The Drazin Inverse of the Sum of Two Bounded Linear Operators and its Applications

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Abstract. Let $P$ and $Q$ be bounded linear operators on a Banach space. The existence of the Drazin inverse of $P + Q$ is proved under some assumptions, and the representations of $(P + Q)^D$ are also given. The results recover the cases $P^2Q = 0, PQ = 0$ studied by Yang and Liu in [19] for matrices, $Q^2P = 0, PQP = 0$ studied by Cvetković and Milovanović in [7] for operators and $PQ^2 = 0, P^2Q = 0$ studied by Shakoor, Yang and Ali in [16] for matrices. As an application, we give representations for the Drazin inverse of the operator matrix $A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

1. Introduction

Let $X$ be a Banach space. The set $\mathcal{B}(X)$ consists of all bounded linear operators on $X$. An operator $T \in \mathcal{B}(X)$ is said to be Drazin invertible, if there exists an operator $T^D \in \mathcal{B}(X)$ such that

$$TT^D = T^D = T(T^D)^2, \quad T^{k+1} = T^k$$

for some integer $k \geq 0$,

where $T^D$ is called the Drazin inverse of $T$. The smallest integer $k$ satisfying the previous system of equations is called the index of $T$, and is denoted by $\text{ind}(T)$. In particular, if $\text{ind}(T) = 1$, $T^D$ is called the group inverse of $T$; if $\text{ind}(T) = 0$, it can be seen that $T$ is invertible and $T^D = T^{-1}$. Note that $T^D$ may not exist, but $T^D$ must be unique if it exists. Moreover, if $T$ is nilpotent, then $T$ is Drazin invertible, and $T^D = 0$.

The Drazin inverse has become a useful tool in the researches of Markov chains, differential and difference equations, optimal control and iterative methods[1, 3].

In [11], M. P. Drazin proves that $(P + Q)^D = P^D + Q^D$ if $PQ = QP = 0$ in an associative ring. In the sequel, many authors begin to consider this problem for matrices and operators, and present explicit representations of $(P + Q)^D$ under the conditions such as

(1) $PQ = QP = 0$ (see [11]),
(2) $PQ = 0$ (see [9, 12]),
(3) $P^2Q = PQ^2 = 0$ (see [5]),

2010 Mathematics Subject Classification. 46C07; 46C05; 15A09.
Keywords. Drazin inverse, bounded linear operator, operator matrix.
Received: 01 December 2015; Accepted: 29 April 2016
Communicated by Dragan S. Djordjević
Research supported by the NNSF of China (11261034, 11461049 and 71561020), and the NSF of Inner Mongolia (2014MS0113).
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Lemma 1.1. Let $T \in \mathcal{B}(X)$, then $T$ is Drazin invertible if and only if $0 \notin \sigma(T) \setminus \{0\}$ and the point zero, provided $0 \in \sigma(T)$, is a pole of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$, and in this case the following representation holds:

$$R(\lambda, T) = \sum_{k=1}^{\text{ind}(T)} \lambda^{-k} T^{k-1} T^n = \sum_{k=0}^{\infty} \lambda^k (T^{D})^{k+1},$$

where $0 < |\lambda| < (r(T^D))^{-1}$.

Remark 1.2. From Lemma 1.1, $T^D$ can be obtained by the coefficient at $\lambda^0$ in the Laurent expansion of the resolvent $R(\lambda, T)$ in a punctured neighborhood of 0, i.e.,

$$T^D = -\frac{1}{2\pi i} \int_{\Gamma} \frac{R(\lambda, T)}{\lambda} d\lambda,$$

where $\Gamma = \{ \lambda \in \mathbb{C} : |\lambda| = \epsilon \}$ with $\epsilon$ being sufficiently small such that $\{ \lambda \in \mathbb{C} : |\lambda| \leq \epsilon \} \cap \sigma(T) = \{0\}$.

Lemma 1.3. Let $A \in \mathcal{B}(X, \mathcal{Y})$ and $B \in \mathcal{B}(\mathcal{Y}, X)$. If $BA$ is Drazin invertible, then $AB$ is also Drazin invertible. Moreover,

$$(AB)^D = A((BA)^D) B, \quad \text{ind}(AB) \leq \text{ind}(BA) + 1.$$ (3)

Lemma 1.4. For the operator matrix $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathcal{B}(X), B \in \mathcal{B}(\mathcal{Y}, X), C \in \mathcal{B}(X, \mathcal{Y})$ and $D \in \mathcal{B}(\mathcal{Y})$. If $A$ is invertible, then $\mathcal{A}$ is invertible if and only if $D - CA^{-1}B$ is invertible.

