h(x,\cdot)|Z_s(z,\cdot)) = \frac{\partial}{\partial k}F(Z_h(x,\cdot) + kZ_s(z,\cdot))\Big|_{k=0} \tag{3}$$

if they exist.

Remark 2.2. *When $h_1(t) = h_2(t) = 1$ on $[0,T]$, $\gamma = 1$ and $\beta = i$, $T^{1,1}_{1,i}(F)$ is the Fourier-Wiener transform introduced by Cameron in [1] and used by Cameron and Martin in [2]. When $h_1(t) = h_2(t) = 1$ on $[0,T]$, $\gamma = \sqrt{2}$ and $\beta = i$, $T^{1,1}_{\sqrt{2},i}(F)$ is the modified Fourier-Wiener transform used by Cameron and Martin in [3]. When $h_1(t) = h_2(t) = 1$, $T^{1,1}_{\gamma,\beta}(F)$ is the the integral transform [4, 5, 9, 13].*

Let $Z_h(x,\cdot)$ be given by the equation (1). Then choose $\{\alpha_1,\cdots,\alpha_n\}$ from $L^2[0,T]$ such that

$$\begin{aligned}&(a)\ \{\alpha_1,\cdots,\alpha_n\} \quad \text{are orthogonal on } [0,T]\\ &(b)\ \{\alpha_1 h,\cdots,\alpha_n h\} \quad \text{are orthonormal on } [0,T]\,.\end{aligned}$$

One way to do this would be to choose $0 = t_0 < t_1 < \cdots < t_n = T$ with

$$\int_{t_{j-1}}^{t_j} h(s)ds > 0$$

for $j = 1,\cdots,n$ and then letting

$$\alpha_j(s) = \Big(\int_{t_{j-1}}^{t_j} h^2(s)ds\Big)^{-1/2}\chi_{[t_{j-1},t_j)}(s).$$

Then the α_j's satisfy (a) and (b) above. Let $f:\mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function and let $F: C_0[0,T] \to \mathbb{C}$ be given by equation

$$F(x) = f(\langle\alpha_1,x\rangle,\cdots\langle\alpha_n,x\rangle).$$

Then we have the following formula

$$\int_{C_0[0,T]} F(Z_h(x,\cdot))dm_w(x) = \int_{C_0[0,T]} f(\langle\alpha_1 h, x\rangle, \cdots, \langle\alpha_n h, x\rangle)dm_w(x)$$
$$= \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, \cdots, u_n) \exp\Big\{-\frac{|\vec{u}|^2}{2}\Big\} d\vec{u}$$

where $|\vec{u}| = \sqrt{u_1^2 + \cdots + u_n^2}$.

We finish this section by describing the class of functionals that we work with in this paper. Let $\mathbb{S}$ be the space of all functionals $F : K_0[0,T] \to \mathbb{C}$ of the form

$$F(x) = f(\langle\alpha_1, x\rangle, \cdots, \langle\alpha_n, x\rangle) \tag{4}$$

for some positive integer n, where $f(\lambda_1, \cdots, \lambda_n)$ is an entire function of n complex variables $\lambda_1, \cdots, \lambda_n$; that is to say,

$$|f(\lambda_1, \cdots, \lambda_n)| \le L \exp\Big\{-M \sum_{j=1}^{n} |\lambda_j|^2\Big\}$$

for some positive constants L and M. To simplify the expressions, we use the following notations

$$f(\langle\vec{\alpha}, x\rangle) \equiv f(\langle\alpha_1, x\rangle, \cdots, \langle\alpha_n, x\rangle)$$

and

$$F^j(x) = f_j(\langle\vec{\alpha}, x\rangle)$$

where $f_j(\vec{\lambda}) = \frac{\partial}{\partial\lambda_j} f(\lambda_1, \cdots, \lambda_n)$ for $j = 1, \cdots, n$.

Remark 2.3. *For any F and G in $\mathbb{S}$, we can always express F by* (4) *and G by*

$$G(x) = g(\langle\alpha_1, x\rangle, \cdots, \langle\alpha_n, x\rangle) \equiv g(\langle\vec{\alpha}, x\rangle) \tag{5}$$

using the same positive integer n, where g is an entire function of exponential type.

