Filomat 31:12 (2017), 3749–3760
https://doi.org/10.2298/FIL1712749K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On $\alpha\beta$-Statistical Convergence for Sequences of Fuzzy Mappings and Korovkin Type Approximation Theorem

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Abstract. Upon prior investigation on statistical convergence of fuzzy sequences, we study the notion of pointwise $\alpha\beta$-statistical convergence of fuzzy mappings of order γ. Also, we establish the concept of strongly $\alpha\beta$-summable sequences of fuzzy mappings and investigate some inclusion relations. Further, we get an analogue of Korovkin-type approximation theorem for fuzzy positive linear operators with respect to $\alpha\beta$-statistical convergence. Lastly, we apply fuzzy Bernstein operator to construct an example in support of our result.

1. Introduction

The concept of statistical convergence for sequence of real numbers was defined by Fast [8] and Steinhaus [24] independently in 1951. Statistical convergence has recently became an area of active research. Over the years, researchers in statistical convergence have devoted their effort to approximation theory, Fourier analysis, number theory, measure theory, trigonometric series and Banach spaces [4, 15–18]. It is well-known that every convergent sequence is statistically convergent but converse is not always true. Also, statistically convergent sequence do not need to be bounded. So, this type convergence is quite effective in approximation theory. First we recall the following definitions:

Let K be a subset of $\mathbb{N}$, the set of natural numbers and $K_n = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n \frac{1}{n}|K_n|$ provided it exists, where $|K_n|$ denotes the cardinality of set K_n. A sequence $x = (x_k)$ is called statistically convergent ($st-$convergent) to the number ℓ, denoted by $st - \lim x = \ell$, for each $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - \ell| \geq \epsilon\}$ has natural density zero, that is

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \epsilon\}| = 0.$$

The idea of $\alpha\beta-$statistical convergence was introduced by Aktuğlu in [1] as follows:

Let $\alpha(n)$ and $\beta(n)$ be two sequences of positive number which satisfy the following conditions:

(i) α and β are both non-decreasing,

2010 *Mathematics Subject Classification*. Primary 40C05, 40G15; Secondary 03E72, 08A72
Keywords. Statistical convergence of fuzzy sequences, $\alpha\beta$-summable sequences, fuzzy Korovkin-type approximation, fuzzy Bernstein operator
Received: 12 December 2015; Accepted: 26 December 2016
Communicated by Dragan S. Djordjević
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(ii) $\beta(n) \geq \alpha(n)$,

(iii) $\beta(n) - \alpha(n) \longrightarrow \infty$ as $n \longrightarrow \infty$.

Here, the set of pairs (α, β) satisfying (i)-(iii) will be denoted by Λ. For each pair $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$ and $K \subset \mathbb{N}$, we define the density $\delta^{\alpha,\beta}(K, \gamma)$ in the following way

$$\delta^{\alpha,\beta}(K,\gamma) = \lim_{n\to\infty} \frac{\left|K \cap P_n^{\alpha,\beta}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}}, \quad (P_n^{\alpha,\beta} = [\alpha(n), \beta(n)]).$$

A sequence $x = (x_k)$ is said to be $\alpha\beta-$statistically convergent of order γ to the number ℓ, if, for each $\epsilon > 0$,

$$\delta^{\alpha,\beta}(\{k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon\}, \gamma) = \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0 \quad (0 < \gamma \leq 1)$$

and denote $st_{\alpha\beta}^{\gamma} - \lim x = \ell$.

Recently, Karakaya and Karaisa [13] have introduced weighted $\alpha\beta-$statistical convergence of order γ, and proved Korovkin type approximation theorems through the weighted $\alpha\beta-$statistical convergence. Very recently, Kadak [9] have introduced the concept of weighted statistical convergence involving $\alpha\beta-$statistical convergence based on (p, q)-integers.

In this study, we introduce the notions of pointwise $\alpha\beta$-statistical convergence of order γ and strongly pointwise $\omega_{\gamma,p}^{\alpha\beta}(F)$-summability ($p \in (0, \infty)$) of fuzzy mappings. Furthermore, we establish some inclusion relations and some related results using newly proposed methods. Based on $\alpha\beta-$statistical convergence, some applications are investigated to obtain fuzzy Korovkin type approximation results.

2. Preliminaries, Background and Notation

In this section, we give some basic notations and fundemental results related to the fuzzy numbers which will be used in this article.

A *fuzzy number* is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e., $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, (Zadeh [26]), where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and called it as *the space of fuzzy numbers*. *λ-level set* $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda := \begin{cases} \{t \in \mathbb{R} : u(t) \geq \lambda\} & , \quad 0 < \lambda \leq 1, \\ \overline{\{t \in \mathbb{R} : u(t) > \lambda\}} & , \quad \lambda = 0. \end{cases}$$

The set $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0, 1]$ which is defined by $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1 & , \quad x = r, \\ 0 & , \quad x \neq r. \end{cases}$$

Theorem 2.1. *[25] Let $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for $u \in E^1$ and for each $\lambda \in [0,1]$. Then the following statements hold:*

(i) The function $u^- : [0,1] \to \mathbb{R}$ is a bounded and non-decreasing left continuous function on $]0,1]$.

(ii) The function $u^+[0,1] \to \mathbb{R}$ is a bounded and non-increasing left continuous function on $]0,1]$.

(iii) The functions u^- and u^+ are right continuous at the point $\lambda = 0$.

(iv) $u^-(1) \le u^+(1)$.

Conversely, if the pair of functions u^- and u^+ satisfies the conditions (i)-(iv), then there exists a unique $u \in E^1$ such that $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$ for each $\lambda \in [0,1]$. The fuzzy number u corresponding to the pair of functions u^- and u^+ is defined by $u : \mathbb{R} \to [0,1]$, $u(x) := \sup\{\lambda : u^-(\lambda) \le x \le u^+(\lambda)\}$.