Remark 1.5. The Lemma above is well known, see, e.g., [15, Lemma 2.1].

2. Main Results

In this section, we investigate the Drazin inverse of the sum of two operators $P, Q \in \mathcal{B}(X)$. It is interesting that the conditions when $n \geq 2$ will share the same representation of the Drazin inverse of $P + Q$.

In order to show that $P + Q$ is Drazin invertible, we need to find out the resolvent of the operator matrix $M = \begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}$ defined on the Banach space $X \times X$. Write $\Delta(\lambda) = \lambda I - R(\lambda, P)Q$. Then, the following two lemmas are necessary.
Lemma 2.1. Let $P, Q \in \mathcal{B}(X)$ be Drazin invertible, $r = \text{ind}(P)$ and $s = \text{ind}(Q)$. If $P^2Q + QPQ = 0$ and $P^nQ = 0$ for some integer $n > 0$, then

$$
\Delta(\lambda)^{-1} = \lambda^{-2}(\lambda^2I + PQ)R(\lambda, Q),
$$

(4)

where $0 < |\lambda| < \min\{|r(P^D)|^{-1}, |r(Q^D)|^{-1}\}$.

Proof. From $P^mQ = 0$ and $P^D = (P^D)^2P$, it follows that $P^DQ = 0$, then $P^mQ = 0$ if the integer $m \geq r$. Moreover, $P^nQ = P^mQ = 0$. By $P^2Q + QPQ = 0$, we have

$$
P^{2k-1}Q = (-1)^{k-1}(Q)^k, \quad P^{2k}Q = (-1)^k(Q)^k, \quad k = 1, 2, \cdots.
$$

(5)

Since there always exists an integer $k_0$ such that $2^{k_0} \leq n \leq 2^{k_0+1} - 1$ for each $n$, we deduce $P^{2^{k_0}+1}Q = 0$ from $P^nQ = 0$. This together with Eq.(5) shows that $PQ$ is Drazin invertible, $(PQ)^D = 0$ and $\text{ind}(PQ) \leq 2^{k_0}$. Thus, using Lemma 1.1, we conclude that

$$
R(\lambda, P)PQ = \left( \sum_{k=1}^{2^{k_0}-1} \lambda^{-k}P^{k-1}P^n - \sum_{k=0}^{\infty} \lambda^k(P^D)^{k+1} \right)PQ
$$

$$
= \sum_{k=1}^{2^{k_0}-1} \lambda^{-k}P^kQ
$$

$$
= \sum_{k=1}^{2^{k_0}-1} \lambda^{-k}P^kQ
$$

$$
= (\lambda I - Q) \sum_{k=1}^{2^{k_0}-1} (-1)^{k-1} \lambda^{-2k}(PQ)^k
$$

$$
= (\lambda I - Q)PQR(\lambda^2; -PQ),
$$

where $0 < |\lambda| < (r(P^D))^{-1}$.

Then,

$$
\Delta(\lambda) = \lambda I - Q - R(\lambda, P)PQ
$$

$$
= (\lambda I - Q)(I - PQR(\lambda^2; -PQ))
$$

$$
= \lambda^2(\lambda I - Q)R(\lambda^2; -PQ).
$$

Therefore, we have

$$
\Delta(\lambda)^{-1} = \lambda^{-2}(\lambda^2I + PQ)R(\lambda, Q),
$$

where $0 < |\lambda| < \min\{|r(P^D)|^{-1}, |r(Q^D)|^{-1}\}$.

Lemma 2.2. Under the assumptions of Lemma 2.1, the representation of the resolvent for the operator matrix

$$
M = \begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}
$$

is given by

$$
R(\lambda, M) = \begin{pmatrix} \lambda^{-2}(\lambda I - Q)(\lambda^2I + PQ)R(\lambda, Q)R(\lambda, P) & \lambda^{-2}(\lambda I - Q)PQR(\lambda, Q) \\ \lambda^{-2}(\lambda^2I + PQ)R(\lambda, Q)R(\lambda, P) & \lambda^{-2}(\lambda^2I + PQ)R(\lambda, Q) \end{pmatrix},
$$

(7)

where $0 < |\lambda| < \min\{|r(P^D)|^{-1}, |r(Q^D)|^{-1}\}$. 

