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Connectedness of Ideal Topological Spaces

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Abstract. In this paper we introduce and study of new types of connectedness in an ideal topological space. We also interrelate these connectedness with connectedness which are already in literature.

1. Introduction

The concept of ideal topological spaces was introduced by Kuratowski [6] and Vaidyanathswamy [10]. An ideal I as we know is a nonempty collection of subsets of X closed with respect to finite union and heredity. (X, τ, I) is an ideal topological space and we call it an ideal space in this paper. For a subset A of X, the local function of A is defined as follows [3, 4]:

$A^* = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x)$ is the collection of all nonempty open sets containing x. In this respect the study of $*$-topology is interesting which had been studied by Jankovic and Hamlett [4, 5], Modak and Bandyopadhyay [7, 8] and many other in detail and its one of the powerful base is $\beta(I, \tau) = \{V - A : V \in \tau, A \in I\}$ [4]. It is also denoted as $\tau^*(I)$ [4,5] and its closure operator is defined as $Cl^*(A) = A \cup A^*$. Again it is happened that $\tau \subset \tau^*(I)$. The theory of ideals gets a new dimension in the case it satisfies $I \cap \tau = \{\emptyset\}$. Such ideals are termed as codense ideals by Dontchev, Ganster and Rose [1]. Compatibility is also an another part of this study. An ideal I is compatible [5] with the topology τ written as $I \sim \tau$ [5] if for any $A \subset X$ there is an open cover Ω_x of A such that for $x \in A$, there is a $U_x \in \Omega_x$ with $U_x \cap A \in I$, then $A \in I$.

The study of connectedness in an ideal topological space was introduced by Ekici and Noiri in [2]. The authors Sathiyasundari and Renukadevi [9] studied it further in detail. We in this paper introduce and study some different types of connectedness with the help of the ideal topological spaces and define types of component. We also characterize these connectedness and interrelate with earlier connectedness.

2. Preliminaries

Definition 2.1. *Nonempty subsets A, B of an ideal space (X, τ, I) are called $*_*$-separated (resp. $*$-separated [2], separated) if $A^* \cap B = A \cap B^* = A \cap B = \emptyset$ (resp. $Cl^*(A) \cap B = A \cap Cl(B) = \emptyset$, $Cl(A) \cap B = A \cap Cl(B) = \emptyset$).*

Definition 2.2. *Nonempty subsets A, B of an ideal space (X, τ, I) are called $*$-Cl-separated (resp. $*$-Cl^*-separated) if $A^* \cap Cl(B) = Cl(A) \cap B^* = A \cap B = \emptyset$(resp. $A^* \cap Cl^*(B) = Cl^*(A) \cap B^* = A \cap B = \emptyset$).*

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Theorem 2.3. *Let (X, τ, I) be a topological space and $A,\ B \subset X$. Then A and B are $*_*$ - separated if and only if A and B are separated in $(X, \tau^*(I))$.*

Proof. Let A and B be $*_*$ - separated, then $A^* \cap B = A \cap B^* = A \cap B = \emptyset$.

Now

$$Cl^*(A) \cap B = (A \cup A^*) \cap B = (A \cap B) \cup (A^* \cap B) = \emptyset;$$
$$A \cap Cl^*(B) = A \cap (B \cup B^*) = (A \cap B) \cup (A \cap B^*) = \emptyset.$$

In consequence $A,\ B$ are separated in $(X, \tau^*(I))$. Reciprocally, if $A,\ B$ are separated in $(X, \tau^*(I))$ we have

$$\emptyset = Cl^*(A) \cap B = (A \cup A^*) \cap B = (A \cap B) \cup (A^* \cap B);$$
$$\emptyset = A \cap Cl^*(B) = A \cap (B \cup B^*) = (A \cap B) \cup (A \cap B^*).$$

Then $A^* \cap B = A \cap B^* = A \cap B = \emptyset$ and $A,\ B$ are $*_*$ - separated. □

Theorem 2.4. *For nonempty subsets of an ideal space (X, τ, I), the following hold:*
(1) Every $*$-Cl^*-separated set is $*_*$-separated.
(2) Every $*$-Cl-separated set is $*$-Cl^*-separated.

