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Geometry of Warped Product Pointwise Semi-Slant Submanifolds of Kaehler Manifolds

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Abstract. In this paper, we study warped product pointwise semi-slant submanifolds of a Kaehler manifold. First, we prove some characterizations results in terms of the tensor fields T and F and then, we obtain a geometric inequality for the second fundamental form in terms of intrinsic invariants. Furthermore, the equality case is also discussed. Moreover, we give some applications for Riemannian and compact Remannian submanifolds as well, i.e., we construct necessary and sufficient conditions for the non-existence of compact warped product pointwise semi-slant submanifold in complex space forms.

1. Introduction

It is well known that the geometry of warped product manifolds provide magnificent setting to supermodel space time near black holes and bodies with large gravitational fields. The idea of warped product manifolds was introduced by Bishop and O'Neill [6] to study manifolds of negative curvature. These manifolds are extension of Riemannian product manifolds with warping functions.

On the other hand, the theory of slant submanifold is still active field of research nowadays which was introduced by Chen in [7] of almost Hermitian manifolds. Among the class of slant manifolds we find that almost complex(holomorphic) and totally real submanifolds are special cases of these submanifolds.

The study of pointwise slant submanifolds of almost Hermitian manifolds got momentum after the work of F. Etayo in [15] which he call them the name of quasi-slant submanifolds. It was proved that the totally geodesic quasi-slant submanifold of Kaehler manifold is slant submanifold. The best exmaple of pointwise slant submanifolds is: every two dimensional submanifold in an almost Hermitian manifold is always a pointwise slant submanifold. Later, these submanifolds in details studied by Chen-Gray [12] in almost Hermitian manifolds. They obtained many interesting results geometric and topological obstructions of almost Hermitian manifolds. It was proved that in [12] a totally geodesic quasi-slant submanifold of a Kaehler manifold is slant submanifold.

Recently, Sahin [29] studied warped product pointwise semi-slant submanifold of Kaehlers. He proved that the there do not exist warped product pointwise semi-slant submanifolds of the form $M = M_\theta \times_f M_T$ of

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Kaehler manifold where M_θ is proper pointwise slant submanifold and M_T is a complex submanifold. Then he considered warped products of the form $M = M_T \times_f M_\theta$ and obtained many interesting results including characterization and inequality. He also provided some examples of pointwise semi-slant submanifolds and their warped products. For the survey of warped product submanifolds we refer to[13].

In the present paper, we extend this study to the warped product pointwise semi-slant submanifolds of Kaehler manifolds. The paper is organised as follows: Section 2, we recall some basic formulas and definitions. Section 3, we give a brief introduction of pointwise semi-slant submanifolds. Section 4, we study warped product pointwise semi-slant submanifolds and obtain some characterization results in terms of the tensor fields. In Section 5, we establish an inequality for the second fundamental form in terms of intrinsic invariants (Chen' Invariants). The equality case is also discussed. Section 6, we give some applications of such inequalities for Riemanian and compact Riemannian submanifolds in complex space forms.

2. Preliminaries

Let $(\widetilde{M}, J, g)$ be an almost Hermitian manifold with almost complex structure J and a Riemannian metric g such that

$$(a)\ J^2 = -I, \qquad (b)\ g(JU, JV) = g(U, V), \tag{2.1}$$

for all vector fields U, V on $\widetilde{M}$, where I is the identity map.

Let $\Gamma(T\widetilde{M})$ denote the set of all vector fields on $\widetilde{M}$ and $\widetilde{\nabla}$ denote the Levi-Civita connection on $\widetilde{M}$. If the almost complex structure J satisfies

$$(\widetilde{\nabla}_U J)V = 0, \tag{2.2}$$

for any $U, V \in \Gamma(T\widetilde{M})$, then $\widetilde{M}$ is called a *Kaehler manifold*.

Let M be a submanifold of an almost Hermitian manifold $\widetilde{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively, then Gauss and Weingarten formula are given by

$$(i)\ \widetilde{\nabla}_U V = \nabla_U V + h(U, V), \quad (ii)\ \widetilde{\nabla}_U N = -A_N U + \nabla^\perp_U N, \tag{2.3}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\widetilde{M}$. They are related as

$$g(h(U, V), N) = g(A_N U, V), \tag{2.4}$$

where g denote the Riemannian metric on $\widetilde{M}$ as well as the metric induced on M. Now for any $U \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, we have

$$(i)\ JU = TU + FU, \quad (ii)\ JN = tN + fN, \tag{2.5}$$

where $TU(tN)$ and $FU(fN)$ are tangential and normal components of $JU(JN)$, respectively. From (2.1) and (2.5)(i), it is easy to observe that for each $U, V \in \Gamma(TM)$, we have

$$(a)\ g(TU, V) = -g(U, TV)\ \text{and}\ (b)\ \|T\|^2 = \sum_{i,j=1}^{n} g^2(Te_i, e_j). \tag{2.6}$$

For a submanifold M of a Riemannian manifold $\widetilde{M}$, the equation of Gauss is given by

$$\widetilde{R}(U, V, Z, W) = R(U, V, Z, W) + g(h(U, Z), h(V, W)) - g(h(U, W), h(V, Z)), \tag{2.7}$$

for any $U, V, Z, W \in \Gamma(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}$ and M respectively. The mean curvature vector H for an orthonormal frame $\{e_1, e_2 \cdots e_n\}$ of tangent space TM on M is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i), \tag{2.8}$$

where $n = dimM$. Also we set

$$h_{ij}^r = g(h(e_i, e_j), e_r) \;\; and \;\; \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.9}$$

The scalar curvature ρ for a submanifold M of an almost complex manifolds $\widetilde{M}$ is given by

$$\rho(TM) = \sum_{1 \le i \ne j \le n} K(e_i \wedge e_j), \tag{2.10}$$

where $K(e_i \wedge e_j)$ is the sectional curvature of plane section spanned by e_i and e_j. Let G_r be a $r-$plane section on TM and $\{e_1, e_2 \cdots e_r\}$ any orthonormal basis of G_r. Then the scalar curvature $\rho(G_r)$ of G_r is given by

$$\rho(G_r) = \sum_{1 \le i \ne j \le r} K(e_i \wedge e_j). \tag{2.11}$$

A submanifold M of an almost Hermitian manifold $\widetilde{M}$ is said to be *totally umbilical* and *totally geodesic* if $h(U, V) = g(U, V)H$ and $h(U, V) = 0$, respectively, for all $U, V \in \Gamma(TM)$ where H is the mean curvature vector of M. Furthermore, if $H = 0$, them M is *minimal* in $\widetilde{M}$. The covariant derivatives of the endomorphism J, T and F are defined respectively as

$$(\widetilde{\nabla}_U J)V = \widetilde{\nabla}_U JV - J\widetilde{\nabla}_U V, \;\; \forall\, U, V \in \Gamma(T\widetilde{M}) \tag{2.12}$$

$$(\widetilde{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \;\; \forall\, U, V \in \Gamma(TM) \tag{2.13}$$

$$(\widetilde{\nabla}_U F)V = \nabla_U^{\perp} FV - F\nabla_U V \;\; \forall\, U, V \in \Gamma(TM). \tag{2.14}$$

On using (2.1), (2.2), (2.3), (2.5) and (2.12)-(2.14), we obtain

$$(a)\; (\widetilde{\nabla}_U T)V = A_{FV}U + th(U, V), \;\; (b)\; (\widetilde{\nabla}_U F)U = fh(U, V) - h(U, TV), \tag{2.15}$$

Assume that the set T^*M containing of all non-zero tangent vectors of submanifold M of an almost Hermitian manifold $\widetilde{M}$. Then for each non-zero vector $X \in \Gamma(T_xM)$ at point $x \in M$, the angle $\theta(X)$ between JX and tangent space T_xM is called the *Wirtinger angle* of X. Globally, the Wirtinger angle become a real-valued function which is defined on T^*M such that $\theta : T^*M \to \mathbb{R}$, is called the *Wirtinger function* . In this case, the submanifold M of almost Hermitian manifolds $\widetilde{M}$ is called pointwise slant submanifold.