The following integration formula is used several times in this paper.

$$\int_{\mathbb{R}} \exp\{-a\eta^2 + b\eta\} d\eta = \Big(\frac{\pi}{a}\Big)^{\frac{1}{2}} \exp\Big\{\frac{b^2}{4a}\Big\} \tag{6}$$

for all complex numbers a and b with $\mathrm{Re}(a) > 0$.

3. Transform with Respect to the Gaussian Process

If $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is the function on $\mathbb{R}^n$ defined by

$$[f]^{\frown}(\vec{\xi}) = \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{i\vec{u}\cdot\vec{\xi}\} d\vec{u}, \quad \vec{u}, \vec{\xi} \in \mathbb{R}^n, \tag{7}$$

where $\vec{u}\cdot\vec{\xi} = u_1\xi_1 + \cdots + u_n\xi_n$. Many mathematicians and physicists study the Fourier transform since the Fourier transform is very useful in physics and various other fields including mathematics. The convolution product with respect to the Fourier transform $\hat{f}$ of a function f is defined by

$$(f * g)(\vec{u}) = \int_{\mathbb{R}^n} f(\vec{v}) g(\vec{u} - \vec{v}) d\vec{v}, \quad \text{for} \quad \vec{u}, \vec{v} \in \mathbb{R}^n. \tag{8}$$

In various fields of mathematics, in particular, functional analysis, convolution is a powerful tool that is used to operate on two functions, f and g, to produce a third function. Mathematically, convolution is described using an integral that expresses the amount of overlap of one function, g, with respect to a second function, f. In view of this description, this classical concept is extremely useful in a variety of research applications. In this section, we introduce the concept of the $*_w$-product. The $*_w$-product preserves useful properties of convolution for the Fourier transform. We then show that the transform with respect to the Gaussian process of the functional F given by equation (4) can be calculated from the $*_w$-product without using the concept of the transform.

We now introduce the concept of the $*_w$-product.

Definition 3.1. *Let F and G in $\mathbb{S}$ be given by equations* (4) *and* (5). *We define their $*_w$-product by*

$$(F *_w G)(x) = (f * g)(\langle \vec{\alpha}, x\rangle)$$

and

$$[F]^\frown(x) = [f]^\frown(\langle \vec{\alpha}, x\rangle).$$

Remark 3.2. *If $F \in \mathbb{S}$, then $[F]^\frown$ exists. This follows from the fact that*

$$\begin{aligned}
|[f]^\frown(\vec{\xi})| &= \Big|\Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{i\vec{u}\cdot\vec{\xi}\} d\vec{u}\Big| \\
&\leq \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} |f(\vec{u})| d\vec{u} \\
&\leq \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} L \exp\Big\{-M\sum_{j=1}^{n} u_j^2\Big\} d\vec{u} = \Big(\frac{L^2}{2M}\Big)^{\frac{n}{2}} < \infty.
\end{aligned}$$

In a similar way it is easy to show that $F *_w G$ exists. Hence, for all $F, G \in \mathbb{S}$, $[F]^\frown$ and $F *_w G$ exist.

Example 3.3. *For $n = 1$, let $f(u) = \exp\{-u^2\}$ and $g(u) = u\exp\{-u^2\}$. Then using equations* (8) *and* (5), *we can easily show that*

$$[f]^\frown(\xi) = \frac{1}{\sqrt{2}} \exp\Big\{-\frac{\xi^2}{4}\Big\}$$

and

$$(f * g)(u) = \frac{\sqrt{\pi}}{2\sqrt{2}} u \exp\Big\{-\frac{u^2}{2}\Big\}.$$

Hence

$$[F]^\frown(x) = [f]^\frown(\langle \alpha_1, x\rangle) = \frac{1}{\sqrt{2}} \exp\Big\{-\frac{\langle \alpha_1, x\rangle^2}{4}\Big\}$$

and

$$(F *_w G)(x) = (f * g)(\langle \alpha_1, x\rangle) = \frac{\sqrt{\pi}}{2\sqrt{2}} \langle \alpha_1, x\rangle \exp\Big\{-\frac{\langle \alpha_1, x\rangle^2}{2}\Big\}.$$