Proof. (1). Let A and B be two $*$-Cl^*-separated sets in (X, τ, I). Then it is obvious that $A^* \cap B = A \cap B^* = \emptyset$ from definition of $*$-Cl^*-separated.
(2). It is obvious from $Cl^*(A) \subseteq Cl(A)$ for any subset A of X. □

Hence from Theorem 2.4 we obtain the following diagram:

$$*\text{-}Cl\text{-separated} \Longrightarrow *\text{-}Cl^*\text{-separated} \Longrightarrow *_*\text{-separated} \Longleftrightarrow \text{separated in } (X, \tau^*(I))$$

Definition 2.5. *A subset A of an ideal space (X, τ, I) is called*
(i). $*_*$-connected (resp. $*_s$-connected [2]) if A is not the union of two $*_*$-separated(resp. $*$-separated) sets in (X, τ, I).
(ii). $*$-connected [2] if A cannot be written as the union of a nonempty open set and a nonempty $*$-open set.

Theorem 2.6. *Let (X, τ, I) be an ideal topological space. Then the space is $*_*$ - connected if and only if $(X, \tau^*(I))$ is connected.*

Definition 2.7. *A subset A of an ideal space (X, τ, I) is called $*$-Cl-connected (resp. $*$-Cl^*-connected) if A is not the union of two $*$-Cl–separated (resp. $*$-Cl^*-separated) sets in (X, τ, I).*

Theorem 2.8. *For any subset of an ideal space (X, τ, I), the following properties hold:*
(1). Every $*_*$-connected set is $*$-Cl^*-connected.
(2). Every $*$-Cl^*-connected set is $*$-Cl-connected.

Proof. This is obvious from the above diagram. □

By Theorem 2.8, we obtain the following diagram:

$$\text{connected in } (X, \tau^*(I)) \Longleftrightarrow *_*\text{-connected} \Longrightarrow *\text{-}Cl^*\text{-connected} \Longrightarrow *\text{-}Cl\text{-connected}.$$

For converse of Theorem 2.4, and Theorem 2.8, we shall give following examples.

Example 2.9. *(i). Let $X = \{a, b\},\ \tau = \{\emptyset, \{b\}, X\}$ and $I = \{\emptyset, \{b\}\}$. Then $(\{a\})^* = \{a\}$ and $(\{b\})^* = \emptyset$. Here $\{a\}$ and $\{b\}$ are $*_*$-separated sets but not $*$-Cl^*-separated because $Cl^*(\{b\}) = X$.*
(ii). Let $X = \{a, b, c\},\ \tau = \{\emptyset, \{a, b\}, X\}$ and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $\tau^*(I) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Here the space X is a $*$-Cl-connected space because, $Cl(\{a\}) = Cl(\{b\}) = Cl(\{a, b\}) = Cl(\{b, c\}) = Cl(\{a, c\}) = X$ and $(\{c\})^* = \{c\}$. But the space is not a $*$-Cl^*-connected space, since $X = \{a, b\} \cup \{c\},\ (\{c\})^* \cap Cl^*(\{a, b\}) = Cl^*(\{c\}) \cap (\{a, b\})^* = \{a\} \cap (\{b, c\} = \emptyset((\{a, b\})^* = \emptyset)$.

3. $*$-Cl-connected spaces

Theorem 3.1. *Let (X, τ, I) be an ideal space. If A is a $*$-Cl-connected subset of X and H, G are $*$-Cl-separated sets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Proof. Let $A \subset H \cup G$. Since $A = (A \cap H) \cup (A \cap G)$, and $(A \cap G)^* \cap Cl(A \cap H) \subset G^* \cap Cl(H)$ (by [2]) $= \emptyset$. By similar way, we have $(A \cap H)^* \cap Cl(A \cap G) = \emptyset$. Moreover $(A \cap H) \cap (A \cap G) \subset H \cap G = \emptyset$. Suppose that $A \cap H$ and $A \cap G$ are nonempty. Then A is not a $*$-Cl-connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$. This implies that $A \subset H$ or $A \subset G$. □