A point x in a pointwise slant submanifold is called a totally real point if its slant function θ satisfies $cos\theta = 0$, at x. In the same way, a point x is called a complex point if its slant function satisfies $sin\theta = 0$ at x. A pointwise slant submanifold M in an almost Hermitian manifold $\widetilde{M}$ is called totally real if every point of M is a totally real point. A pointwise slant submanifold of an almost Hermitian manifold is called pointwise proper slant if it contains no totally real points. A pointwise slant submanifold M is called slant when its slant function θ is globally constant, i.e., θ is also independent of the choice of the point on M. It is clear that pointwise slant submanifolds include holomorphic, totally real and slant submanifolds. It clear that CR-submanifold and slant submanifold are particular case of semi-slant submanifolds with $\theta = \pi/2$ and $\mathcal{D} = 0$, respectively.

Recently, Chen and Garay in [12] proved the following theorem for pointwise slant submanifolds such as:

Theorem 2.1. *Let M be a submanifold of an almost Hermitian manifold $\widetilde{M}$. Then M is pointwise slant if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$T^2 = -\cos^2\theta I. \tag{2.16}$$

for some real-valued function θ defined on the tangent bundle TM of M (cf. [12]).

Hence, for a pointwise slant submanifold M of an almost Hermitian manifold $\widetilde{M}$, we have the following relations which are consequences of the Theorem 2.1,

$$g(TU, TV) = \cos^2\theta g(U, V), \tag{2.17}$$

$$g(FU, FV) = \sin^2\theta g(U, V), \tag{2.18}$$

for any $U, V \in \Gamma(TM)$. For differential function φ on M, the gradient $grad\varphi$ and Laplacian $\nabla\varphi$ of φ are defined respectively as

$$g(grad\varphi, X) = X\varphi \ \ and \ \ \nabla X = \sum_{i=1}^{n}\{(\nabla_{e_i}e_i)\varphi - e_ie_i\}. \tag{2.19}$$

The Laplacian of f is defined by

$$\Delta f = \sum_{i=1}^{n}\{(\nabla_{e_i}e_i)f - e_i(e_i(f))\} = -\sum_{i=1}^{n} g(\nabla_{e_i}grad f, e_i). \tag{2.20}$$

For a compact orientable Riemannian manifold M without boundary. Thus from the integration theory on manifolds, we have

$$\int_M \Delta f dV = 0, \tag{2.21}$$

such that dV denote the volume element of M (see [4]).■

3. Pointwise semi-slant submanifolds

The concept of semi-slant submanifolds were defined and studied by N. Papaghiuc (cf. [27]) as natural extension of CR-submanifolds of almost Hermitian manifolds in terms of slant immersion. Moreover, as a generalisation of semi-slant submanifolds, the pointwise semi-slant submanifolds were studied by Sahin [29]. He defined these submanifolds as follows:

Definition 3.1. *Let M be a submanifold of Kaehler manifold $\widetilde{M}$ is said to be a pointwise semi-slant submanifold if there exists two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that*

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\theta$,
(ii) $\mathcal{D}$ *is holomorphic, i.e.,* $J(\mathcal{D}) \subseteq \mathcal{D}$,
(iii) $\mathcal{D}^\theta$ *is pointwise slant distribution with slant function* $\theta : T^*M \to \mathbb{R}$.

On a pointwise semi-slant submanifold, If we denote the dimensions of $\mathcal{D}$ and $\mathcal{D}^\theta$ by d_1 and d_2, then M is *invariant* if $d_2 = 0$ and *pointwise slant* if $d_1 = 0$. Also, if θ is constant then M is proper semi-slant submanifold with slant angle *theta*. We say that a pointwise semi-slant submanifold is proper if $d_1 \neq 0$ and θ is not constant. Moreover, if ν is an *invariant* subspace under J of normal bundle $T^\perp M$, then in case of pointwise semi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as:

$$T^\perp M = F\mathcal{D}^\theta \oplus \nu$$

.

Let us denotes the orthogonal projections on $\mathcal{D}$ and $\mathcal{D}^\theta$ by B and C respectively. then we can write

$$U = BU + CU, \tag{3.1}$$

where $BU \in \Gamma(\mathcal{D})$ and $BU \in \Gamma(\mathcal{D}^\theta)$. From (2.5) and (3.1), we obtain

$$JBU \in \Gamma(\mathcal{D}), \quad FBU = 0, \tag{3.2}$$

$$TBU \in \Gamma(\mathcal{D}^\theta), \quad FBU \in \Gamma(T^\perp M). \tag{3.3}$$

On a pointwise semi-slant submanifold M of Kaehler manifold $\widetilde{M}$, the following are straightforward observations

$$\left.\begin{array}{ll} (i)\ F\mathcal{D} = 0, & (ii)\ T\mathcal{D} = \mathcal{D}, \\ (iii)\ t(T^\perp M) \subseteq \mathcal{D}^\theta, & (iv)\ T\mathcal{D}^\theta \subseteq \mathcal{D}^\theta. \end{array}\right\} \tag{3.4}$$

For the integrability conditions of distributions involved in the definition pointwise semi-slant submanifold, we refer to (cf. [29]). Now. we have the following useful result which is important for a Section 4.

Theorem 3.1. *Let M be a pointwise semi-slant submanifold M of a Kaehler manifold $\widetilde{M}$. Then the distribution $\mathcal{D}$ is define as totally geodesic foliations if and only if*

$$h(X, JY) \in \Gamma(\nu),$$

for any $X, Y \in \Gamma(\mathcal{D})$.

Proof. Let $X, Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\theta)$, we have $g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(J\widetilde{\nabla}_X Y, JZ)$. Using (2.5)(i) and (2.12), we obtain $g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X JY, TZ) + g(\widetilde{\nabla}_X JY, FZ) - g((\widetilde{\nabla}_X J)Y, JZ)$. From (2.2), (2.3)(i) and the definition of totally geodesic foliation we get required result. □

4. Warped Product Pointwise Submanifolds

Bishop and O'Neill defined in [6] the notion of warped product manifolds. They defined these manifolds as: Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f : M_1 \to (0, \infty)$ and $\pi_1 : M_1 \times M_2 \to M_1$, $\pi_2 : M_1 \times M_2 \to M_2$, the projection maps given by $\pi_1(p, q) = p$ and $\pi_2(p, q) = q$ for any $(p, q) \in M_1 \times M_2$. Then the warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_1{*}X, \pi_1{*}Y) + (f \circ \pi_1)^2 g_2(\pi_2{*}X, \pi_2{*}Y) \tag{4.1}$$

for any $X, Y \in TM$, where $*$ is the symbol for the tangent maps. The function f is called the warping function of M. In particular, a warped product manifold is said to be *trivial* if its warping function is constant. In such a case, we call the warped product manifold a Riemannian product manifold.

It was proved in [6] that for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, the following holds

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z \tag{4.2}$$

where ∇ denotes the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ is a warped product manifold, then M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifolds of M.

Recall that B. Sahin proved in [29] that there do not exist warped product pointwise semi-slant of the form $M = M_\theta \times_f M_T$ of a Kaehler manifold $\widetilde{M}$. Then he considered the warped products of the form $M = M_T \times_f M_\theta$. In the following we have the following results for both types of warped products.

Theorem 4.1. *[29] There do not exist proper warped product pointwise semi-slant submanifold $M = M_\theta \times_f M_T$ in a Kaehler manifold $\widetilde{M}$ such that M_θ is a proper pointwise slant submanifold and M_T is a holomorphic submanifold of $\widetilde{M}$.*

Lemma 4.1. *[29] Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$, where M_T and M_θ are holomorphic and pointwise slant submanifolds of $\widetilde{M}$ respectively. Then*

$$g(h(X,Z), FTW) = -(JX\ln f)g(Z,TW) - (X\ln f)cos^2\theta g(Z,W),$$

$$g(h(Z,JX), FW) = (X\ln f)g(Z,W) + (JX\ln f)(Z,TW),$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$

Lemma 4.2. *On a non-trivial warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ of a Kaehler manifold $\widetilde{M}$, we have*

$$(i)\ \ (\widetilde{\nabla}_X T)Z = 0,\ \ (ii)(\widetilde{\nabla}_Z T)X = (JX\ln f)Z - (X\ln f)TZ,$$

$$(iii)\ \ (\widetilde{\nabla}_{TZ} T)X = (JX\ln f)TZ + cos^2\theta(X\ln f)Z,$$

for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$

Proof. From (2.13) and (4.2), we derive

$$(\widetilde{\nabla}_X T)Z = \nabla_X TZ - T\nabla_X Z = (X\ln f)TZ - (X\ln f)TZ = 0,$$

for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$. Again from (2.13) and (4.2), we obtain

$$(\widetilde{\nabla}_Z T)X = \nabla_Z TX - T\nabla_Z X = (JX\ln f)Z - (X\ln f)TZ,$$

which is the second result of lemma. If we replace Z by TZ in(ii) and using Theorem 2.1, we get the last result of lemma, which proves the lemma. □