Remark 3.4. *Let F and G be as in Definition 3.1. Let $H \in \mathbb{S}$ be given by the formula*

$$H(x) = h(\langle \vec{\alpha}, x\rangle).$$

Then the $*_w$-product preserves the following useful properties of convolution for the Fourier transform:

(1) The $*_w$-product is commutative; that is to say

$$(F *_w G) = (G *_w F).$$

(2) The $*_w$-product is associative; that is to say

$$F *_w (G *_w H) = (F *_w G) *_w H.$$

(3) The $*_w$-product is distributive; that is to say

$$F *_w (G + H) = (F *_w G) + (F *_w H).$$

(4) The Fourier transform of a $*_w$-product is the product of their Fourier transform; that is to say

$$[(F *_w G)]^{\frown} = (2\pi)^{\frac{n}{2}} [F]^{\frown} [G]^{\frown}.$$

(5) The Fourier transform of a product is the $*_w$-product of the Fourier transform; that is to say

$$[FG]^{\frown} = (2\pi)^{-\frac{n}{2}} [F]^{\frown} *_w [G]^{\frown}.$$

The following lemma was established in [10].

Lemma 3.5. *Let $F \in \mathbb{S}$ be given by equation* (4)*. Then for all non-zero complex numbers γ and β with $Re(\frac{1}{\gamma^2}) > 0$,*

$$T^{h_1,h_2}_{\gamma,\beta}(F)(y) = W(\beta y)$$

for all $y \in K_0[0, T]$, where $W(y) = w(\gamma; \langle \vec{\alpha} h_2, y\rangle)$ and

$$w(\gamma; \vec{v}) = \Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{ -\frac{1}{2\gamma^2}|\vec{u} - \vec{v}|^2\Big\} d\vec{u}. \tag{9}$$

For convenience, throughout the rest of this paper for all non-zero complex numbers a and b, let

$$\Phi_{a,b}(y) = \phi_{a,b}(\langle \vec{\alpha}, y\rangle), \quad \text{for all} \quad y \in K_0[0, T] \tag{10}$$

where

$$\phi_{a,b}(\vec{u}) = \Big(\frac{1}{a^2}\Big)^{\frac{n}{2}} \exp\Big\{ -\frac{1}{2b^2}|\vec{u}|^2\Big\}.$$

Using Lemma 3.5, the transform with respect to the Gaussian process of the functional $F \in \mathbb{S}$ can be expressed in terms of the $*_w$-product.

Theorem 3.6. *Let $F \in \mathbb{S}$ be given by equation* (4)*. Then for all non-zero complex numbers γ and β with $Re(\frac{1}{\gamma^2}) > 0$,*

$$T^{h_1,h_2}_{\gamma,\beta}(F)(y) = \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} [\Phi_{\gamma,\gamma} *_w F](\beta Z_{h_2}(y, \cdot)) \tag{11}$$

for all $y \in K_0[0, T]$, where $\Phi_{\gamma,\gamma}(y) = \phi_{\gamma,\gamma}(\langle \vec{\alpha}, y\rangle)$ is given by equation (10)*.*

Proof. Since

$$w(\gamma; \vec{v}) = \Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{ -\frac{|\vec{u} - \vec{v}|^2}{2\gamma^2}\Big\} d\vec{u},$$

we have

$$\begin{aligned}[w]^{\frown}(\gamma;\vec{\xi}) &= \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n} w(\gamma;\vec{v})\exp\{i\vec{v}\cdot\vec{\xi}\}d\vec{v}\\
&= \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n}\Big[\Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n} f(\vec{u})\exp\Big\{-\frac{|\vec{u}-\vec{v}|^2}{2\gamma^2}\Big\}d\vec{u}\Big]\exp\{i\vec{v}\cdot\vec{\xi}\}d\vec{v}\\
&= \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}\Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\vec{u})\exp\Big\{-\frac{|\vec{u}-\vec{v}|^2}{2\gamma^2}+i\vec{v}\cdot\vec{\xi}\Big\}d\vec{u}d\vec{v}.\end{aligned}$$