Let $u, v, w \in E^1$ and $\alpha \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on E^1 by

$$\begin{aligned} u \oplus v = w \quad &\Leftrightarrow \quad [w]_\lambda = [u]_\lambda + [v]_\lambda \text{ for all } \lambda \in [0,1] \\ &\Leftrightarrow \quad w^-(\lambda) = u^-(\lambda) + v^-(\lambda) \text{ and } w^+(\lambda) = u^+(\lambda) + v^+(\lambda) \text{ for all } \lambda \in [0,1], \\ & \quad\quad [\alpha \odot u]_\lambda = \alpha[u]_\lambda \text{ for all } \lambda \in [0,1], \\ & \quad\quad u \odot v = w \Leftrightarrow [w]_\lambda = [u]_\lambda \odot [v]_\lambda \text{ for all } \lambda \in [0,1], \end{aligned}$$

where it is immediate that

$$\begin{aligned} w^-(\lambda) &= \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}, \\ w^+(\lambda) &= \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\} \end{aligned}$$

for all $\lambda \in [0,1]$. Also, for $u, v \in E^1$, define $u \le v \Leftrightarrow u^-(\lambda) \le v^-(\lambda)$ and $u^+(\lambda) \le v^+(\lambda)$ for all $\lambda \in [0,1]$.

Now, we can define the metric D on E^1 by means of the Hausdorff metric d as

$$D(u,v) := \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) := \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$

It is noticed that (E^1, D) is complete metric space (see [22]). Let $f, g : [a,b] \to E^1$ be fuzzy number valued functions. Then the distance between f and g is given by

$$D^*(f,g) := \sup_{\lambda \in [0,1]} \sup_{x \in [a,b]} \max\{|f^-(\lambda) - g^-(\lambda)|, |f^+(\lambda) - g^+(\lambda)|\}.$$

3. Fuzzy Summability Results

In this section, we define pointwise $\alpha\beta$-statistical convergence and strongly pointwise $\omega_\gamma^{\alpha\beta}(F)$-summability with γ order for fuzzy mappings. Also, we examine some inclusion relations and some summbability results for our new methods.

Definition 3.1. *Let $(\alpha, \beta) \in \Lambda$ and $\gamma \in (0,1]$. A sequence (u_k) of fuzzy numbers is said to be pointwise $\alpha\beta$-statistically convergent of order γ to a fuzzy number $\ell \in E^1$, on a set H, and denoted by $st_\gamma^{\alpha\beta}(F)\text{-}\lim u_k = u$, if for every $\varepsilon > 0$,*

$$\delta^{\alpha,\beta}(K_D, \gamma) = \lim_{n\to\infty} \frac{\left|\{k \in P_n^{\alpha,\beta} : D(u_k(x), \ell(x)) \ge \varepsilon \text{ for all } x \in H\}\right|}{(\beta(n) - \alpha(n) + 1)^\gamma} = 0$$

where $K_D = \{k : D(u_k(x), \ell(x)) \ge \varepsilon$ for all $x \in H\}$. For $\gamma = 1$, we write that u is pointwise $\alpha\beta$-statistically convergent to ℓ, and denoted by $st^{\alpha\beta}(F)\text{-}\lim u_k = \ell$.

That is, for every $\varepsilon > 0$, we may write

$$\delta_{\pm}^{\alpha,\beta}(K_D^{\pm},\gamma) = \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |u_k^{\pm}(\lambda) - \ell^{\pm}(\lambda)| \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0, \text{ uniformly in } \lambda,$$

where $K_D^{\pm} = \{k : |u_k^{\pm}(\lambda) - \ell^{\pm}(\lambda)| \geq \varepsilon\}$. The set of all $\alpha\beta$-statistically convergent sequences of fuzzy numbers of order γ will be denoted by $S_{\gamma}^{\alpha,\beta}(F)$.

In particular, by taking into account Definition 3.1 we give the following special cases:

(a) Taking $\alpha(n) = 1$, $\beta(n) = n$ and $\gamma \in (0,1]$ i.e., $P_n^{\alpha,\beta} = [1,n]$. In this case, pointwise $\alpha\beta$-statistically convergence is reduced to ordinary statistical convergence of fuzzy numbers with γ order . Similarly take $\gamma = 1$, then $\alpha\beta$-statistically convergence reduces to the ordinary statistical convergence of fuzzy numbers given in [20].

(b) Given non-decreasing sequence (λ_n) of positive real numbers such that $\lim_n \lambda_n = \infty$, $\lambda_1 = 1$, $\lambda_{n+1} \leq \lambda_n + 1$. Then, if we take $\alpha(n) = n - \lambda_n + 1$, $\beta(n) = n$ and $\gamma \in (0,1]$, pointwise $\alpha\beta$-statistically convergence reduces to the λ-statistical convergence of order γ of fuzzy numbers defined by [6]. Analogously, take $\gamma = 1$ this convergence reduces to the λ-statistical convergence of fuzzy numbers defined by [23] .

(c) Let $\theta = (k_r)$ be a lacunary sequence. If we take $\alpha(r) = k_{r-1} + 1$, $\beta(r) = k_r$ and $\gamma \in (0,1]$, pointwise $\alpha\beta$-statistically convergence can be expressed by lacunary statistical convergence of order γ of fuzzy numbers. Similarly, choose $\gamma = 1$, this convergence reduces to the lacunary statistical convergence of fuzzy numbers defined by [21].

Lemma 3.2. *(see [1]) Given two subsets K and M of natural numbers and $0 < \gamma \leq \eta \leq 1$. The following statements hold;*

(a) $\delta^{\alpha,\beta}(\emptyset,\gamma) = 0$ *and* $\delta^{\alpha,\beta}(\mathbb{N},1) = 1$,

(b) *if K_D is finite, then $\delta^{\alpha,\beta}(K_D,\gamma) = 0$,*

(c) *if $K_D \subset M_D$, then $\delta^{\alpha,\beta}(K_D,\gamma) \leq \delta^{\alpha,\beta}(M_D,\gamma)$ where $M_D = \{m : D(u_k(x),\ell(x)) \geq \varepsilon$ for all $x \in H\}$,*

(d) $\delta^{\alpha,\beta}(K_D,\eta) \leq \delta^{\alpha,\beta}(K_D,\gamma)$.