\qed
Proof. Let \( \rho(\Delta) \) denote the set of all \( \lambda \in \mathbb{C} \) such that \( \Delta(\lambda) \) is invertible in \( \mathcal{B}(\lambda) \). By Lemma 1.4, we obtain \( \rho(M) \cap \rho(P) = \rho(P) \cap \rho(\Delta) \). If \( \lambda \in \rho(M) \cap \rho(P) \), then

\[
R(\lambda, M) = \begin{pmatrix}
R(\lambda, P) + R(\lambda, P)PQ\Delta(\lambda)^{-1}R(\lambda, P) & R(\lambda, P)PQ\Delta(\lambda)^{-1} \\
\Delta(\lambda)^{-1}R(\lambda, P) & \Delta(\lambda)^{-1}
\end{pmatrix},
\]

where \( 0 < |\lambda| < \min\{\|(r(P^D))^{-1}, (r(Q^D))^{-1}\}. \) By (4) and (6), we immediately have the expression

\[
R(\lambda, P)PQ\Delta(\lambda)^{-1} = (\lambda I - Q)PQR(\lambda^2, -PQ)\Delta(\lambda)^{-1} = \lambda^{-2}(\lambda I - Q)PQR(\lambda, Q).
\]

Then, we further have

\[
R(\lambda, P) + R(\lambda, P)PQ\Delta(\lambda)^{-1}R(\lambda, P) = (I + R(\lambda, P)PQ\Delta(\lambda)^{-1})R(\lambda, P)
\]

\[
= (I + \lambda^{-2}(\lambda I - Q)PQR(\lambda, Q))R(\lambda, P)
\]

\[
= \lambda^{-2}(\lambda I - Q)(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P).
\]

Moreover,

\[
\Delta(\lambda)^{-1}R(\lambda, P) = \lambda^{-2}(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P).
\]

The proof is completed. \( \square \)

We will give other two necessary lemmas in order to obtain the representation of \((P + Q)^D\).

**Lemma 2.3.** Under the assumptions of Lemma 2.1, the following statements are true:

1. The coefficients \( \alpha_i \) at \( \lambda^i \) \((i = -1, 0, 1, 2)\) of \( R(\lambda, Q)R(\lambda, P) \) are given by

\[
\alpha_{-1} = -(Q^\tau \delta P^D + Q^D \tau P^\tau),
\]

\[
\alpha_0 = -(Q^\tau \delta(P^D)^2 + (P^D)^2 \tau P^\tau) + Q^D P^D,
\]

\[
\alpha_1 = -Q^\tau \delta(P^D)^3 + (P^D)^3 \tau P^\tau + Q^D (P^D)^2 + (Q^D)^2 P^D,
\]

\[
\alpha_2 = -Q^\tau \delta(P^D)^4 + (P^D)^4 \tau P^\tau + Q^D (P^D)^3 + (Q^D)^2 (P^D)^2 + (Q^D)^3 P^D,
\]

where \( \delta = \sum_{k=0}^{r-1} Q^k P^D k \), \( r = \sum_{k=0}^{r-1} (Q^D)^k P^k \).

2. \( \alpha_{-1} = Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2 \) and \( \alpha_0 + PQ\alpha_2 \) are the coefficients at \( \lambda^2 \) of \( (\lambda I - Q)(\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P) \) and \( (\lambda^2 I + PQ)R(\lambda, Q)R(\lambda, P) \), respectively.

3. \( -PQ^D - P^2 (Q^D)^2 \) and \( -Q^D - P(Q^D)^2 \) are the coefficients at \( \lambda^2 \) of \( (\lambda I - Q)PQR(\lambda, Q) \) and \( (\lambda^2 I + PQ)R(\lambda, Q) \), respectively.