Lemma 4.3. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then*

$$(\widetilde{\nabla}_U T)X = (JX\ln f)CU - (X\ln f)TCU, \tag{4.3}$$

$$(\widetilde{\nabla}_U T)Z = g(CU,Z)J\nabla\ln f - g(CU,TZ)\nabla\ln f, \tag{4.4}$$

$$(\widetilde{\nabla}_U T)TZ = g(CU,TZ)J\nabla\ln f + \cos^2\theta g(CU,Z)\nabla\ln f, \tag{4.5}$$

for any $U \in \Gamma(TM)$, $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Thus from using (2.15)(a), it follows that

$$(\widetilde{\nabla}_X T)Y = th(X,Y),$$

for $X, Y \in \Gamma(TM_T)$. Since for warped product submanifold, M_T is totally geodesic in M, then using these fact we get $th(X,Y) = 0$, which implies that $h(X,Y) \in \Gamma(\nu)$. Thus the above relation becomes

$$(\widetilde{\nabla}_X T)Y = 0. \tag{4.6}$$

Now we applying (3.1) into $(\widetilde{\nabla}_U T)X$ to derive another relation

$$(\widetilde{\nabla}_U T)X = (\widetilde{\nabla}_{BU} T)X + (\widetilde{\nabla}_{CU} T)X,$$

for $U \in \Gamma(TM)$. The first part of right hand side in the above equation should be zero by virtue (4.6). Thus the second part of the above equation follows from Lemma 4.2(ii). Again from (3.1), we have

$$(\widetilde{\nabla}_U T)Z = (\widetilde{\nabla}_{BU} T)Z + (\widetilde{\nabla}_{CU} T)Z,$$

for $Z \in \Gamma(TM_\theta)$ and $U \in \Gamma(TM)$. Taking the inner product with $X \in \Gamma(TM_T)$ and using (2.13), we obtain

$$\begin{aligned} g((\widetilde{\nabla}_U T)Z, X) &= g(\nabla_{CU} TZ, X) - g(T\nabla_{CU} Z, X) \\ &= g(\nabla_{CU} Z, JX) - g(\nabla_{CU} X, TZ) \\ &= -g(\nabla_{CU} JX, Z) - g(\nabla_{CU} X, TZ). \end{aligned}$$

From (4.2), we get

$$\begin{aligned} g((\widetilde{\nabla}_U T)Z, X) &= -(JX \ln f) g(CU, Z) - (X \ln f) g(CU, TZ) \\ &= g(CU, Z) g(J\nabla \ln f, X) - g(CU, TZ) g(\nabla \ln f, X), \end{aligned}$$

which implies that

$$(\widetilde{\nabla}_U T)Z = g(CU, Z) J\nabla \ln f - g(CU, TZ)\nabla \ln f,$$

which is (4.4). Replacing Z by TZ in (4.3) and using Theorem 2.1 for pointwise slant submanifold M_θ. Then the above equation takes the form

$$(\widetilde{\nabla}_U T)TZ = g(CU, TZ) J\nabla \ln f + \cos^2\theta g(CU, Z)\nabla \ln f,$$

which is (4.5). Hence, the lemma is proved completely. □

In a sequel, now we prove characterization results in terms of the tensor fields.

Theorem 4.2. *Let M be a pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$ with pointwise slant distribution $\mathcal{D}^\theta$ is integrable. Then M is locally a warped product submanifold if and only if*

$$(\widetilde{\nabla}_U T)V = (JBV\lambda)CU - (BV\lambda)TCU + g(CU, CV) J\vec{\nabla}\lambda - g(CU, TCV)\vec{\nabla}\lambda, \tag{4.7}$$

for each $U, V \in \Gamma(TM)$ and a C^∞-function μ on M with $Z\lambda = 0$ for each $Z \in \Gamma(\mathcal{D}^\theta)$.

Proof. Suppose that M be a warped product pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then using (3.1), we obtain

$$(\widetilde{\nabla}_U T)V = (\widetilde{\nabla}_U T)BV + (\widetilde{\nabla}_U T)CV,$$

for $U, V \in \Gamma(TM)$. Thus the first part directly follows by Lemma 4.3 (4.3)-(4.4) in the above equation. Let us prove the converse part that M be a pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$ such that the given condition (4.7) holds. It is easy to obtain the following condition

$$(\widetilde{\nabla}_X T)Y = 0.$$

by consider $U = X$ and $V = Y$ in (4.7), for $X, Y \in \Gamma(\mathcal{D})$. Taking the inner product with $TZ \in \Gamma(\mathcal{D}^\theta)$ and using (2.13), we derive

$$g(\widetilde{\nabla}_X JY, TZ) = g(T\nabla_X Y, TZ).$$

Since TZ and JY are orthogonal then from property of Riemannian connection and from (2.6), we derive

$$g(\widetilde{\nabla}_X TZ, JY) = -g(\nabla_X Y, T^2 Z).$$

From the covariant derivative of an almost complex structure J and Theorem 2.1, it is easily seen that

$$g((\widetilde{\nabla}_X J)TZ, Y) - g(\widetilde{\nabla}_X JTZ, Y) = \cos^2\theta g(\nabla_X Y, Z).$$

Thus using the structure equation of Kaehler manifold and (2.5)(i), we arrive at

$$g(\widetilde{\nabla}_X T^2 Z, Y) + g(\widetilde{\nabla}_X FTZ, Y) = \cos^2\theta g(\nabla_X Y, Z).$$

Then using Theorem 2.1, in the first part of the above equation for pointwise slant function θ and also from (2.3)(ii), we obtain

$$\sin 2\theta X(\theta)g(Z,Y) - \cos^2\theta g(\nabla_X Z, Y) = g(h(X,Y), FTZ) + \cos^2\theta g(\nabla_X Y, Z),$$

which implies that

$$g(h(X,Y), FTZ) = 0.$$

It is indicates that $h(X,Y) \in \Gamma(\nu)$ for all $X, Y \in \Gamma(\mathcal{D})$. Then from Theorem 3.1, i.e., the distribution $\mathcal{D}$ is defines a totally geodesic foliations and its leaves are totally geodesic in M. Furthermore, we set $U = Z$ and $V = W$ in (4.7), we derive

$$(\widetilde{\nabla}_Z T)W = g(Z,W)J\nabla\lambda + g(TZ,W)\nabla\lambda$$

for $Z, W \in \Gamma(\mathcal{D}^\theta)$. Taking the inner product with $X \in \Gamma(\mathcal{D})$ and using (2.5)(i), we obtain

$$g(\nabla_Z TW, X) - g(T\nabla_Z W, X) = -(X\lambda)g(Z,TW) - (JX\lambda)g(Z,W).$$

By hypothesis of the theorem, as we have considered that the pointwise slant distribution is integrable. It is obvious that, let M_θ be a leaf of $\mathcal{D}^\theta$ in M and h^θ be the second fundamental form of M_θ in M. Then

$$g(h^\theta(Z,TW), X) + g(h^\theta(Z,W), JX) = -(X\lambda)g(Z,TW) - (JX\lambda)g(Z,W). \tag{4.8}$$

Replacing W by TW and X by JX in (4.7) and from the Theorem 2.1, we derive

$$-\cos^2\theta g(h^\theta(Z,W), JX) - g(h^\theta(Z,TW), X) = \cos^2\theta(JX\lambda)g(Z,W) + (X\lambda)g(Z,TW). \tag{4.9}$$

Thus from (4.8) and (4.9), it follows that

$$\sin^2\theta g(h^\theta(Z,W), JX) = -sin^2\theta(JX\lambda)g(Z,W).$$

which implies that

$$g(h^\theta(Z,W), JX) = -(JX\lambda)g(Z,W).$$

From the gradient definition. Finally, we get

$$h^\theta(Z,W) = -g(Z,W)\nabla\lambda,$$

From the above relation, we conclude that M_θ is totally umbilical in M such that $H^\theta = -\nabla\lambda$ is the mean curvature vector of M_θ. Now, we can easily show that the mean curvature vector H^θ is parallel corresponding to the normal connection ∇' of M_θ in M. This means that M_θ is an extrinsic spheres in M. Hence from result of Hiepko (cf. [16]), M is called a warped product submanifold of integral manifolds M_T and M_θ of $\mathcal{D}$ and $\mathcal{D}^\theta$, respectively. Its complete proof of the theorem. □

Theorem 4.3. *Let M be a pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$ such that the pointwise slant distribution $\mathcal{D}^\theta$ is integrable. Then M is locally a warped product submanifold if and only if*

$$(\widetilde{\nabla}_U F)V = fh(U,BV) - h(U,TCV) - (BV\lambda)FCU \tag{4.10}$$

for each $U, V \in \Gamma(TM)$ and a C^∞-function μ on M with $Z\lambda = 0$, for each $Z \in \Gamma(\mathcal{D}^\theta)$.