Lemma 3.3. *Suppose that the sequence $u = (u_k)$ of fuzzy numbers converges to $\ell \in E^1$ in the ordinary fuzzy sense, then $st_{\gamma}^{\alpha\beta}(F)$-$\lim u_k = \ell$.*

Proof. The proof is straightforward, hence is omitted. □

Definition 3.4. *Let $(\alpha,\beta) \in \Lambda$ and $\gamma \in (0,1]$. Then a sequence $u = (u_k)$ of fuzzy numbers is said to be pointwise $\alpha\beta$-statistically Cauchy of order γ, if there exists a natural number $N = N(\varepsilon)$ such that*

$$\lim_{n\to\infty} \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \left\{\left|\{k \in P_n^{\alpha,\beta} : D(u_k(x),u_N(x)) \geq \varepsilon \textit{ for all } x \in H\}\right|\right\} = 0 \tag{1}$$

for every $\varepsilon > 0$.

Definition 3.5. *A sequence $u = (u_k)$ of fuzzy numbers is said to be strongly pointwise $\alpha\beta$-summable of order γ if there is a fuzzy number ℓ such that*

$$\lim_{n\to\infty} \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \sum_{k\in P_n^{\alpha,\beta}, x\in H} (D(u_k(x),\ell(x))^p = 0 \textit{ for some } \ell \in E^1, \tag{2}$$

where p is positive real number and $\gamma \in (0,1]$. The number ℓ is unique when it exists. By $\omega_{\gamma,p}^{\alpha\beta}(F)$, we denote the set of all strongly pointwise $\alpha\beta$-summable of fuzzy sequences of order γ. In this case, we write $\omega_{\gamma,p}^{\alpha\beta}(F)$- $\lim u_k = \ell$.

Lemma 3.6. *Let $(\alpha,\beta)\in\Lambda$ and $\gamma\in(0,1]$. Given two sequences (u_k) and (w_k) of fuzzy numbers and $u_0,w_0\in E^1$.*

(a) *If $st_\gamma^{\alpha\beta}(F)$-$\lim_n u_k(x)=u_0(x)$ and $c\in\mathbb{R}^+$, then*

$$st_\gamma^{\alpha\beta}(F)-\lim_n cu_k(x)=cu_0(x).$$

(b) *If $st_\gamma^{\alpha\beta}(F)$-$\lim u_k(x)=u_0(x)$ and $st_\gamma^{\alpha\beta}(F)$-$\lim w_k(x)=w_0(x)$, then*

$$st_\gamma^{\alpha\beta}(F)-\lim(u_k(x)+w_k(x))=u_0(x)+w_0(x).$$

Proof. **(a)** It is obvious that this case holds for $c=0$. Assume that $c\neq 0$, then the following inequality holds:

$$\frac{\left|\{k\in P_n^{\alpha,\beta}:D(cu_k(x),cu_0(x))\geq\varepsilon\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}\leq\frac{\left|\{k\in P_n^{\alpha,\beta}:D(u_k(x),u_0(x))\geq\varepsilon/c\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}. \tag{3}$$

We derive by passing to the limit in (3) as $n\to\infty$ that

$$\left(\frac{\left|\{k\in P_n^{\alpha,\beta}:D(cu_k(x),cu_0(x))\geq\varepsilon\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}\right)\to 0 \text{ for all } x\in H.$$

Hence $st_\gamma^{\alpha\beta}(F)-\lim cu_k(x)=cu_0(x)$.

(b) Let us assume that $st_\gamma^{\alpha\beta}(F)$-$\lim u_k(x)=u_0(x)$ and $st_\gamma^{\alpha\beta}(F)$-$\lim w_k(x)=w_0(x)$. It is clear here that

$$D(u_k(x)+w_k(x),u_0(x)+w_0(x))\leq D(u_k(x),u_0(x))+D(w_k(x),w_0(x)).$$

Therefore, we obtain

$$\begin{aligned}&\frac{\left|\{k\in P_n^{\alpha,\beta}:D(u_k(x)+w_k(x),u_0(x)+w_0(x))\geq\varepsilon\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}\\ &\leq\frac{\left|\{k\in P_n^{\alpha,\beta}:D(u_k(x),u_0(x))\geq\varepsilon/2\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}+\frac{\left|\{k\in P_n^{\alpha,\beta}:D(w_k(x),w_0(x))\geq\varepsilon/2\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}.\end{aligned} \tag{4}$$

Similarly, by passing to the limit in (4) as $n\to\infty$ we get $st_\gamma^{\alpha\beta}(F)-\lim(u_k(x)+w_k(x))=u_0(x)+w_0(x)$. □

Theorem 3.7. *Let γ,η be real numbers such that $0<\gamma\leq\eta\leq 1$ and $0<p<\infty$. Then, we have $\omega_{\gamma,p}^{\alpha\beta}(F)\subseteq S_\eta^{\alpha,\beta}(F)$.*

Proof. Suppose that $u=(u_k)\in\omega_{\gamma,p}^{\alpha\beta}(F)$. For $\epsilon>0$, we get

$$\begin{aligned}\sum_{k\in P_n^{\alpha,\beta},\,x\in H}D(u_k(x),\ell(x))^p &= \sum_{\substack{k\in P_n^{\alpha,\beta},\,x\in H\\ k:D(u_k(x),\ell(x))\geq\epsilon}}D(u_k(x),\ell(x))^p+\sum_{\substack{k\in P_n^{\alpha,\beta},\,x\in H\\ k:D(u_k(x),\ell(x))<\epsilon}}D(u_k(x),\ell(x))^p\\ &\geq \sum_{\substack{k\in P_n^{\alpha,\beta},\,x\in H\\ k:D(u_k(x),\ell(x))\geq\epsilon}}D(u_k(x),\ell(x))^p\\ &\geq \left|\left\{k\in P_n^{\alpha,\beta}:D(u_k(x),\ell(x))\geq\epsilon\right\}\right|\epsilon^p.\end{aligned} \tag{5}$$