Next, carrying out the integration with respect to $v_1,\cdots,v_n$ in the above expression we obtain that

$$\begin{aligned}[w]^{\frown}(\gamma;\vec{\xi}) &= \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n} f(\vec{u})\exp\Big\{-\frac{\gamma^2}{2}|\vec{\xi}|^2+i\vec{u}\cdot\vec{\xi}\Big\}d\vec{u}\\
&= \exp\Big\{-\frac{\gamma^2}{2}|\vec{\xi}|^2\Big\}\Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n} f(\vec{u})\{i\vec{u}\cdot\vec{\xi}\}d\vec{u}\\
&= \exp\Big\{-\frac{\gamma^2}{2}|\vec{\xi}|^2\Big\}[f]^{\frown}(\vec{\xi}).\end{aligned}$$

Since

$$[\phi_{\gamma,\gamma}]^{\frown}(\vec{\xi}) = \exp\Big\{-\frac{\gamma^2}{2}|\vec{\xi}|^2\Big\},$$

the last expression in the above equation is equal to

$$[w]^{\frown}(\gamma;\vec{\xi}) = [\phi_{\gamma,\gamma}]^{\frown}(\vec{\xi})[f]^{\frown}(\vec{\xi}) = \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}[\phi_{\gamma,\gamma}*f]^{\frown}(\vec{\xi})$$

and so $[w](\gamma;\vec{\xi}) = (2\pi)^{-\frac{n}{2}}[\phi_{\gamma,\gamma}*f](\vec{\xi})$. Hence, by using Definition 3.1, we have

$$T_{\gamma,\beta}^{h_1,h_2}(F)(y) = \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}[\Phi_{\gamma,\gamma}*_w F](\beta Z_{h_2}(y,\cdot))$$

which establishes equation (11). □

Our next theorem tells us that the transform with respect to the Gaussian process of $[F]^{\frown}$ is expressed in terms of the $*_w$-product, without using the concepts of the transform.

Theorem 3.7. *Let $F\in\mathbb{S}$ be given by equation* (4). *Then for all non-zero complex numbers γ and β*

$$T_{\gamma,\beta}^{h_1,h_2}([F]^{\frown})(y) = \Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}}\exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2,y\rangle|^2\Big\}[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}}*_w F]\Big(\frac{i\beta}{\gamma^2}Z_{h_2}(y,\cdot)\Big) \tag{12}$$

for all $y\in K_0[0,T]$, where $\Phi_{\gamma,\gamma}$ is given by equation (10).

Proof. First using equations (2), (6) and (7), we obtain that

$$\begin{aligned}T_{\gamma,\beta}^{h_1,h_2}([F]^{\frown})(y) &= \int_{C_0[0,T]}[f]^{\frown}(\gamma\langle\vec{\alpha}h_1,x\rangle+\beta\langle\vec{\alpha}h_2,y\rangle)dm_w(x)\\
&= \Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n}[f]^{\frown}(\vec{u}+\beta\langle\vec{\alpha}h_2,y\rangle)\exp\Big\{-\frac{|\vec{u}|^2}{2\gamma^2}\Big\}d\vec{u}\\
&= \Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n}[f]^{\frown}(\vec{u})\exp\Big\{-\frac{1}{2\gamma^2}|\vec{u}-\beta\langle\vec{\alpha}h_2,y\rangle|^2\Big\}d\vec{u}\\
&= \Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}}\Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\vec{v})\exp\Big\{-\frac{1}{2\gamma^2}|\vec{u}-\beta\langle\vec{\alpha}h_2,y\rangle|^2+i\vec{u}\cdot\vec{v}\Big\}d\vec{u}d\vec{v}.\end{aligned}$$

Next, we carry out the integrations with respect to $u_1, \cdots, u_n$ using (6) and obtain