Proof. From the first case, suppose that M be a warped product pointwise semi-slant submanifold in a Kaehler manifold $\widetilde{M}$. Then using (3.1) in $(\widetilde{\nabla}_U F)X$, we derive

$$(\widetilde{\nabla}_U F)X = (\widetilde{\nabla}_{BU} F)X + (\widetilde{\nabla}_{CU} F)X,$$

for $U \in \Gamma(TM)$ and $X \in \Gamma(TM_T)$. The first term of the above equation identically zero by using the fact that M_T is totally geodesic on M. Last term follows from (2.14) and (4.2), we obtain

$$\begin{aligned}(\widetilde{\nabla}_U F)X &= -F\nabla_{CU}X \\ &= -(X \ln f)FCU. \end{aligned} \tag{4.11}$$

From (2.15)(b), we derive

$$(\widetilde{\nabla}_U F)Z = fh(U, Z) - h(U, TZ), \tag{4.12}$$

for $Z \in \Gamma(TM_\theta)$. Furthermore, again from (3.1), we obtain

$$(\widetilde{\nabla}_U F)V = (\widetilde{\nabla}_U F)BV + (\widetilde{\nabla}_U F)CV. \tag{4.13}$$

Hence, from (4.11), (4.12) in (4.13), we get desired result (4.10).

Conversely, suppose that M be a pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$ with integrable distribution $\mathcal{D}^\theta$ and (4.10) holds. Then for $X, Y \in \Gamma(\mathcal{D})$, it follows from (4.10), we get $-F\nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(\mathcal{D})$, thus the leaves of $\mathcal{D}$ are totally geodesic in M. On the other hand, the pointwise slant distribution $\mathcal{D}^\theta$ is assumed to be integrable. Then we can consider M_θ to be a leaf of $\mathcal{D}^\theta$ and h^θ be the second fundamental form of immersion into M. Thus replacing $U = Z$ and $V = X$, in (4.10) for $Z \in \Gamma(\mathcal{D}^\theta)$ and $X \in \Gamma(\mathcal{D})$ and using the fact that $CX = 0$, we derive

$$(\widetilde{\nabla}_Z F)X = -(X\lambda)FZ. \tag{4.14}$$

Taking inner product in (4.14) with FW for $W \in \Gamma(\mathcal{D}^\theta)$ and using relation (2.18), then equation (4.14) can be modified as:

$$g((\widetilde{\nabla}_Z F)X, FW) = -\sin^2\theta(X\lambda)g(Z, W).$$

Apply (2.14) in left hand side in the above equation, we obtain

$$g(-F\nabla_Z X, FW) = -\sin^2\theta(X\lambda)g(Z, W).$$

Thus by virtue (2.18) and definition of gradient of $\ln f$, we arrive at

$$-\sin^2\theta g(\nabla_Z X, W) = -\sin^2\theta g(\nabla\lambda, X)g(Z, W),$$

which implies that

$$g(h^\theta(Z, W), X) = -g(X, \nabla\lambda)g(Z, W).$$

Finally, we obtain

$$h^\theta(Z, W) = -g(Z, W)\nabla\lambda.$$

The above relation shows that the leaf M_θ (of $\mathcal{D}^\theta$) is totally umbilical in M such that $H^\theta = -\nabla\lambda$, is the mean curvature vector of M_θ. Moreover, the condition $Z\lambda = 0$, for any $Z \in \Gamma(\mathcal{D}^\theta)$ implies that the leaves of $\mathcal{D}^\theta$ are extrinsic spheres in M, i.e., the integral manifold M_θ of $\mathcal{D}^\theta$ is totally umbilical and its mean curvature vector is non zero and parallel along M_θ. Thus from result of Hiepko (cf. [16]), i.e., $M = M_T \times_f M_\theta$ is locally a warped product submanifold, where M_T is an integral manifold of $\mathcal{D}$ and f is a warping function. It completes proof of the theorem. □

5. Inequalities for Warped Product Pointwise Semi-slant Submanifolds

In this section, we construct some geometric properties of the mean curvature for warped product semi-slant submanifolds and using these result to derive a general inequality for the second fundamental form in terms of Chen's invariants. Similar inequality has been obtained in [22] for the squared norm of the second fundamental form for warped product submanifolds such that the base manifold is invariant (holomorphic) submanifold of a Kenmotsu manifold.

Let $\phi : M = M_1 \times_f M_2 \to \widetilde{M}$ be isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold of $\widetilde{M}$ of constant section curvature c. Let $n_1,\ n_2$ and n be the dimension of $M_1,\ M_2$, and $M_1 \times_f M_2$ respectively. Then for unit vector X, Z tangent to $M_1,\ M_2$ respectively, we have

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{5.1}$$

If we consider the local orthonormal frame $\{e_1, e_2 ....e_n)$ such that $e_1, e_2.....e_{n_1}$ tangent to M_1 and $e_{n_1+1}.......e_n$ are tangent to M_2. Then in view of Guass equation (2.7), we derive

$$\rho(TM) = \widetilde{\rho}(TM) + \sum_{r=1}^{2m} \sum_{1 \le i \ne j \le n} (h^r_{ii} h^r_{jj} - (h^r_{ij})^2), \tag{5.2}$$

for each $j = n_1 + 1.....n$. Now we are ready to prove the general inequality. For this we need to define a frame and obtain some preparatory lemmas. To prove the general inequality, we need the following frame fields and some preparatory results.

Let $M = M_T \times_f M_\theta$ be an $n = n_1 + n_2$-dimensional warped product pointwise semi-slant submanifold of a $2m$-dimensional Kaehler manifold $\widetilde{M}$ such that $\dim M_T = n_1 = 2d_1$ and $\dim_{\mathbf{R}} M_\theta = n_2 = 2d_2$. Let us consider the tangent spaces of M_T and M_θ by $\mathcal{D}$ and $\mathcal{D}^\theta$ respectively. Assume that $\{e_1, e_2, \cdots, e_{d_1}, e_{d_1+1} = Je_1, \cdots, e_{2d_1} = Je_{d_1}\}$ is a local orthonormal frame of $\mathcal{D}$ and $\{e_{2d_1+1} = e^*_1, \cdots, e_{2d_1+d_2} = e^*_{d_2}, e_{2d_1+d_2+1} = e^*_{d_2+1} = \sec\theta T e^*_1, \cdots, e_{n_1+n_2} = e^*_{n_2} = \sec\theta T e^*_{d_2}\}$ is a local orthonormal frame of $\mathcal{D}^\theta$. Thus the orthonormal frames of the normal sub bundles, $F\mathcal{D}^\theta$ and ν respectively are,$\{e_{n+1} = \widetilde{e}_1 = \csc\theta F e^*_1, \cdots, e_{n+d_2} = \widetilde{e}_{d_2} = \csc\theta F e^*_1, e_{n+d_2+1} = \widetilde{e}_{d_2+1} = \csc\theta \sec\theta FT e^*_1, \cdots, e_{n+2d_2} = \widetilde{e}_{2d_2} = \csc\theta\sec\theta FT e^*_{d_2}\}$ and $\{e_{n+2d_2+1}, \cdots, e_{2m}\}$.

Lemma 5.1. *Let M be a non-trivial warped product pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then*

$$g(h(X, X), FZ) = g(h(X, X), FTZ) = 0, \tag{5.3}$$

$$g(h(JX, JX), FZ) = g(h(JX, JX), FTZ) = 0, \tag{5.4}$$

$$g(h(X, X), \xi) = -g(h(JX, JX), \xi), \tag{5.5}$$

for any $X \in \Gamma(TM_T)$, $Z \in \Gamma(TM_\theta)$ and $\xi \in \Gamma(\nu)$.