Using (5), we obtain

$$\sum_{k\in P_n^{\alpha,\beta},\, x\in H} \frac{D(u_k(x),\ell(x))^p}{(\beta(n)-\alpha(n)+1)^\gamma} \geq \frac{\left|\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}\epsilon^p$$
$$\geq \frac{\left|\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\beta(n)-\alpha(n)+1)^\eta}\epsilon^p.$$

This implies that $u=(u_k)\in S_\eta^{\alpha,\beta}(F)$. □

Theorem 3.8. *Let $(\alpha,\beta),(\psi,\xi)\in\Lambda$ such that $\psi(n)\leq\alpha(n)$ and $\beta(n)\leq\xi(n)$ for all $n\in\mathbb{N}$, and $0<\gamma\leq\eta\leq1$.*

(a) *If*

$$\liminf_{n\to\infty}\left(\frac{(\beta(n)-\alpha(n)+1)^\gamma}{(\xi(n)-\psi(n)+1)^\eta}\right)>0, \tag{6}$$

then $S_\eta^{\psi,\xi}(F)\subseteq S_\gamma^{\alpha,\beta}(F)$,

(b) *If*

$$\lim_{n\to\infty}\left(\frac{\xi(n)-\psi(n)+1}{(\beta(n)-\alpha(n)+1)^\eta}\right)=1 \tag{7}$$

then $S_\gamma^{\alpha,\beta}(F)\subseteq S_\eta^{\psi,\xi}(F)$.

Proof.

(a) Since $\psi(n)\leq\alpha(n)$ and $\beta(n)\leq\xi(n)$ for all $n\in\mathbb{N}$, we have

$$\left\{k\in P_n^{\psi,\xi} : D(u_k(x),\ell(x))\geq\epsilon,\ x\in H\right\}\supset\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon,\ x\in H\right\}$$

By our assumption, we get $\beta(n)-\alpha(n)+1\leq\xi(n)-\psi(n)+1$ for all $n\in\mathbb{N}$. Using this fact, one can see that

$$\frac{\left|\left\{k\in P_n^{\psi,\xi} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\xi(n)-\psi(n)+1)^\eta}\geq\frac{(\beta(n)-\alpha(n)+1)^\gamma}{(\xi(n)-\psi(n)+1)^\eta}\frac{\left|\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\alpha(n)-\beta(n)+1)^\gamma}$$

for each $x\in H$. Taking the limit of both sides of the aforementioned inequality as $n\longrightarrow\infty$ and by (6), we have $S_\eta^{\psi,\xi}(F)\subseteq S_\gamma^{\alpha,\beta}(F)$.

(b) Let $u=(u_k)\in S_\gamma^{\alpha,\beta}(F)$ and (7) be satisfied. As $\psi(n)\leq\alpha(n)$ and $\beta(n)\leq\xi(n)$ for all $n\in\mathbb{N}$, we obtain

$$\frac{\left|\left\{k\in P_n^{\psi,\xi} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\xi(n)-\psi(n)+1)^\eta}=\frac{\left|\{\psi(n)\leq k\leq\alpha(n)-1 : D(u_k(x),\ell(x))\geq\epsilon\}\right|}{(\xi(n)-\psi(n)+1)^\eta}$$
$$+\frac{\left|\{\alpha(n)\leq k\leq\beta(n) : D(u_k(x),\ell(x))\geq\epsilon\}\right|}{(\xi(n)-\psi(n)+1)^\eta}+\frac{\left|\{\beta(n)+1\leq k\leq\xi(n) : D(u_k(x),\ell(x))\geq\epsilon\}\right|}{(\xi(n)-\psi(n)+1)^\eta}$$
$$\leq\frac{\alpha(n)-\psi(n)}{(\xi(n)-\psi(n)+1)^\eta}+\frac{\xi(n)-\beta(n)}{(\xi(n)-\psi(n)+1)^\eta}+\frac{\left|\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon,\ x\in H\right\}\right|}{(\xi(n)-\psi(n)+1)^\eta}$$

$$= \frac{(\xi(n)-\psi(n)+1)-(\beta(n)-\alpha(n)+1)}{(\xi(n)-\psi(n)+1)^{\eta}} + \frac{\left|\left\{k \in P_n^{\alpha,\beta} : D(u_k(x),\ell(x)) \geq \epsilon,\ x \in H\right\}\right|}{(\xi(n)-\psi(n)+1)^{\eta}}$$

$$\leq \left(\frac{\xi(n)-\psi(n)+1}{(\beta(n)-\alpha(n)+1)^{\eta}} - 1\right) + \frac{\left|\left\{k \in P_n^{\alpha,\beta} : D(u_k(x),\ell(x)) \geq \epsilon,\ x \in H\right\}\right|}{(\beta(n)-\alpha(n)+1)^{\gamma}}$$

Taking the limit of both sides of above inequality as $(n \to \infty)$ and using (6), $st_{\gamma}^{\alpha\beta}(F)$-$\lim u_k(x) = \ell(x)$ on H, we get $st_{\eta}^{\psi\xi}(F)$-$\lim u_k(x) = \ell(x)$ on H. That is $u = (u_k) \in S_{\eta}^{\psi,\xi}(F)$. This step concludes the proof. □

Theorem 3.9. *Let $(\alpha,\beta),(\psi,\xi) \in \Lambda$ such that $\psi(n) \leq \alpha(n)$ and $\beta(n) \leq \xi(n)$ for all $n \in \mathbb{N}$, and $0 < \gamma \leq \eta \leq 1$.*

(a) *Let* (6) *holds, then we have $\omega_{\eta,p}^{\psi\xi}(F) \subset \omega_{\gamma,p}^{\alpha\beta}(F)$.*

(b) *Let* (7) *holds and $u = (u_k)$ be bounded sequence of fuzzy mappings on H then, we have $\omega_{\gamma,p}^{\alpha\beta}(F) \subset \omega_{\eta,p}^{\psi\xi}(F)$.*

Proof.