$$\begin{aligned}
T_{\gamma,\beta}^{h_1,h_2}([F]\widehat{\ })(y) &= \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{v}) \exp\Big\{-\frac{\gamma^2}{2}|\vec{v}|^2 + i\beta\langle\vec{\alpha}h_2, y\rangle\cdot\vec{v}\Big\} d\vec{v} \\
&= \gamma^{-n} \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2, y\rangle|^2\Big\}\Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f\Big(\frac{1}{\gamma}\vec{v} + \frac{i\beta}{\gamma^2}\langle\vec{\alpha}h_2, y\rangle\Big) \exp\Big\{-\frac{|\vec{v}|^2}{2}\Big\} d\vec{v} \\
&= \gamma^{-n} \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2, y\rangle|^2\Big\} \int_{C_0[0,T]} f\Big(\frac{1}{\gamma}\langle\vec{\alpha}h_1, x\rangle + \frac{i\beta}{\gamma^2}\langle\vec{\alpha}h_2, y\rangle\Big) dm_w(x) \\
&= \gamma^{-n} \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2, y\rangle|^2\Big\} \int_{C_0[0,T]} F\Big(\frac{1}{\gamma}Z_{h_1}(x,\cdot) + \frac{i\beta}{\gamma^2}Z_{h_2}(y,\cdot)\Big) dm_w(x) \\
&= \gamma^{-n} \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2, y\rangle|^2\Big\} T_{\frac{1}{\gamma},\frac{i\beta}{\gamma^2}}^{h_1,h_2}(F)(y).
\end{aligned}$$

Hence, applying Theorem 3.6 to the last expression in the above equation, we obtain

$$T_{\gamma,\beta}^{h_1,h_2}([F]\widehat{\ })(y) = \Big(\frac{1}{2\pi\gamma^2}\Big)^{\frac{n}{2}} \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2, y\rangle|^2\Big\}[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w F]\Big(\frac{i\beta}{\gamma^2}Z_{h_2}(y,\cdot)\Big),$$

which completes the proof of Theorem 3.7. □

In the table below we express transforms with respect to the Gaussian process in terms of $*_w$-product only.

	Expressions
$T_{\gamma,\beta}^{h_1,h_2}(F *_w G)(y)$	$\big(\frac{1}{2\pi}\big)^{\frac{n}{2}}[\Phi_{\gamma,\gamma} *_w (F *_w G)](\beta Z_{h_2}(y,\cdot))$
$T_{\gamma,\beta}^{h_1,h_2}([F *_w G]\widehat{\ })(y)$	$\big(\frac{1}{2\pi\gamma^2}\big)^{\frac{n}{2}} \exp\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2, y\rangle|^2\}$ $[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w (F *_w G)](\frac{i\beta}{\gamma^2}Z_{h_2}(y,\cdot))$
$[T_{\gamma,\beta}^{h_1,h_2}(F) *_w T_{\gamma,\beta}^{h_1,h_2}(G)](y)$	$\big(\frac{1}{2\pi}\big)^{n}[\Phi_{\frac{\gamma}{\sqrt{\pi}},\sqrt{2}\gamma} *_w (F *_w G)](\beta Z_{h_2}(y,\cdot))$
$[T_{\gamma,\beta}^{h_1,h_2}(F) *_w (T_{\gamma,\beta}^{h_1,h_2}(G) + T_{\gamma,\beta}^{h_1,h_2}(H))](y)$	$\big(\frac{1}{2\pi}\big)^{n}[\Phi_{\frac{\gamma}{\sqrt{\pi}},\sqrt{2}\gamma} *_w (F *_w G)](\beta Z_{h_2}(y,\cdot))$ $+\big(\frac{1}{2\pi}\big)^{n}[\Phi_{\frac{\gamma}{\sqrt{\pi}},\sqrt{2}\gamma} *_w (F *_w H)](\beta Z_{h_2}(y,\cdot))$

Table 1: Transform involving the $*_w$-product

In Table 1 above, $\Phi_{\gamma,\gamma}$ and $\Phi_{\frac{\gamma}{\sqrt{\pi}},\sqrt{2}\gamma}$ are given by equation (10).

In Theorem 3.8, we establish that the transform of the first variation equals the first variation of the transform with respect to the Gaussian process.

Theorem 3.8. *Let $F \in \mathbb{S}$ be given by equation* (4). *Let $\frac{1}{l} \in L^\infty[0,T]$. Assume that $l(t)s(t) = m(t)h(t)h_2(t)$ on $[0,T]$. Then for all non-zero complex numbers γ and β*

$$\begin{aligned}
T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_z(s,\cdot)))(y) &= \delta T_{\gamma,\beta}^{hh_1,\frac{hh_2}{l}}(F)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)) \\
&= \sum_{j=1}^{n}\langle\alpha_j s, z\rangle\Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}[\Phi_{\gamma,\gamma} *_w F^j](\beta Z_{hh_2}(y,\cdot))
\end{aligned} \tag{13}$$

for all $y \in K_0[0,T]$, *where* $\Phi_{\gamma,\gamma}$ *is given by equation* (10).