Theorem 3.2. *Let (X, τ, I) be an ideal space. If A is a $*$-Cl^*-connected subset of X and H, G are $*$-Cl^*-separated sets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Proof. The proof is similar with Theorem 3.1. □

Theorem 3.3. *If A is a $*$-Cl-connected subset of (X, τ, I) and $A \subset B \subset A^*$, then B is also a $*$-Cl-connected subset of X.*

Proof. Suppose B is not a $*$-Cl-connected subset of (X, τ, I) then there exist $*$-Cl-separated sets H and G such that $B = H \cup G$. This implies that H and G are nonempty and $G^* \cap Cl(H) = \emptyset = Cl(G) \cap H^* = G \cap H$. By Theorem 3.2, we have that either $A \subset H$ or $A \subset G$. Suppose that $A \subset H$. Then $A^* \subset H^*$. This implies that $G \subset B \subset A^*$ and $Cl(G) = A^* \cap Cl(G) \subset H^* \cap Cl(G) = \emptyset$. Thus G is an empty set. Since G is nonempty, this is a contradiction. Hence, B is $*$-Cl-connected. □

Theorem 3.4. *If A is a $*$-Cl^*-connected subset of (X, τ, I) and $A \subset B \subset A^*$, then B is also a $*$-Cl^*-connected subset of X.*

Proof. Suppose B is not a $*$-Cl^*-connected subset of (X, τ, I) then there exist $*$-Cl^*-separated sets H and G such that $B = H \cup G$. This implies that H and G are nonempty and $G^* \cap Cl^*(H) = \phi = Cl^*(G) \cap H^* = G \cap H$. By Theorem 3.2, we have that either $A \subset H$ or $A \subset G$. Suppose that $A \subset H$. Then $A^* \subset H^*$. This implies that $G \subset B \subset A^*$ and $Cl^*(G) = Cl^*(A^*) \cap Cl^*(G) \subset Cl(A^*) \cap Cl^*(G) \subset A^* \cap Cl^*(G) \subset H^* \cap Cl^*(G) = \emptyset$. Thus G is an empty set. Since G is nonempty, this is a contradiction. Hence, B is $*$-Cl^*-connected. □

Corollary 3.5. *(a) If A is a $*$-Cl-connected set in an ideal space (X, τ, I), then A^* is $*$-Cl-connected.*
(b) If A is a $*$-Cl^*-connected set in an ideal space (X, τ, I), then A^* is $*$-Cl^*-connected.

Corollary 3.6. *(a) If $I \cap \tau = \{\emptyset\}$ in (X, τ, I), then for any nonempty open, $*$-Cl-connected set V, $Cl(V)$ is also a $*$-Cl-connected set.*
(b) If $I \cap \tau = \{\emptyset\}$ and $I \sim \tau$ in (X, τ, I), then for any nonempty open, $*$-Cl-connected set G, $Cl(G)$ and $Cl^*(G)$ are also $*$-Cl-connected.

Proof. (a) It is obvious from Note 3.2 of [7].
(b) It is obvious from Lemma 2.2 of [8] and the fact that every open set is $*$-open. □

Corollary 3.7. *(a) If $I \cap \tau = \{\emptyset\}$ in (X, τ, I), then for any nonempty open, $*$-Cl^*-connected set V, $Cl(V)$ is also a $*$-Cl^*-connected set.*
(b) If $I \cap \tau = \{\emptyset\}$ and $I \sim \tau$ in (X, τ, I), then for any nonempty open, $*$-Cl^*-connected set G, $Cl(G)$ and $Cl^*(G)$ are also $*$-Cl^*-connected.