(a) We can prove this part in a similar way used in the proof of Theorem 3.8(a).

(b) Let $u = (u_k) \in \omega_{\gamma,p}^{\alpha\beta}(F)$ be bounded sequence of fuzzy mappings on H and (7) holds. Since (u_k) is bounded sequence of fuzzy mappings on H, then there exists some $M > 0$ such that $D(u_k(x),\ell(x)) \leq M$ for all $k \in \mathbb{N}$ and $x \in H$. For a given $\epsilon > 0$, we obtain

$$\begin{aligned}
\sum_{k \in P_n^{\xi,\psi},\, x\in H} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^{\eta}} &= \sum_{k \in P_n^{\psi,\xi} \setminus P_n^{\alpha,\beta}} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^{\eta}} + \sum_{k \in P_n^{\alpha,\beta}} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^{\eta}} \\
&\leq \left(\frac{(\xi(n)-\psi(n)+1)-(\beta(n)-\alpha(n)+1)}{(\xi(n)-\psi(n)+1)^{\eta}}\right) M^p + \sum_{k \in P_n^{\alpha,\beta}} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^{\eta}} \\
&\leq \frac{(\xi(n)-\psi(n)+1)-(\beta(n)-\alpha(n)+1)^{\eta}}{(\xi(n)-\psi(n)+1)^{\eta}} M^p + \sum_{k \in P_n^{\alpha,\beta}} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^{\eta}} \\
&\leq \left(\frac{\xi(n)-\psi(n)+1}{(\beta(n)-\alpha(n)+1)^{\eta}} - 1\right) M^p + \sum_{k \in P_n^{\alpha,\beta},\, x\in H} \frac{D(u_k(x),\ell(x))^p}{(\beta(n)-\alpha(n)+1)^{\gamma}}.
\end{aligned}$$

Since $u \in \omega_{\gamma,p}^{\alpha\beta}(F)$, by passing to the limit as $n \to \infty$ in the last inequality and by (7) we get $u \in \omega_{\eta,p}^{\psi\xi}(F)$. Hence $\omega_{\gamma,p}^{\alpha\beta}(F) \subset \omega_{\eta,p}^{\psi\xi}(F)$. □

Theorem 3.10. *Let $(\alpha,\beta),(\psi,\xi) \in \Lambda$ such that $\psi(n) \leq \alpha(n)$ and $\beta(n) \leq \xi(n)$ for all $n \in \mathbb{N}$, and $0 < \gamma \leq \eta \leq 1$.*

(a) *Let* (6) *holds, if a sequence $u = (u_k)$ of fuzzy numbers is strongly pointwise $\psi\xi$-summable to $\ell \in E^1$ it is pointwise $\alpha\beta$-statistically convergent of order γ to the same limit ℓ.*

(b) *Let* (7) *holds and (u_k) be a bounded sequence o fuzzy numbers, then if a sequence is pointwise $\alpha\beta$-statistically convergent of order γ to the fuzzy limit ℓ, then it is strongly $\psi\xi$–summable to the same limit ℓ.*

i.e. if $(u_k) \in \ell_\infty(F)$ and $st_{\gamma}^{\alpha\beta}(F)$-$\lim u_k = \ell$, then $\omega_{\eta,p}^{\psi\xi}(F)$- $\lim u_k = \ell$.

Proof. **(a)** Let the condition (6) holds and $\omega_{\eta,p}^{\psi\xi}(F) - \lim u_k = \ell$. Given a sequence (u_k) of fuzzy numbers and $\varepsilon > 0$, we have

$$\sum_{k \in P_n^{\psi,\xi},\, x\in H} D(u_k(x),\ell(x))^p \geq \left|\left\{k \in P_n^{\alpha,\beta} : D(u_k(x),\ell(x)) \geq \epsilon\right\}\right| \varepsilon^p.$$

Therefore, we obtain that

$$\sum_{k\in P_n^{\xi,\psi},\, x\in H} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^\eta} \geq \frac{\left|\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\xi(n)-\psi(n)+1)^\eta}\,\varepsilon^p$$
$$\geq \left(\frac{(\beta(n)-\alpha(n)+1)^\gamma}{(\xi(n)-\psi(n)+1)^\eta}\right)\frac{\left|\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}\,\varepsilon^p$$

for each $x\in H$. By passing to the limit as $n\to\infty$ in the last inequality and by (6), we obtain $st_\gamma^{\alpha\beta}(F)\text{-}\lim u_k=\ell$.

(b) Consider $st_\gamma^{\alpha\beta}(F)-\lim u_k=\ell$ where (u_k) is bounded and (7) holds. Then there exists some $M>0$ such that $D(u_k(x),\ell(x))\leq M$ for all $k\in\mathbb{N}$ and $x\in H$. Since $\psi(n)\leq\alpha(n)$ and $\beta(n)\leq\xi(n)$ we have