Proof. By using equations (3) and (4), we have

$$\begin{aligned} &T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_z(s,\cdot)))(y) \\ &= \int_{C_0[0,T]} \frac{\partial}{\partial k} f(\gamma\langle\vec{\alpha}hh_1,x\rangle + \beta\langle\vec{\alpha}hh_2,y\rangle + k\langle\vec{\alpha}s,z\rangle)\Big|_{k=0} dm_w(x) \\ &= \sum_{j=1}^{n}\langle\alpha_j s,z\rangle \int_{C_0[0,T]} f_j(\gamma\langle\vec{\alpha}hh_1,x\rangle + \beta\langle\vec{\alpha}hh_2,y\rangle)dm_w(x) \\ &= \sum_{j=1}^{n}\langle\alpha_j s,z\rangle T_{\gamma,\beta}^{hh_1,hh_2}(F^j)(y). \end{aligned} \tag{14}$$

On the other hand, by using the same method as used to establish (14) above we obtain that

$$\begin{aligned} \delta T_{\gamma,\beta}^{hh_1,\frac{hh_2}{l}}(F)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)) &= \sum_{j=1}^{n}\Big\langle\alpha_j\frac{mhh_2}{l},z\Big\rangle T_{\gamma,\beta}^{hh_1,hh_2}(F^j)(y) \\ &= \sum_{j=1}^{n}\langle\alpha_j s,z\rangle T_{\gamma,\beta}^{hh_1,hh_2}(F^j)(y). \end{aligned} \tag{15}$$

From equations (14) and (15), we have

$$T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_z(s,\cdot)))(y) = \delta T_{\gamma,\beta}^{hh_1,\frac{hh_2}{l}}(F)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)). \tag{16}$$

Hence, applying Theorem 3.6 to the last expression in the equations (14) and (15) above, we establish equation (13). □

4. Relationships Involving Two Concepts

In Section 3, we introduced the concept of the $*_w$-product. We showed that the transform with respect to the Gaussian process of F can be expressed in terms of the $*_w$-product. In this section, we establish the various relationships involving the two concepts.

Formula 1. Let $F \in \mathbb{S}$ be given by equation (4). Let $\frac{1}{l} \in L^\infty[0,T]$. Assume that $l(t)s(t) = m(t)h(t)h_2(t)$ on $[0,T]$. Then for all non-zero complex numbers γ and β

$$[\Phi_{\gamma,\gamma} *_w \delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))](\beta Z_{h_2}(y,\cdot)) = \delta[\Phi_{\gamma,\gamma} *_w F](\beta Z_{hh_2}(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)) \tag{17}$$

for all $y \in K_0[0,T]$, where $\Phi_{\gamma,\gamma}$ is given by equation (10).

Proof. From Theorem 3.8, we have the following formula

$$T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y) = \delta T_{\gamma,\beta}^{hh_1,\frac{hh_2}{l}}(F)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)). \tag{18}$$

Hence, applying Theorem 3.6 to both sides of the expression in equation (18) above, we obtain

$$T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y) = \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}[\Phi_{\gamma,\gamma} *_w \delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))](\beta Z_{h_2}(y,\cdot))$$

and

$$\delta T_{\gamma,\beta}^{hh_1,\frac{hh_2}{l}}(F)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)) = \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}\delta[\Phi_{\gamma,\gamma} *_w F](\beta Z_{hh_2}(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)).$$

Thus we have the desired result. □

Formula 2. Let $F \in \mathcal{S}$ be given by equation (4). Let $\frac{1}{l} \in L^\infty[0,T]$. Assume that $l(t)s(t) = m(t)h(t)h_2(t)$ on $[0,T]$. Then for all non-zero complex numbers γ and β

$$\begin{aligned}&[\Phi_{\gamma,\gamma} *_w \delta[F]^{\frown}(Z_h(\cdot,\cdot)|Z_s(z,\cdot))](\beta Z_{h_2}(y,\cdot))\\&= \Big(\frac{1}{\gamma}\Big)^n \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}\frac{hh_2}{l},y\rangle|^2\Big\}\delta[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w F]\Big(\frac{i\beta}{\gamma^2}Z_{hh_2}(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)\Big)\end{aligned} \tag{19}$$

where $\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}}$ is given by equation (10).