Proof. From relation (2.3), we have

$$g(h(X, X), FTZ) = g(\widetilde{\nabla}_X X, FTZ) = -g(\widetilde{\nabla}_X FTZ, X).$$

Thus from relation (2.5) and the covariant derivative of almost complex structure J, we obtain

$$g(h(X, X), FTZ) = g(\widetilde{\nabla}_X TZ, JX) + g((\widetilde{\nabla}_X J)TZ, X) + g(\widetilde{\nabla}_X T^2 Z, X),$$

Using the structure equation of Kaehler manifolds and Theorem 2.1 for pointwise semi-slant submanifold, we get

$$g(h(X, X), FTZ) = -g(\nabla_X JX, TZ) + \sin 2\theta X(\theta) g(Z, X) - \cos^2\theta g(\nabla_X X, Z)$$

Since, M_T is totally geodesic in M, with this fact we get result (5.3). On other part, interchanging Z by TZ and X by JX in the above equation we get the required result (5.4). Now for (5.5), from Kaehler manifold, we have $\widetilde{\nabla}_X JX = J\widetilde{\nabla}_X X$, this relation reduced to

$$\nabla_X JX + h(JX, X) = J\nabla_X X + Jh(X, X).$$

Taking the inner product with $J\xi$ in the above equation for any $\xi \in \Gamma(\nu)$, we obtain

$$g(h(JX, X), J\xi) = g(h(X, X), \xi). \tag{5.6}$$

Interchanging X by JX in (5.6) and making use of (2.1)(i). Furthermore, the fact ν is an invariant normal bundle of $T^{\perp}M$ under an almost complex structure J, we get

$$-g(h(X, JX), J\xi) = g(h(JX, JX), \xi). \tag{5.7}$$

From (5.6) and (5.7), we get (5.5). Its complete proof of lemma. □

Lemma 5.2. *Let ϕ be an isometrically pointwise immersion $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}$ such that M_T is invariant submanifold of $\widetilde{M}$ and M_θ is pointwise slant submanifold of $\widetilde{M}$. Then the squared norm of mean curvature of M is given by*

$$||H||^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m} \left[h^r_{n_1+1n_1+1} + ... + h^r_{nn}\right]^2,$$

where H is the mean curvature vector. Moreover, and n_1, n_2, n and $2m$ are dimensions of M_T, M_θ, $M_T \times_f M_\theta$ and $\widetilde{M}$ respectively.

Proof. From the definition of the mean curvature vector, we have

$$||H||^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m} (h^r_{11} + \cdots + h^r_{nn})^2,$$

Thus from consideration of dimension $n = n_1 + n_2$ of $M_T \times_f M_\theta$ such that n_1 and n_2 are dimensions of M_T and M_θ respectively, we arrive at

$$||H||^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m} (h^r_{11} + \cdots h^r_{n_1n_1} + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2.$$

Using the frame of $\mathcal{D}$ and coefficient of n_1 in right hand side of the above equation, we get

$$\left(h^r_{11} + \cdots h^r_{n_1n_1} + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\right)^2 = \left(h^r_{11} + \cdots h^r_{d_1d_1} + h^r_{d_1+1d_1+1} + \cdots h^r_{2d_12d_1} + h^r_{n_1+1n_1+1} + ... + h^r_{nn}\right)^2.$$

From the relation $h^r_{ij} = g(h(e_i, e_j), e_r)$, for $1 \leq i, j \leq n$ and $n+1 \leq r \leq 2m$ and frame for $\mathcal{D}$, the above equation take the form

$$\left(h^r_{11} + \cdots h^r_{n_1n_1} + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\right)^2$$
$$= \{g(h(e_1, e_1), e_r) + ... + g(h(e_{d_1}, e_{d_1}), e_r) + g(h(Je_1, Je_1), e_r) + \cdots + g(h(Je_{d_1}, Je_{d_1}), e_r) + ... + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\}^2. \tag{5.8}$$

It well known that e_r belong to normal bundle $T^{\perp}M$ for ever $r \in \{n+1 \cdots 2m\}$, it mean that there two cases such that e_r belong to $F(TM_\theta)$ or ν.

Case 1: If $e_r \in \Gamma(F\mathcal{D}^\theta)$, then from using frame in (5.6) of normal components for pointwise slant distribution $\mathcal{D}^\theta$ which is defined in frame. Then equation (5.8) can be written as

$$\begin{aligned}
&\left(h^r_{11} + \cdots h^r_{n_1n_1} + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\right)^2 \\
&= \Big\{ \csc\theta g(h(e_1, e_1), Fe^*_1) + .. + \csc\theta g(h(e_{d_1}, e_{d_1}, Fe^*_{d_2}) + \csc\theta \sec\theta g(h(e_1, e_1), FTe^*_1) \\
&\quad + \cdots + \csc\theta \sec\theta g(h(e_{d_1}, e_{d_1}), FTe^*_{d_2}) + \csc\theta g(h(Je_1, Je_1), Fe^*_1) + \\
&\quad \cdots + \csc\theta g(h(Je_{d_1}, Je_{d_1}), Fe^*_{d_2}) + \csc\theta \sec\theta g(h(Je_1, Je_1), FTe^*_1) + \\
&\quad \cdots + \csc\theta \sec\theta g(h(Je_{d_1}, Je_{d_1}), FTe^*_{d_2}) + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\Big\}^2.
\end{aligned}$$

Now from virtue (5.1) and (5.2) of Lemma 5.1, finally we get

$$(h^r_{11} + \cdots h^r_{n_1n_1} + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2 = (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2. \tag{5.9}$$

Case 2: If $e_r \in \Gamma(\nu)$, then from relation (5.4) of Lemma 5.1, the equation (5.8) simplifies as

$$\Big(h^r_{11} + ...h^r_{n_1n_1} + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\Big)^2$$

$$= \Big\{g(h(e_1, e_1), e_r) + \cdots + g(h(e_{d_1}), e_{d_1}), e_r) - g(h(e_1, e_1), e_r) \cdots - g(h(e_{d_1}, e_{d_1}), e_r) + \cdots + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\Big\}^2,$$

which implies that

$$\Big(h^r_{11} + \cdots h^r_{n_1n_1} + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\Big)^2 = \Big(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\Big)^2. \tag{5.10}$$

From (5.7) and (5.9) for every normal vector e_r belong to the normal bundle $T^\perp M$ and taking the summing, we can deduce that

$$\sum_{r=n+1}^{2m} \Big(h^r_{11} + \cdots h^r_{n_1n_1} + h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\Big)^2 = \sum_{r=n+1}^{2m} \Big(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\Big)^2.$$

Hence, the above relation proves our assertion. It completes proof of the lemma. □

Theorem 5.1. *Let $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}$ be an isometrically immersion of an n-dimensional non-trivial warped product pointwise semi-slant submanifold M into $2m$-dimensional Kaehler manifold $\widetilde{M}$ such that M_θ is pointwise slant submanifold and M_T is invariant submanifold of $\widetilde{M}$. Then*

(i) The squared norm of the second fundamental form of M is given by

$$\|h\|^2 \geq 2\left(\widetilde{\rho}(TM) - \widetilde{\rho}(TM_T) - \widetilde{\rho}(TM_\theta) - \frac{n_2 \nabla f}{f}\right), \tag{5.11}$$

where n_2 is the dimension of pointwise slant subamnifold M_θ.

(ii) The equality holds in the above inequality, if and only if M_T is totally geodesic and M_θ is totally umbilical submanifolds of $\widetilde{M}$.