$$\begin{aligned}
&\sum_{k\in P_n^{\xi,\psi},\, x\in H} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^\eta}\\
&= \sum_{k\in P_n^{\psi,\xi}\setminus P_n^{\alpha,\beta}} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^\eta} + \sum_{k\in P_n^{\alpha,\beta}} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^\eta}\\
&\leq \left(\frac{(\xi(n)-\psi(n)+1)-(\beta(n)-\alpha(n)+1)^\eta}{(\xi(n)-\psi(n)+1)^\eta}\right)M^p + \sum_{k\in P_n^{\alpha,\beta},\, x\in H} \frac{D(u_k(x),\ell(x))^p}{(\xi(n)-\psi(n)+1)^\eta}\\
&= \left(\frac{\xi(n)-\psi(n)+1}{(\beta(n)-\alpha(n)+1)^\eta}-1\right)M^p + \sum_{\substack{k\in P_n^{\alpha,\beta},\, x\in H\\ D(u_k(x),\ell(x))\geq\epsilon}} \frac{D(u_k(x),\ell(x))^p}{(\beta(n)-\alpha(n)+1)^\eta} + \sum_{\substack{k\in P_n^{\alpha,\beta},\, x\in H\\ D(u_k(x),\ell(x))<\epsilon}} \frac{D(u_k(x),\ell(x))^p}{(\beta(n)-\alpha(n)+1)^\eta}\\
&\leq \left(\frac{\xi(n)-\psi(n)+1}{(\beta(n)-\alpha(n)+1)^\eta}-1\right)M^p + \frac{\left|\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}M^p + \frac{\beta(n)-\alpha(n)+1}{(\beta(n)-\alpha(n)+1)^\gamma}\,\varepsilon^p\\
&\leq \left(\frac{\xi(n)-\psi(n)+1}{(\beta(n)-\alpha(n)+1)^\eta}-1\right)M^p + \frac{\left|\left\{k\in P_n^{\alpha,\beta} : D(u_k(x),\ell(x))\geq\epsilon\right\}\right|}{(\beta(n)-\alpha(n)+1)^\gamma}M^p + \frac{\xi(n)-\psi(n)+1}{(\beta(n)-\alpha(n)+1)^\gamma}\,\varepsilon^p
\end{aligned}$$

for all $n\in\mathbb{N}$ and $x\in H$. Therefore $\omega_{\eta,p}^{\psi\xi}(F)\text{- }\lim u_k=\ell$.

□

4. Some Applications

In this section, we get an analogue of fuzzy Korovkin theorem associating with $\alpha\beta-$statistical convergence. Also, we apply classical fuzzy Bernstein operator to construct an example in support of our result. For these type of approaches and related concepts, one may refer to [5, 14, 19]. Before proceeding further, let us give some basic definitions and notations which will be used in this section.

Let $f:[a,b]\to E^1$ be fuzzy number valued functions. Then, f is said to be fuzzy continuous at x_0 in $[a,b]$ if and only if whenever $x_n\to x_0$, then $f(x_n)\to f(x_0)$ as $n\to\infty$. Also we say that f is fuzzy continuous if it is continuous at every point $x\in[a,b]$. The set of all fuzzy-valued continuous functions on the interval $[a,b]$ is denoted by $C_{\mathcal{F}}[a,b]$ [2]. Through the paper we denote $[f]_\lambda=[f_\lambda^+,f_\lambda^-]$ and $[f(x)]_\lambda=[f_\lambda^-(x),f_\lambda^+(x)]$ for all $\lambda\in[0,1]$ and $\forall x\in[a,b]$

The space $C_{\mathcal{F}}[a,b]$ is only cone not a vector space and let $L:C_{\mathcal{F}}[a,b]\to C_{\mathcal{F}}[a,b]$ be operator. Also L is called fuzzy linear operator if for every $\upsilon,\mu\in\mathbb{R}$, $f,g\in C_{\mathcal{F}}[a,b]$ and $x\in[a,b]$,

$$L(\upsilon\odot f\oplus\mu\odot f;x)=\upsilon\odot L(f;x)\oplus\mu\odot L(g;x)$$

holds. Then, L is said to be fuzzy positive linear operator if it is fuzzy linear and $L(f;x) \preceq L(g;x)$ holds whenever for $f,g \in C_{\mathcal{F}}[a,b]$ and $x \in [a,b]$, $f(x) \preceq g(x)$. Throughout the paper we use the test functions $f_i(x) = x^i$, $i = 0,1,2$ for $x \in [a,b]$.

Theorem A [3] Let $(L_n)_{n\geq 1}$ be a of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Assume that there exists a corresponding sequence $(\widetilde{L}_n)_{n\geq 1}$ of positive linear operators from $C[a,b]$ into itself with property

$$\left(L_n(f;x)_\lambda^\pm\right) = \widetilde{L}_n(f_\lambda^\pm;x) \tag{8}$$

for all $\lambda \in [0,1]$, $\forall x \in [a,b]$, $n \in \mathbb{N}$ and $f \in C_{\mathcal{F}}[a,b]$. Assume that

$$\lim_{n\to\infty} \|\widetilde{L}_n(e_i) - e_i\| = 0, \text{ for each } i = 0,1,2.$$

Then, for all $f \in C_{\mathcal{F}}[a,b]$, we have

$$\lim_{n\to\infty} D^*\left(L_n(f), f\right) = 0.$$

Similarly one can prove $\alpha\beta$ fuzzy statistical version of Theorem A. Now, we prove the following stronger version of Theorem A with the help of fuzzy $\alpha\beta$-statistical convergence.

Theorem 4.1. *Let $(T_k)_{k\geq 1}$ be a of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Assume that there exists a corresponding sequence $(\widetilde{T}_k)_{k\geq 1}$ of positive linear operators from $C[a,b]$ into itself with the property* (8)*. Assume that*

$$st_\gamma^{\alpha\beta}(F) - \lim_n \|\widetilde{T}_k(e_i) - e_i\| = 0, \text{ for each } i = 0,1,2. \tag{9}$$

Then, for all $f \in C_{\mathcal{F}}[a,b]$, we have

$$st_\gamma^{\alpha\beta}(F) - \lim_n D^*\left(T_k(f), f\right) = 0.$$

Proof. Let the conditions (9) hold, $\lambda \in [0,1]$, $\forall x \in [a,b]$, $k \in \mathbb{N}$ and $f \in C_{\mathcal{F}}[a,b]$. By fuzzy continuity of f at x, it follows that, for given $\varepsilon > 0$, there exists δ such that for all $t \in [a,b]$

$$|f_\lambda^\pm(t) - f_\lambda^\pm(x)| < \varepsilon, \text{ whenever } \forall |t-x| < \delta. \tag{10}$$

Since f is fuzzy bounded, we get $|f_\lambda^\pm(x)| \leq M_\lambda^\pm$, $a < x < b$. Hence

$$|f_\lambda^\pm(t) - f_\lambda^\pm(x)| \leq 2M_\lambda^\pm, \ a < x, t < b. \tag{11}$$