Proof. (1) Note that \( P, Q \) are Drazin invertible. Applying Eq.(1) for \( P, Q \) in a punctured neighborhood of 0, we have

\[
R(\lambda, Q) = \sum_{k=1}^{\infty} \lambda^{-k} Q^{k-1} Q^\tau - \sum_{k=0}^{\infty} \lambda^k (Q^D)^{k+1}
\]

and

\[
R(\lambda, P) = \sum_{k=1}^{\infty} \lambda^{-k} P^{k-1} P^\tau - \sum_{k=0}^{\infty} \lambda^k (P^D)^{k+1}.
\]

Then the coefficients \( \alpha_i \) at \( \lambda^i \) \((i = -1, 0, 1, 2)\) of \( R(\lambda, Q)R(\lambda, P) \) can be easily obtained.
Thus, by Lemma 2.3 (1), \( \alpha_{-1} - Q\alpha_0 + PQ\alpha_1 - QPQ\alpha_2 \) is the coefficient at \( \lambda^2 \) of \((\lambda I - Q)(\lambda I + PQ)R(\lambda, Q)R(\lambda, P)\). Analogously, (3) can be proved. \( \square \)

**Lemma 2.4.** Under the assumptions of Lemma 2.1, the following statements are valid:

1. \( \tau Q = Q \), and hence \( \tau P^2Q = P^2Q \).
2. \( \tau PQ = PQ + Q^Dp^2Q \), and hence \( \tau PQ^D = PQ^D + Q^Dp^2Q^D \).
3. \( \tau\delta = \tau + \delta - I \).
4. \( \alpha_{-1}PQ = \alpha_0PQ = \alpha_1PQ = Q\alpha_2PQ = 0 \).
5. \( \alpha_{-1}Q = -Q\alpha_0Q = -(Q^D)^{i+1}, \quad i = 0, 1, 2, 3 \).
6. \( \alpha_{i+1} = -\alpha_{i+1}, \quad i = -1, 0, 1, 2, 3 \).
7. \( \alpha_i P^2(Q^D)^2 = -(Q^D)^{i+2}P^2(Q^D)^2, \quad i = -1, 0, 1, 2 \).

Here
\[
\alpha_3 = -(Q^D\delta(P^D)^5 + (Q^D)^5\tau P^5) + Q^D(P^D)^4 + (Q^D)^3(P^D)^3 + (Q^D)^2P^2 + (Q^D)^3(P^D)^2,
\]
and \( \delta, \tau \) are defined as in Lemma 2.3.

**Proof.** (1) By \( \tau = \sum_{k=0}^{r-1} (Q^D)^k p^k \), we have \( \tau Q = \sum_{k=0}^{r-1} (Q^D)^k p^k Q \). If \( r \) is odd, then, by Eq.(5), we get
\[
\tau Q = Q + \sum_{k=1}^{r-1} ((Q^D)^{2k-1}p^{2k-1}Q + (Q^D)^{2k}p^{2k}Q)
= Q + \sum_{k=1}^{r-1} ((-1)^{k-1}(Q^D)^{2k-1}(PQ)^k + (-1)^k(Q^D)^{2k}(PQ)^k)
= Q + \sum_{k=1}^{r-1} ((-1)^{k-1}(Q^D)^{2k-1}(PQ)^k + (-1)^k(Q^D)^{2k-1}(PQ)^k)
= Q.
\]
If \( r \) is even, then
\[
\tau Q = Q + \sum_{k=1}^{r-1} ((Q^D)^{2k-1}P^{2k-1}Q + (Q^D)^{2k}P^{2k}Q) + (Q^D)^{r-1}P^{r-1}Q
= Q + (Q^D)^{r-1}P^{r-1}Q
= Q + (-1)^{r-1}(Q^D)^{r-1}(PQ)^r
= Q + (-1)^{r-1}(Q^D)^{r-1}(PQ)^r
= Q - (Q^D)^rP^r Q
= Q
\]
since \( P^rQ = P^{r+1}P^DQ = 0 \), and hence \( \tau Q = Q \). Thus, (1) is proved.
(2) Obviously, \(\tau PQ = \sum_{n=0}^{r-1} (Q^n)p^nQ\). If \(r\) is even, then

\[
\tau PQ = PQ + Q^2p^2Q + \sum_{k=1}^{r-1} ((Q^2)^{2k}p^{2k+1}Q + (Q^2)^{2k+1}p^{2k+2}Q)
\]

\[
= PQ + Q^2p^2Q + \sum_{k=1}^{r-1} ((-1)^k(Q^2)^{2k}(PQ)^{k+1} + (-1)^{k+1}(Q^2)^{2k+1}(Q^2)^{k+1})
\]

\[
= PQ + Q^2p^2Q + \sum_{k=1}^{r-1} ((-1)^k(Q^2)^{2k}(PQ)^{k+1} + (-1)^{k+1}(Q^2)^{2k}(PQ)^{k+1})
\]

\[
= PQ + Q^2p^2Q.
\]