Proof. Putting $X = W = e_i$, and $Y = Z = e_j$ in Gauss equation (2.7), we obtain

$$\widetilde{R}(e_i, e_j, e_j, e_i) = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_j, e_i) - g(h(e_i, e_e), h(e_j, e_j)).$$

Over $1 \leq i, j \leq n (i \neq j)$, taking summation in above equation, we obtain

$$2\widetilde{\rho}(TM) = 2\rho(TM) - n^2\|H\|^2 + \|h\|^2.$$

Then from (2.11), we derive

$$\|h\|^2 = n^2\|H\|^2 + 2\widetilde{\rho}(TM) - 2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i \wedge e_j) - 2\rho(TM_T) - 2\rho(TM_\theta).$$

The fourth and fifth terms of the above equation can be obtained by using (5.2), then we get

$$\|h\|^2 = n^2\|H\|^2 + 2\widetilde{\rho}(TM) - 2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i \wedge e_j)$$

$$-2\widetilde{\rho}(TM_T) - 2\sum_{r=n+1}^{2m}\sum_{1\le i\neq t\le n_1}(h^r_{ii}h^r_{tt} - (h^r_{it})^2) - 2\widetilde{\rho}(TM_\theta) - 2\sum_{r=n+1}^{2m}\sum_{n_1+1\le j\neq l\le n}(h^r_{jj}h^r_{ll} - (h^r_{jl})^2). \tag{5.12}$$

Now we using the following formula obtained by Chen (cf. [10]) for general warped product submanifold, i.e.,

$$\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i \wedge e_j) = \frac{n_2\nabla f}{f}.$$

Then equation (5.12) implies that

$$\|h\|^2 = n^2\|H\|^2 + 2\widetilde{\rho}(TM) - 2\frac{n_2\Delta f}{f} - 2\widetilde{\rho}(TM_\theta)$$

$$-2\widetilde{\rho}(TM_T) - 2\sum_{r=n+1}^{2m}\sum_{1\le i\neq t\le n_1}(h^r_{ii}h^r_{tt} - (h^r_{it})^2) - 2\sum_{r=n+1}^{2m}\sum_{n_1+1\le j\neq l\le n}(h^r_{jj}h^r_{ll} - (h^r_{jl})^2).$$

We adding and subtracting the same terms in the above equation, we find that

$$\|h\|^2 = n^2\|H\|^2 + 2\widetilde{\rho}(TM) - 2\frac{n_2\Delta f}{f} - 2\widetilde{\rho}(TM_\theta) - 2\widetilde{\rho}(TM_T)$$

$$-2\sum_{r=n+1}^{2m}\sum_{1\le i\neq t\le n_1}(h^r_{ii}h^r_{tt} - (h^r_{it})^2) - \sum_{r=n+1}^{2m}((h^r_{11})^2 + ... + (h^r_{nn})^2)$$

$$+\sum_{r=n+1}^{2m}((h^r_{11})^2 + \cdots + (h^r_{nn})^2) - 2\sum_{r=n+1}^{2m}\sum_{n_1+1\le j\neq l\le n}(h^r_{jj}h^r_{ll} - (h^r_{jl})^2).$$

The above equation is equivalent to the new form

$$\|h\|^2 = n^2\|H\|^2 + 2\widetilde{\rho}(TM) - 2\frac{n_2\Delta f}{f} - 2\widetilde{\rho}(TM_\theta) - 2\widetilde{\rho}(TM_T)$$

$$+2\sum_{r=n+1}^{2m}\sum_{i,t=1}^{n_1}(h^r_{it})^2 - \sum_{r=n+1}^{2m}(h^r_{11} + \cdots + h^r_{nn})^2 - 2\sum_{r=n+1}^{2m}\sum_{n_1+1\le j\neq l\le n}(h^r_{jj}h^r_{ll} - (h^r_{jl})^2).$$

Again we adding and subtracting the same terms for last term in the above equation. Then we modified as

$$\|h\|^2 = n^2\|H\|^2 + 2\widetilde{\rho}(TM) - 2\frac{n_2\Delta f}{f} - 2\widetilde{\rho}(TM_\theta) - 2\widetilde{\rho}(TM_T) + 2\sum_{r=n+1}^{2m}\sum_{i,t=1}^{n_1}(h^r_{it})^2$$

$$-\sum_{r=n+1}^{2m}(h^r_{11} + \cdots + h^r_{nn})^2 - \sum_{r=n+1}^{2m}((h^r_{n_1+1n_1+1})^2 + \cdots + (h^r_{nn})^2)$$

$$-2\sum_{r=n+1}^{2m}\sum_{n_1+1\le j\neq l\le n}(h^r_{jj}h^r_{ll} - (h^r_{jl})^2) + \sum_{r=n+1}^{2m}((h^r_{n_1+1n_1+1})^2 + \cdots + (h^r_{nn})^2).$$

After using the Lemma 5.2. The above equation turn into the new form, i.e.,

$$\|h\|^2 = 2\widetilde{\rho}(TM) - 2\frac{n_2.\Delta f}{f} - 2\widetilde{\rho}(TM_\theta) - 2\widetilde{\rho}(TM_T) + 2\sum_{r=n+1}^{2m}\sum_{i,t=1}^{n_1}(h^r_{it})^2 + 2\sum_{r=n+1}^{2m}\sum_{j,l=n_1+1}^{n}(h^r_{jl})^2. \tag{5.13}$$

Thus the equation (5.13) implies the inequality (5.11). If equality sign in (5.11) holds if and only if we have

$$(i)\ \sum_{r=n+1}^{2m}\sum_{i,t=1}^{n_1}(g(h(e_i,e_t),e_r))^2=0,$$

$$(ii)\ \sum_{r=n+1}^{2m}\sum_{j,l=n_1+1}^{n_1}(g(h(e_j,e_l),e_r))^2=0. \tag{5.14}$$

As the fact that M_T is totally geodesic in M, from (5.3) and (5.4), it implies that M_T is totally geodesic in $\widetilde{M}$. On the other hand, (5.14) implies that h vanishes on $\mathcal{D}^\theta$. Moreover, $\mathcal{D}^\theta$ is a spherical distribution in M, then it follows that M_θ is totally umbilical in $\widetilde{M}$. Its complete proof of the theorem. □

Now, we are able to prove the following theorem by using the above result for a complex space form as follows:

Theorem 5.2. *Assume that $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}$ be an isometrically immersion of an n-dimensional non-trivial warped product pointwise semi-slant submanifold M into a 2m-dimensional complex space form $\widetilde{M}(c)$ with constant holomorphic sectional curvature c such that M_θ is a proper pointwise slant submanifold and M_T is an invariant submanifold of $\widetilde{M}$. Then*

(i) The squared norm of the second fundamental form of M is given by

$$||h||^2 \geq 2n_2\left(||\nabla(\ln f)||^2+\frac{n_1c}{4}-\Delta(\ln f)\right) \tag{5.15}$$

where n_2 is the dimension of pointwise slant subamnifold M_θ.

(ii) The equality holds in the above inequality if and only if M_T is totally geodesic and M_θ is totally umbilical submanifolds of $\widetilde{M}$. Moreover, M is minimal submanifold in $\widetilde{M}$.

Proof. The Remannian curvature of complex space form with constant holomorphic sectional curvature c is given by

$$\widetilde{R}(X,Y,Z,W)=\frac{c}{4}\Big\{g(Y,Z)g(X,W)-g(Y,W)g(X,Z)+g(X,JZ)g(JY,W)-g(Y,JZ)g(JX,W)+2g(X,JY)g(JZ,W)\Big\},$$

for any $X,Y,Z,W \in \Gamma(T\widetilde{M})$. Now substituting $X=W=e_i$ and $Y=Z=e_j$ in the above equation, we get

$$\widetilde{R}(e_i,e_j,e_j,e_i)=\frac{c}{4}\Big\{g(e_i,e_i)g(e_j,e_j)-g(e_i,e_j)g(e_i,e_j)+g(e_i,Je_j)g(je_j,e_i)-g(e_i,Je_i)g(e_j,Je_j)+2g^2(Je_j,e_i)\Big\}.$$

Taking summation over basis vector of TM such that $1\leq i\neq j\leq n$, it is easy to obtain that

$$2\widetilde{\rho}(TM)=\frac{c}{4}\left(n(n-1)+3\sum_{1\leq i\neq j\leq n}g^2(Te_i,e_j)\right). \tag{5.16}$$

Let M be a proper pointwise semi-slant submanifold of complex space form $\widetilde{M}(c)$. Thus we set the following frame, i.e.,

$$e_1, e_2=Je_1,\cdots e_{2d_1-1}, e_{2d_1}=Je_{2d_1-1},$$

$$e_{2d_1+1}, e_{2d_1+2}=\sec\theta Te_{2d_1+1},\cdots e_{2d_1+2d_2-1}e_{2d_1+2d_2}=\sec\theta Te_{d_1-1}.$$