By using (10) and (11), we have

$$|f_\lambda^\pm(t) - f_\lambda^\pm(x)| \leq \varepsilon + \frac{2M_\lambda^\pm}{\delta^2}(t-x)^2, \ \forall |t-x| < \delta.$$

This implies that

$$-\varepsilon - \frac{2M_\lambda^\pm}{\delta^2}(t-x)^2 < f_\lambda^\pm(t) - f_\lambda^\pm(x) < \varepsilon + \frac{2M_\lambda^\pm}{\delta^2}(t-x)^2.$$

By using the positivity and linearity of $\{\widetilde{T}_k\}$, we get

$$\widetilde{T}_k(e_0,x)\left(-\varepsilon - \frac{2M_\lambda^\pm}{\delta^2}(t-x)^2\right) < \widetilde{T}_k(e_0,x)\left(f_\lambda^\pm(t) - f_\lambda^\pm(x)\right) \leq \widetilde{T}_k(e_0,x)\left(\varepsilon + \frac{2M_\lambda^\pm}{\delta^2}(t-x)^2\right)$$

where x is fixed and so $f_\lambda^\pm(x)$ is constant number. Therefore,

$$\begin{aligned} -\varepsilon\widetilde{T}_k(e_0,x) - \frac{2M_\lambda^\pm}{\delta^2}\widetilde{T}_k((t-x)^2,x) \quad &< \quad \widetilde{T}_k(f_\lambda^\pm(t),x) - f_\lambda^\pm(x)T_k(e_0,x) < \varepsilon\widetilde{T}_k(e_0,x) \\ &+ \quad \frac{2M_\lambda^\pm}{\delta^2}\widetilde{T}_k((t-x)^2,x). \end{aligned}$$

On the other hand

$$\widetilde{T}_k(f_\lambda^\pm(t),x)-f_\lambda^\pm(x)=\widetilde{T}_k(f_\lambda^\pm(t),x)-f_\lambda^\pm(x)\widetilde{T}_k(1,x)+f_\lambda^\pm(x)[\widetilde{T}_k(1,x)-1].$$

Then, we get

$$\widetilde{T}_k(f_\lambda^\pm,x)-f_\lambda^\pm(x)<\varepsilon\widetilde{T}_k(1,x)+\frac{2M_\lambda^\pm}{\delta^2}\widetilde{T}_k((t-x)^2,x)+f_\lambda^\pm(x)+f_\lambda^\pm(x)[\widetilde{T}_k(1,x)-1].$$

Let us compute second moment

$$\begin{aligned}\widetilde{T}_k((t-x)^2,x) &= \widetilde{T}_k(x^2-2xt+t^2,x)\\ &= x^2\widetilde{T}_k(1,x)-2x\widetilde{T}_k(t,x)+\widetilde{T}_k(t^2,x)\\ &= [\widetilde{T}_k(t^2,x)-x^2]-2x[\widetilde{T}_k(t,x)-x]+x^2[\widetilde{T}_k(1,x)-1].\end{aligned}$$

Using above equality, one can see that

$$\begin{aligned}\widetilde{T}_k(f_\lambda^\pm,x)-f_\lambda^\pm(x) &< \varepsilon+\frac{2M_\lambda^\pm}{\delta^2}\{[\widetilde{T}_k(t^2,x)-x^2]-2x[\widetilde{T}_k(t,x)-x]\\ &\quad +x^2[\widetilde{T}_k(e_0,x)-1]\}+f_\lambda^\pm(x)(\widetilde{T}_k(1,x)-1)\\ &= \varepsilon[\widetilde{T}_n(1,x)-1]+\varepsilon+\frac{2M_\lambda^\pm}{\delta^2}\{[\widetilde{T}_k(t^2,x)-x^2]\\ &\quad -2x[\widetilde{T}_k(t,x)-x]+x^2[\widetilde{T}_k(1,x)-1]\}+f_\lambda^\pm(x)(\widetilde{T}_k(1,x)-1).\end{aligned}$$

Because of ε is arbitrary, we obtain

$$\begin{aligned}|\widetilde{T}_k(f_\lambda^\pm,x)-f_\lambda^\pm(x)| &\le \varepsilon+\left(\varepsilon+M_\lambda^\pm+\frac{2M_\lambda^\pm c^2}{\delta^2}\right)|\widetilde{T}_k(e_0,x)-e_0|+\frac{4M_\lambda^\pm c}{\delta^2}|\widetilde{T}_k(e_1,x)-e_1|\\ &\quad +\frac{2M_\lambda^\pm}{\delta^2}|\widetilde{T}_k(e_2,x)-e_2|\end{aligned}$$

where $c=\max\{|a|,|b|\}$. Then, letting $R_\lambda^\pm(\varepsilon)=\max\left(\varepsilon+M_\lambda^\pm+\frac{2Mb^2}{\delta^2},\frac{4M_\lambda^\pm b}{\delta^2}\right)$ and taking supremum over $x\in[a,b]$ in the last inequality, we get

$$\|\widetilde{T}_k(f_\lambda^\pm)-f_\lambda^\pm\| \le \varepsilon+R_\lambda^\pm(\varepsilon)\left\{\|\widetilde{T}_k(e_0,x)-e_0\|+\|\widetilde{T}_k(e_1,x)-e_1\|+\|\widetilde{T}_k(e_2,x)-e_2\|\right\} \tag{12}$$

Besides, it follows from (8) that

$$\begin{aligned}D^*(T_k(f),f) &= \sup_{x\in[a,b]} D(T_k(f(t);x),f(x))\\ &= \sup_{x\in[a,b]}\sup_{\lambda\in[0,1]}\max\left\{\left|\widetilde{T}_k(f_\lambda^-(t);x)-f_\lambda^-(x)\right|,\left|\widetilde{T}_k(f_\lambda^+(t);x)-f_\lambda^+(x)\right|\right\}\\ &= \sup_{\lambda\in[0,1]}\max\left\{\|\widetilde{T}_k(f_\lambda^-)-f_\lambda^-\|,\|\widetilde{T}_k(f_\lambda^+)-f_\lambda^+\|\right\}.\end{aligned} \tag{13}$$