Proof. First, by using equation (11), the left hand side of equation (12) is equal to

$$T_{\gamma,\beta}^{h_1,h_2}([F]^{\frown})(y) = \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}[\Phi_{\gamma,\gamma} *_w [F]^{\frown}](\beta Z_{h_2}(y,\cdot))$$

and so

$$[\Phi_{\gamma,\gamma} *_w [F]^{\frown}](\beta Z_{h_2}(y,\cdot)) = \Big(\frac{1}{\gamma}\Big)^n \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2,y\rangle|^2\Big\}[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w F]\Big(\frac{i\beta}{\gamma^2}Z_{h_2}(y,\cdot)\Big) \tag{20}$$

for all $y \in K_0[0,T]$, where $\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}}$ is given by equation (10). From equations (17) and (20), we have

$$\begin{aligned}&[\Phi_{\gamma,\gamma} *_w \delta[F]^{\frown}(Z_h(\cdot,\cdot)|Z_s(z,\cdot))](\beta Z_{h_2}(y,\cdot))\\&= \delta[\Phi_{\gamma,\gamma} *_w [F]^{\frown}](\beta Z_{hh_2}(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))\\&= \Big(\frac{1}{\gamma}\Big)^n \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}\frac{hh_2}{l},y\rangle|^2\Big\}\delta[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w F]\Big(\frac{i\beta}{\gamma^2}Z_{hh_2}(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)\Big),\end{aligned}$$

which completes the proof of Formula 2. □

Formula 3. Let $F \in \mathcal{S}$ be given by equation (4). Let $\frac{1}{l} \in L^\infty[0,T]$. Assume that $l(t)s(t) = m(t)h(t)h_2(t)$ on $[0,T]$. Then for all non-zero complex numbers γ and β

$$\begin{aligned}&[\Phi_{\gamma,\gamma} *_w [\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))]^{\frown}](\beta Z_{h_2}(y,\cdot))\\&= \Big(\frac{1}{\gamma}\Big)^n \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2,y\rangle|^2\Big\}\delta[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w F]\Big(\frac{i\beta}{\gamma^2}Z_{hh_2}(y,\cdot)\Big| -\frac{i\gamma}{\beta}Z_m(z,\cdot)\Big)\end{aligned} \tag{21}$$

where $\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}}$ is given by equation (10).

Proof. By using equations (12) and (17), we have

$$\begin{aligned}&[\Phi_{\gamma,\gamma} *_w [\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))]^{\frown}](\beta Z_{h_2}(y,\cdot))\\&= \Big(\frac{1}{\gamma}\Big)^n \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2,y\rangle|^2\Big\}[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w \delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))]\Big(\frac{i\beta}{\gamma^2}Z_{h_2}(y,\cdot)\Big)\\&= \Big(\frac{1}{\gamma}\Big)^n \exp\Big\{-\frac{\beta^2}{2\gamma^2}|\langle\vec{\alpha}h_2,y\rangle|^2\Big\}\delta[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w F]\Big(\frac{i\beta}{\gamma^2}Z_{hh_2}(y,\cdot)\Big| -\frac{i\gamma}{\beta}Z_m(z,\cdot)\Big),\end{aligned}$$

which completes the proof of Formula 3. □

The following simple example illustrates the Formulas 1-3 in Section 4.

Let $f(u) = \exp\{-u^2\}$ and let

$$F(x) = f(\langle\alpha, x\rangle). \tag{22}$$

Then for all non-zero complex numbers γ, direct calculations show that

$$\begin{aligned}(\Phi_{\gamma,\gamma} *_w F)(x) &= (\phi_{\gamma,\gamma} * f)(\langle\alpha, x\rangle)\\ &= \Big(\frac{2\pi}{1+2\gamma^2}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{\langle\alpha, x\rangle^2}{1+2\gamma^2}\Big\}\end{aligned} \tag{23}$$

where $\Phi_{\gamma,\gamma}$ is given by equation (10). Equation (23) now follows from equation (8).