Obviously, we derive

$$g^2(Je_i,e_{i+1})=1,\quad for\ i\in\{1,.....2d_1-1\}$$

$$= \cos^2\theta \;\; for \;\; i \in \{2d_1+1, ...2d_1+2d_2-1\}.$$

Thus it is easily seen that

$$\sum_{i,j=1}^{n} g^2(Te_i, e_j) = 2(d_1 + d_2.\cos\theta). \tag{5.17}$$

From (5.16) and (5.17), it follows that

$$2\widetilde{\rho}(TM) = \frac{c}{4}n(n-1) + \frac{3c}{2}(d_1 + d_2\cos\theta). \tag{5.18}$$

Similarly, for TM_T, we derive

$$2\widetilde{\rho}(TM_T) = \frac{c}{4}\left[n_1(n_1-1) + 3n_1\right] = \frac{c}{4}\left[n_1(n_1+2)\right]. \tag{5.19}$$

Now using fact that $\|T\|^2 = n_2.cos^2\theta$, for pointwise slant submanifold TM_θ, we derive

$$2\widetilde{\rho}(TM_\theta) = \frac{c}{4}\left[n_2(n_2-1) + 3n_2\cos^2\theta\right] = \frac{c}{4}\left[n_2^2 + n_2(3\cos^2\theta - 1)\right]. \tag{5.20}$$

Therefore using (5.18), (5.19) and (5.20) in (5.11), we get the required result and the equality case directely comes from Theorem 5.1(ii). It completes proof of the theorem. □

Corollary 5.1. *Assume that $\phi : M = M_T \times_f M_\perp \to \widetilde{M}$ be an isometrically immersion of an n-dimensional non-trivial CR-warped product submanifold M into a $2m$-dimensional complex space form $\widetilde{M}(c)$ with constant holomorphic sectional curvature c such that $M_\perp$ is totally real submanifold and M_T is invariant submanifold of $\widetilde{M}$. Then*

(i) The squared norm of the second fundamental form of M is given by

$$\|h\|^2 \geq \frac{n_1 n_2 c}{2} - \frac{2n_2\Delta f}{f}, \tag{5.21}$$

where n_2 is the dimension of totally real submanifold subamnifold $M_\perp$.

(ii) The equality holds in the above inequality if and only if M_T and $M_\perp$ are totally umbilical and totally geodesic submanifolds of $\widetilde{M}$, respectively. Moreover, M is minimal submanifold $\widetilde{M}$.

Proof. The proof follows from the Theorem 5.2, if the slant function θ becomes globally constant and using $\theta = \frac{\pi}{2}$, for totally real submanifolds, we get required result. □

6. Applications to Compact Warped Product Submanifolds in Complex Space Forms

Theorem 6.1. *Let $M = M_T \times_f M_\theta$ be a compact warped product pointwise semi-slant submanifold of complex space form $\widetilde{M}(c)$. Then M is a Riemannian product if*

$$\|h\|^2 \geq \frac{n_1.n_2.c}{2}, \tag{6.1}$$

where n_1 and n_2 are dimensions of M_T is invariant and M_θ is proper pointwise slant submanifolds, respectively.

Proof. Let us consider that, the inequality holds in Theorem 5.2, we get

$$\frac{n_1 n_2 c}{2} + n_2\|\nabla \ln f\|^2 - \|h\|^2 \leq n_2\Delta(\ln f). \tag{6.2}$$

From the integration theory on manifolds, i.e., compact orient-able Riemannian manifold without boundary on M, we obtain

$$\int_M \left(\frac{n_1 n_2 c}{2} + n_2\|\nabla \ln f\|^2 - \|h\|^2\right) dV \leq n_2 \int_M \Delta(\ln f) dV = 0.$$

If the following inequality holds

$$\|h\|^2 \geq \frac{n_1 n_2 c}{2}.$$

Then

$$\int_M (\|\nabla \ln f\|^2) dV \leq 0.$$

Since integration always be positive for positive functions. Hence, we derive $\|\nabla \ln f\|^2 \leq 0$, but $\|\nabla \ln f\|^2 \geq 0$, which implies that $\nabla \ln f = 0$, i.e., f is a constant function on M. Thus M becomes simply Riemannian product manifold. □

Theorem 6.2. *Let $M = M_T \times_f M_\theta$ be a compact warped product proper pointwise semi-slant submanifold in a complex space form $\widetilde{M}(c)$ such that M_T is invariant submanifold of dimension n_1 and M_θ is pointwise slant submanifold of dimension n_2 in $\widetilde{M}(c)$. Then M is simply a Riemannian product if and only if*

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\nu(e_i, e_j)\|^2 = \frac{n_1.n_2.c}{4}, \tag{6.3}$$

where θ is a real value function define on T^*M is called a slant function and h_ν is a components of h in $\Gamma(\nu)$.

Proof. Suppose that the equality sign holds in (5.15), then we have

$$\|h(\mathcal{D}, \mathcal{D})\|^2 + \|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2 + 2\|h(\mathcal{D}, \mathcal{D}^\theta)\|^2 = \frac{n_1.n_2.c}{2} + 2n_2\{\|\nabla \ln f\|^2 - \Delta(\ln f)\}.$$

Following the equality case of the inequality in (5.15) implies from Theorem 5.2 (ii) that M_T is totally geodesic in $\widetilde{M}$ and this means that $h(e_i, e_j) = 0$, for any $1 \leq i, j \leq 2d_1$. Also and M_θ is totally umbilical submanifolds into $\widetilde{M}$ and it can be written as $h(e_t^*, e_s^*) = g(e_t^*, e_s^*)H$, for any $1 \leq t, s \leq 2d_2$. Since M is minimal submanifold in $\widetilde{M}$ by hypothesis, then its mean curvature vector H identically zero, i.e., $H = 0$. Hence $h(e_t^*, e_s^*) = 0,\ for\ every\ \ 1 \leq t, s \leq 2d_2$ by minimality of M_T.Thus above equation takes the new form

$$\frac{n_1.n_2.c}{4} = n_2\Delta(\ln f) + \|h(\mathcal{D}, \mathcal{D}^\theta)\|^2 - n_2\|\nabla \ln f\|^2.$$

Suppose that M is compact submanifold, then M is closed and bounded. Hence taking integration over the volume element dV of M and from (2.21), we derive

$$\int_M \Big(\frac{n_1.n_2.c}{4}\Big) dV = \int_M \left(\|h(\mathcal{D}, \mathcal{D}^\theta)\|^2 + n_2\|\nabla \ln f\|^2\right) dV \tag{6.4}$$

Let us assume that $X = e_i$ and $Z = e_j$ for $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$, respectively, then we have

$$h(e_i, e_j) = \sum_{r=n+1}^{n+n_2} g(h(e_i, e_j), e_r)e_r + \sum_{r=n+n_2+1}^{2m} g(h(e_i, e_j), e_r)e_r.$$

The first term in the right hand side of the above equation is $F\mathcal{D}^\theta$-component and the second term is ν-component. Taking summation over the vector fields on M_T and M_θ and using adapted frame fields, we get

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j), h(e_i, e_j)) =$$

$$= \csc^2\theta \sum_{i=1}^{d_1}\sum_{j,k=1}^{d_2} g(h(e_i,e_j^*),Fe_k^*)^2 + \csc^2\theta\sec^2\theta \sum_{i=1}^{d_1}\sum_{j,k=1}^{d_2} g(h(e_i,Te_j^*),Fe_k^*)^2$$

$$+ \csc^2\theta\sec^2\theta \sum_{i=1}^{d_1}\sum_{j,k=1}^{d_2} g(h(\varphi e_i,e_j^*),FTe_k^*)^2 + \csc^2\theta\sec^2\theta \sum_{i=1}^{d_1}\sum_{j,k=1}^{d_2} g(h(Je_i,e_j^*),FTe_k^*)^2$$

$$+ \csc^2\theta\sec^4\theta \sum_{i=1}^{d_1}\sum_{j,k=1}^{d_2} g(h(Je_i,Te_j^*),FTe_k^*)^2 + \csc^2\theta\sec^2\theta \sum_{i=1}^{d_1}\sum_{j,k=1}^{d_2} g(h(Je_i,Te_j^*),Fe_k^*)^2$$

$$+ \csc^2\theta \sum_{i=1}^{d_1}\sum_{j,k=1}^{d_2} g(h(Je_i,e_j^*),Fe_k^*)^2 + \csc^2\theta\sec^4\theta \sum_{i=1}^{d_1}\sum_{j,k=1}^{d_2} g(h(e_i,Te_j^*),FTe_r^*)^2 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{r=n+n_2+1}^{2m} g(h(e_i,e_j),e_r)^2.$$