Proof. (a) It is obvious from Note 3.2 of [7].
(b) It is obvious from Lemma 2.2 of [8] and the fact that every open set is $*$-open. □

Theorem 3.8. *If $\{M_i : i \in I\}$ is a nonempty family of $*$-Cl-connected sets of an ideal space (X, τ, I) with $\cap_{i \in I} M_i \neq \emptyset$, then $\cup_{i \in I} M_i$ is $*$-Cl-connected.*

Proof. Suppose $\cup_{i\in I}M_i$ is not $*$-Cl-connected. Then we have $\cup_{i\in I}M_i = H\cup G$, where H and G are $*$-Cl-separated sets in X. Since $\cap_{i\in I}M_i \neq \emptyset$, we have a point $x \in \cap_{i\in I}M_i$. Since $x \in \cup_{i\in I}M_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_i$ for each $i \in I$, then M_i and H intersect for each $i \in I$. By Theorem 3.1, $M_i \subset H$ or $M_i \subset G$. Since H and G are disjoint, $M_i \subset H$ for all $i \in I$ and hence $\cup_{i\in I}M_i \subset H$. This implies that G is empty. This is a contradiction. Suppose that $x \in G$. By the similar way, we have that H is empty. This is a contradiction. Thus, $\cup_{i\in I}M_i$ is $*$-Cl-connected. □

Theorem 3.9. *If $\{M_i : i \in I\}$ is a nonempty family of $*$-Cl^*-connected sets of an ideal space (X, τ, I) with $\cap_{i\in I}M_i \neq \emptyset$, then $\cup_{i\in I}M_i$ is $*$-Cl^*-connected.*

Proof. The proof is similar with Theorem 3.8. □

Corollary 3.10. *(a) If A is a $*$-Cl-connected subset of the ideal space (X, τ, I) and $A \cap A^* \neq \emptyset$, then $Cl^*(A)$ is a $*$-Cl-connected set.*

(b) If A is a $*$-Cl^*-connected subset of the ideal space (X, τ, I) and $A \cap A^* \neq \emptyset$, then $Cl^*(A)$ is a $*$-Cl^*-connected set.

Theorem 3.11. *Let (X, τ, I) be an ideal space, $\{A_\alpha : \alpha \in \triangle\}$ be a family of $*$-Cl-connected subsets of X and A be a $*$-Cl-connected subset of X. If $A \cap A_\alpha \neq \emptyset$ for every α, then $A \cup (\cup A_\alpha)$ is $*$-Cl-connected.*

Proof. Since $A \cap A_\alpha \neq \emptyset$ for each $\alpha \in \triangle$, by Theorem 3.8, $A \cup A_\alpha$ is $*$-Cl-connected for each $\alpha \in \triangle$. Moreover, $A \cup (\cup A_\alpha) = \cup(A \cup A_\alpha)$ and $\cap(A \cup A_\alpha) \supset A \neq \emptyset$. Thus by Theorem 3.8, $A \cup (\cup A_\alpha)$ is $*$-Cl-connected. □

Theorem 3.12. *Let (X, τ, I) be an ideal space, $\{A_\alpha : \alpha \in \triangle\}$ be a family of $*$-Cl^*-connected subsets of X and A be a $*$-Cl^*-connected subset of X. If $A \cap A_\alpha \neq \emptyset$ for every α, then $A \cup (\cup A_\alpha)$ is $*$-Cl^*-connected.*

Proof. The proof is similar with Theorem 3.11. □

Recall that a subset A of (X, τ, I) is called $*$-dense-in-itself [3] if $A \subset A^*$.

Theorem 3.13. *Let (X, τ, I) be an ideal space. If A and B are $*$-Cl-separated and $*$-dense-in-itself subsets of X, and $A \cup B \in \tau$, then A and B are open and hence $*$-open.*

Proof. Since A and B are $*$-Cl-separated in X, then $A = (A \cup B) \cap (X - Cl(B))$. Since $A \cup B \in \tau$ and $Cl(B)$ is closed in X, then A is open in X. By the similar way, we obtain that B is open. □

Theorem 3.14. *Let (X, τ, I) be an ideal space. If A and B are $*$-Cl^*-separated and $*$-dense-in-itself subsets of X, and $A \cup B \in \tau$, then A and B are open and hence $*$-open.*