Example using the Formula 1. Let F be given by equation (22). Then by using equations (3) and (23)

$$\delta[\Phi_{\gamma,\gamma} *_w F](\beta Z_{hh_2}(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)) = \frac{-2\sqrt{2\pi}}{(1+2\gamma^2)^{3/2}}\langle\alpha hh_2, y\rangle\langle\alpha m, z\rangle \exp\Big\{-\frac{\beta^2}{1+2\gamma^2}\langle\alpha hh_2, y\rangle^2\Big\}$$

and hence from Formula 1,

$$[\Phi_{\gamma,\gamma} *_w \delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))](\beta Z_{h_2}(y,\cdot)) = \frac{-2\sqrt{2\pi}}{(1+2\gamma^2)^{3/2}}\langle\alpha hh_2, y\rangle\langle\alpha m, z\rangle \exp\Big\{-\frac{\beta^2}{1+2\gamma^2}\langle\alpha hh_2, y\rangle^2\Big\}.$$

Example using the Formula 2. Let F be given by equation (22). Then by using equations (3) and (23)

$$\begin{aligned}&\frac{1}{\gamma}\exp\Big\{-\frac{\beta^2}{2\gamma^2}\langle\alpha\frac{hh_2}{l}, y\rangle^2\Big\}\delta[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w F]\Big(\frac{i\beta}{\gamma^2}Z_{hh_2}(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)\Big)\\ &= \frac{-2i\sqrt{2\pi}}{(2+\gamma^2)^{3/2}}\langle\alpha hh_2, y\rangle\langle\alpha m, z\rangle \exp\Big\{\frac{\beta^2}{(2+\gamma^2)\gamma^2}\langle\alpha hh_2, y\rangle^2 - \frac{\beta^2}{2\gamma^2}\langle\alpha\frac{hh_2}{l}, y\rangle^2\Big\}\end{aligned}$$

and hence from Formula 2,

$$\begin{aligned}&[\Phi_{\gamma,\gamma} *_w \delta[F]^\frown(Z_h(\cdot,\cdot)|Z_s(z,\cdot))](\beta Z_{h_2}(y,\cdot))\\ &= \frac{-2i\sqrt{2\pi}}{(2+\gamma^2)^{3/2}}\langle\alpha hh_2, y\rangle\langle\alpha m, z\rangle \exp\Big\{\frac{\beta^2}{(2+\gamma^2)\gamma^2}\langle\alpha hh_2, y\rangle^2 - \frac{\beta^2}{2\gamma^2}\langle\alpha\frac{hh_2}{l}, y\rangle^2\Big\}.\end{aligned}$$

Example using the Formula 3. Let F be given by equation (22). Then by using equations (3) and (23)

$$\begin{aligned}&\frac{1}{\gamma}\exp\Big\{-\frac{\beta^2}{2\gamma^2}\langle\alpha h_2, y\rangle^2\Big\}\delta[\Phi_{\frac{1}{\gamma},\frac{1}{\gamma}} *_w F]\Big(\frac{i\beta}{\gamma^2}Z_{hh_2}(y,\cdot)\Big| -\frac{i\gamma}{\beta}Z_m(z,\cdot)\Big)\\ &= \frac{-\gamma\sqrt{2\pi}}{(2+\gamma^2)^{3/2}}\langle\alpha hh_2, y\rangle\langle\alpha m, z\rangle \exp\Big\{\frac{\beta^2}{(2+\gamma^2)\gamma^2}\langle\alpha hh_2, y\rangle^2 - \frac{\beta^2}{2\gamma^2}\langle\alpha h_2, y\rangle^2\Big\}\end{aligned}$$

and hence from Formula 2,

$$\begin{aligned}&[\Phi_{\gamma,\gamma} *_w [\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))]^\frown](\beta Z_{h_2}(y,\cdot))\\ &= \frac{-\gamma\sqrt{2\pi}}{(2+\gamma^2)^{3/2}}\langle\alpha hh_2, y\rangle\langle\alpha m, z\rangle \exp\Big\{\frac{\beta^2}{(2+\gamma^2)\gamma^2}\langle\alpha hh_2, y\rangle^2 - \frac{\beta^2}{2\gamma^2}\langle\alpha h_2, y\rangle^2\Big\}.\end{aligned}$$

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