From (12) and (13), one can see that

$$D^*(T_k(f),f)\le\varepsilon+R(\varepsilon)\left\{\|\widetilde{T}_k(e_0,x)-e_0\|+\|\widetilde{T}_k(e_1,x)-e_1\|+\|\widetilde{T}_k(e_2,x)-e_2\|\right\},$$

where $R(\varepsilon) = \sup_{\lambda\in[0,1]} \max\{|R_\lambda^-(\varepsilon)|, |R_\lambda^+(\varepsilon)|\}$. For a given $\varepsilon' > 0$, choose a number $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$. Then, setting

$$\begin{aligned}
\mathcal{A} : &= \left\{k \le n : D^*(T_k(f), f) \ge \varepsilon'\right\},\\
\mathcal{A}_1 : &= \left\{k \le n : \| \widetilde{T}_k(e_0, x) - e_0 \| \ge \frac{\varepsilon' - \varepsilon}{3R(\varepsilon)}\right\},\\
\mathcal{A}_2 : &= \left\{k \le n : \| \widetilde{T}_k(e_1, x) - e_1 \| \ge \frac{\varepsilon' - \varepsilon}{3R(\varepsilon)}\right\},\\
\mathcal{A}_3 : &= \left\{k \le n : \| \widetilde{T}_k(e_2, x) - e_2 \| \ge \frac{\varepsilon' - \varepsilon}{3R(\varepsilon)}\right\}.
\end{aligned}$$

Then, $\mathcal{A} \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, so we have $\delta^{\alpha,\beta}(\mathcal{A}) \le \delta^{\alpha,\beta}(\mathcal{A}_1) + \delta^{\alpha,\beta}(\mathcal{A}_2) + \delta^{\alpha,\beta}(\mathcal{A}_3)$. Thus, by condition (9), we obtain $st_\gamma^{\alpha\beta}(F) - \lim_n D^*(T_k(f), f) = 0$, whence the result. □

We remark that our Theorem 4.1 is stronger than that of Theorem A. For this purpose, we consider the following example:

Let the sequence $v = (v_k)$ of fuzzy numbers defined by

$$v_k = \begin{cases} \overline{k}, & 1 \le k \le (\beta(n) - \alpha(n) + 1)^{\gamma/2}, \\ \overline{1}, & \text{otherwise}, \end{cases} \tag{14}$$

where $(\beta(n) - \alpha(n) + 1)^{\gamma/2}$ is the first integer in $[\alpha(n), \beta(n)]$. Considering the sequence of fuzzy Bernstein operators

$$B_k^{\mathcal{F}}(f, x) = \bigoplus_{i=0}^{k} \binom{k}{i} x^i (1-x)^{k-i} \odot f\left(\frac{k}{i}\right),$$

where $\forall x \in [0,1]$, $k \in \mathbb{N}$ and $f \in C_{\mathcal{F}}[0,1]$. This is a fuzzy positive linear operator and we write

$$\left\{B_k^{\mathcal{F}}(f; x)\right\}_\lambda^{\pm} = \widetilde{B}_k(f_\lambda^{\pm}; x) = \sum_{i=0}^{k} \binom{k}{i} x^i (1-x)^{k-i} f_\lambda^{\pm}\left(\frac{k}{i}\right),$$

where $f_\lambda^{\pm} \in C[0,1]$ for all $\lambda \in [0,1]$. Observe easily that (see [3])

$$\widetilde{B}_k(e_0; x) = 1, \quad \widetilde{B}_k(e_1; x) = x \quad \text{and} \quad \widetilde{B}_k(e_2; x) = x^2 + \frac{x - x^2}{k}.$$

Considering the sequence $\{\widetilde{M}_k\}$ of positive fuzzy linear operators given by

$$M_k^{\mathcal{F}}(f(t); x) = (\overline{1} \oplus v_k) \odot B_k^{\mathcal{F}}(f(t); x). \tag{15}$$

Using the facts that $\widetilde{M}_k(f_\lambda^{\pm}; x) = (1 + v_k)\widetilde{B}_k(f_\lambda^{\pm}; x)$ and $st_\gamma^{\alpha\beta} - \lim_k v_k = 0$, we conclude that

$$\begin{aligned}
st_\gamma^{\alpha\beta}(F) &- \lim_n \|\widetilde{M}_k(1; x) - 1\| = 0,\\
st_\gamma^{\alpha\beta}(F) &- \lim_n \left\|\widetilde{M}_k(t; x) - x\right\| = 0,\\
st_\gamma^{\alpha\beta}(F) &- \lim_n \left\|\widetilde{M}_k(t^2; x) - x^2\right\| = 0.
\end{aligned}$$

So, by Theorem 4.1, for all $f \in C_{\mathcal{F}}[0,1]$,

$$st_\gamma^{\alpha\beta}(F) - \lim_k D^*\left(M_k^{\mathcal{F}}(f), f\right) = 0.$$

However, the fuzzy sequence $(v_k)_{k\ge1}$ defined (14) is not fuzzy convergent, the sequence $\{M_k^{\mathcal{F}}(f, x)\}_{k\ge1}$ given by (15) does not satisfy the Theorem 4.1 for all $f \in C_{\mathcal{F}}[0,1]$.

Concluding Remarks

In this article, certain results on statistical convergence of fuzzy mappings have been extended to the $\alpha\beta$-statistical convergence of order γ. Given results in this article not only generalize the earlier works done by several authors [7, 10, 11, 13, 21] but also give a new perspective concerning the development of statistical convergence of fuzzy mappings and Korovkin type approximation theorems for fuzzy positive linear operators. As a future work we will study the rates of $\alpha\beta$-statistical fuzzy convergence of the operators by means of the fuzzy modulus of continuity.

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