Then using Lemma 4.1, we derive

$$\|h(\mathcal{D},\mathcal{D}^\theta)\|^2 = n_2(\csc^2\theta + \cot^2\theta)\|\nabla \ln f\|^2 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\nu(e_i,e_j)\|^2. \tag{6.5}$$

Then from (6.4) and (6.5), it follow that

$$\int_M \left[\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i,e_j)\|^2 + 2n_2\cot^2\theta\|\nabla \ln f\|^2\right] dV = \int_M \Big(\frac{n_1.n_2 c}{4}\Big) dV. \tag{6.6}$$

If (6.3) is satisfied, then (6.6) implies that f is constant function on proper pointwise semi-slant submanifold M. Thus M is a Riemannian product of invariant and pointwise slant submanifolds M_T and M_θ respectively. *Conversely*, suppose that M is simply a Riemannian product then warping function f must be constant, i.e., $\nabla \ln f = 0$. Thus from (6.6) implies the equality (6.3). Its complete proof of the theorem. □

We immediately obtain the following corollaries by using $\theta = \frac{\pi}{2}$, for totally real submanifold as:

Corollary 6.1. *Let $M = M_T \times_f M_\perp$ be a compact CR-warped product submanifold of complex space form $\widetilde{M}(c)$. Then M is a Riemannian product if*

$$\|h\|^2 \geq \frac{n_1.n_2.c}{2}$$

where n_1 and n_2 are dimensions of M_T and $M_\perp$ respectively.

Corollary 6.2. *Let $M = M_T \times_f M_\perp$ be a compact CR-warped product submanifold in a complex space form $\widetilde{M}(c)$ such that M_T is invariant submanifold of dimension n_1 and $M_\perp$ is totally real submanifold of dimension n_2 into $\widetilde{M}(c)$. Then M is simply a Riemannian product if and only if*

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\nu(e_i,e_j)\|^2 = \frac{n_1.n_2.c}{4}.$$

where h_ν is a components of h in $\Gamma(\nu)$.

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References

[1] A. Ali and C. Ozel, *Geometry of warped product pointwise semi-slant submanifolds of cosymplectic manifolds and its applications*, Int. J. Geom. Methods Mod. Phys. 14, 1750042 (2017).
[2] N. S. Al-Luhaibi, F. R. Al-Solamy, and V. A. Khan, *CR-warped product submanifolds of nearly Kaehler manifolds*, J. Korean Math. Soc. 46 (2009), no-5, pp. 979-995.
[3] N. Aktan, M. Z. Sarikaya and E. Ozusaglam, *B.Y. Chen's inequality for semi-slant submanifolds in $T-$space forms*, Balkan J. Geom. Its Appl., Vol.13, No.1, 2008, pp. 1-10.
[4] M. Atceken, *Contact CR-warped product submanifolds in cosymplectic manifolds*, Collect. Math. 62 (2011) , 17-26.
[5] A. Bejancu, *Geometry of CR-Submanifolds*, Kluwer Academic Publishers, Dortrecht, 1986.
[6] R. L. Bishop and B. O'Neil, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. 145 (1969), 1-9.
[7] B.Y. Chen, *Slant immersions*, Bull. Austral. Math. Soc. 41 (1990), 135-147.
[8] B. Y. Chen, *Geometry of warped product CR-submanifold in Kaehler manifolds*, Monatsh. Math. 133 (2001), no. 03, 177-195.
[9] B. Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds* II, Monatsh. Mat. 134 (2001), 103-119.
[10] B.Y. Chen, *On isometric minimal immersions from warped products into real space forms*, Proc. Edinb. Math. Soc., 45 (2002), 579-587.
[11] B. Y. Chen, *Another general inequality for warped product CR-warped product submanifold in complex space forms*, Hokkaido Math. J., 32 (2003), 415-444.
[12] B.Y. Chen, O. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds*, Turk. J. Math., 36 (2012), 630-640.
[13] B. Y. Chen, *Geometry of warped product submanifolds*: A survey, J. Adv. Math. Stud., 6 (2), (2013), 1-43.
[14] D. Cioroboiu, *B. Y. Chen inequalities for semi-slant submanifolds in Sasakian space forms*, International. J. Math and Maths. Sci., 27(2003), 1731–1738.
[15] F. Etayo, *On quasi-slant submanifolds of an almost Hermitian manifold*, Pub. Math. Debrecen. 53 (1998), 217-223.
[16] S. Hiepko, *Eine inner kennzeichungder verzerrten produckt*, Math. Ann. 244 (1979), 209-215.
[17] V. A. Khan, K. A. Khan and S. Uddin, *CR-warped product submanifolds in a Kaehler manifold*, South. Asian. Bull. Math. 33 (2009), 865-874.
[18] K. A. Khan, V. A. Khan and S. Uddin, *Warped product submanifolds of cosymplectic manifolds*, Balkan J. Geom. Its Appl., 13 (2008), 55-65.
[19] V. A. Khan and K. Khan, *Semi-slant warped product submanifolds of a nearly Kaehler manifold*, Differential Geometry - Dynamical Systems, Vol.16, (2014), pp. 168-182.
[20] V. A. Khan and M. Shuaib, *Some warped product submanifolds of a Kenmutso manifold*, Bull Korean. Math. Soc.51 (2014), No-3, 863-881
[21] A. Mustafa, S. Uddin and B. R. Wong, *Generalized inequalities on warped product submanifolds in nearly trans-Sasakian manifolds*, Journal of Inequalities and Applications 2014, 2014:346.
[22] A. Mustafa, A. De and S. Uddin, *Characterization of warped product submanifolds in Kenmotsu manifolds*, Balkan J. Geom. Its Appl., Vol.20, No.1, (2015), pp. 86-97.
[23] J. Mikeš, *Equidistant Kähler spaces*, Maths. Notes 38, (1985), 855-858. Zbl0594.53024.
[24] J. Mikeš and G. A. Starko, *Hyperbolically Sasakian and equidistant hyperbolically Kählerian spaces*, J. Sov. Math 59, No. 2, (1992), 756-760. Zbl0741.53034.
[25] S. M. Minčić, M. S. Stanković and Lj. S. Velmirović, *Generalized Kählerian spaces*, Filomat 15 (2001), 167-174.
[26] K. S. Park, *Pointwise slant and semi-slant submanifolds of almost contact manifolds*, Int. J. Math., 26, 1550099 (2015).
[27] N. Papaghiuc, *Semi-slant submanifold of Kaehler manifold*, An. St. Univ. Al. I. Cuza. Iasi, 40 (1994), 55-61.
[28] B. Sahin, *Non-existence of warped product submanifolds of Kaehler manifolds*, Geometriae Dedicata, 117 (2006), 195-202.
[29] B. Sahin, *Warped product pointwise semi-slant submanifold of Kaehler manifold*, Portugaliae Mathematica. Volume 70, Issue 3, (2013), pp. 251-268.
[30] M. Sikha and J. Mikeš, *On equidistant parabolically Kählerian spaces*, Tr. Geom. Semi. 22, (1994) 97-107. Zbl0863.53047.
[31] M. S. Stanković, S. M. Minčić and Lj. S. Velmirovič, *On equitorsion holomorphically projective mappings of generalised Kahlerian spaces*, Czech. Math. J. 54 (129), No. 3, (204) 701-715.
[32] S. S. Shukla and P. K. Rao, *B. Y. Chen inequalities for bi-slant submanifolds in generalized complex space forms*, J. Nonlinear Sci. Appl., 3 (2010), no. 4, 282-293.
[33] S. Uddin, Falleh R. Al-Solamy and V.A. Khan *Geometry of warped product semi-slant submanifolds of nearly Kaehler manifolds*, Result. Math. To appear.
[34] S. Uddin and L.S. Alqahtani *B.-Y. Chen type inequality for warped product immersions in cosymplectic space forms*, (submitted).
[35] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, 1984.
[36] M. Zlatanović, I. Hinterleiter and M. Najdanović, *Geodesic mapping onto Kählerian spaces of the first kind*, Czech. Math. J. 64, No. 4, (2014), 1113-1122. Zblo6433717.