Proof. The proof is similar with Theorem 3.13. □

Definition 3.15. *Let X be an ideal space and $x \in X$. The union of all $*$-Cl-connected (resp. $*$-Cl^*-connected) subsets of X containing x is called the $*$-Cl-component (resp. $*$-Cl^*-component) of X containing x.*

Theorem 3.16. *Each $*$-Cl-component of an ideal space (X, τ, I) is a maximal $*$-Cl-connected set of X.*

Theorem 3.17. *Each $*$-Cl^*-component of an ideal space (X, τ, I) is a maximal $*$-Cl^*-connected set of X.*

Theorem 3.18. *The set of all distinct $*$-Cl-components of an ideal space (X, τ, I) forms a partition of X.*

Proof. Let A and B be two distinct $*$-Cl-components of X. Suppose A and B intersect. Then, by Theorem 3.8, $A \cup B$ is $*$-Cl-connected in X. Since $A \subset A \cup B$, then A is not maximal. Thus A and B are disjoint. □

Theorem 3.19. *The set of all distinct $*$-Cl^*-components of an ideal space (X, τ, I) forms a partition of X.*

Proof. The proof is similar with Theorem 3.18. □

Theorem 3.20. *Let (X, τ, I) be an ideal space where I is codense. Then, each $*$-Cl-connected subset of X which is both open and $*$-closed is $*$-Cl-component of X.*

Proof. Let A be a $*$-Cl-connected subset of X such that A is both open and $*$-closed. Let $x \in A$. Since A is a $*$-Cl-connected subset of X containing x, if C is the $*$-Cl-component containing x, then $A \subset C$. Let A be a proper subset of C. Then C is nonempty and $C \cap (X - A) \neq \emptyset$. Since A is open and $*$-closed, $X - A$ is closed and $*$-open and $(A \cap C) \cap ((X - A) \cap C) = \emptyset$. Also $(A \cap C) \cup ((X - A) \cap C) = (A \cup (X - A)) \cap C = C$. Again A and $X - A$ are two nonempty disjoint open set and $*$-open set respectively, such that $A \cap Cl(X - A) = \emptyset = Cl^*(A) \cap (X - A)$. This implies that $A^* \cap Cl(X - A) = \emptyset = Cl(A) \cap (X - A)^*$, since I is codense and $(X - A)^* \subset Cl(X - A)$. This shows that $(A \cap C)$ and $((X - A) \cap C)$ are $*$-Cl-separated sets. This is a contradiction. Hence, A is not a proper subset of C and $A = C$. This completes the proof. □

Theorem 3.21. *Let (X, τ, I) be an ideal space. Then, each $*$-Cl^*-connected subset of X which is both open and $*$-closed is $*$-Cl^*-component of X.*

Proof. Let A be a $*$-Cl^*-connected subset of X such that A is both open and $*$-closed. Let $x \in A$. Since A is a $*$-Cl^*-connected subset of X containing x, if C is the $*$-Cl^*-component containing x, then $A \subset C$. Let A be a proper subset of C. Then C is nonempty and $C \cap (X - A) \neq \emptyset$. Since A is open and $*$-closed, $X - A$ is closed and $*$-open and $(A \cap C) \cap ((X - A) \cap C) = \emptyset$. Also $(A \cap C) \cup ((X - A) \cap C) = (A \cup (X - A)) \cap C = C$. Again A and $X - A$ are two nonempty disjoint open set and $*$-open set respectively, such that $A \cap Cl(X - A) = \emptyset = Cl^*(A) \cap (X - A)$. This implies that $A^* \cap Cl^*(X - A) \subset Cl^*(A) \cap Cl(X - A) = Cl^*(A) \cap (X - A) = \emptyset$ and $Cl^*(A) \cap (X - A)^* \subset A \cap Cl(X - A) = \emptyset$. This shows that $(A \cap C)$ and $((X - A) \cap C)$ are $*$-Cl^*-separated sets. This is a contradiction. Hence, A is not a proper subset of C and $A = C$. This completes the proof. □

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