Similarly, if \(r\) is odd, then

\[
\tau PQ = PQ + Q^2p^2Q + \sum_{k=1}^{r-1} ((Q^2)^{2k}p^{2k+1}Q + (Q^2)^{2k+1}p^{2k+2}Q) + (Q^2)^{-1}P^rQ
\]

\[
= PQ + Q^2p^2Q + (Q^2)^{-1}P^rQ
\]

\[
= PQ + Q^2p^2Q.
\]

Therefore, the relation \(\tau PQ = PQ + Q^2p^2Q\) is proved.

On the other hand, by \(P^2Q = -Q^2P\), it is obvious that

\[
\tau P^2Q^2 = -\tau Q^2PQ^2 = -Q^2PQ^2 = P^2Q^2.
\]

(3) In view of \(\tau Q = Q\), we clearly have

\[
\tau\delta = \tau\sum_{k=0}^{\infty} Q^k(P^D)^k
\]

\[
= \tau + (Q^D + Q^2(P^D)^2 + \cdots + Q^{r-1}(P^D)^{r-1})
\]

\[
= \tau + \delta - 1.
\]

(4) We only prove \(\alpha_{-1}PQ = 0\), and the proof of others are similar.

Since \(P^nQ = PQ, \tau PQ = PQ + Q^2P^2Q\) and \(P^2Q + PQ = 0\), it follows that

\[
\alpha_{-1}PQ = -Q^D\tau PQ
\]

\[
= -Q^D(Q^2PQ + Q^3p^2Q)
\]

\[
= -Q^D PQ + (Q^2)^2QPQ
\]

\[
= 0.
\]

(5) The conclusion can be immediately obtained from \(P^DQ = 0\), \(P^nQ = Q\) and \(\tau Q = Q\).

(6) We only prove the case \(i = -1\), and other cases are similar.

Note that \(P^DQ^n = P^D, P^nQ^D = Q^D, P^2Q^D = 0\) and \(P^nQ^n = P^n - Q^D\), so

\[
\alpha_{-1}\alpha_{-1} = (Q^nD^D + Q^n\tau P^n)^2,
\]

\[
= Q^nD^D\delta P^D + (Q^D\tau Q^D\tau P^n + Q^D\tau P^n\delta P^D - Q^D\tau Q^D\delta P^D).
On the other hand, the relation $P D Q = 0$ implies $P D = P D$ and $P D = \delta - PP D$. Also, $\tau Q D = Q D$ can be obtained based on $\tau Q = Q$. Therefore, we have

$$a - 1 \alpha - 1 = Q^\tau \delta (P D)^2 + (Q D)^2 \tau P n + Q D (\delta - P P D) P D - Q D Q D \delta P D$$

$$= Q^\tau \delta (P D)^2 + (Q D)^2 \tau P n - Q D P D$$

$$= -\alpha_0,$$

since, by Lemma 2.4 (3),

$$Q D \tau (\delta - P P D) P D = Q D (\tau \delta - \tau P P D) P D$$

$$= Q D (\tau + \delta - I - \tau P P D) P D$$

$$= Q D (\tau + \delta - I) P D - Q D \tau P D$$

$$= Q D (\delta - I) P D.$$

(7) Note that $\tau P D Q^2 = P D Q^2$. Then, the claim follows from $P D Q D = 0$ and $P n P D (Q D)^2 = P D (Q D)^2$. □

The following is the main result of this section.

**Theorem 2.5.** Let $P, Q \in B(\lambda)$ be Drazin invertible, $r = \text{ind}(P)$ and $s = \text{ind}(Q)$. If $P D + Q P Q = 0$ and $P n Q = 0$ for some integer $n > 0$, then $P + Q$ is Drazin invertible, and

$$(P + Q D) = -\alpha_0 P - PQ \alpha_2 P + (Q D)^2 + Q D,$$  \hspace{1cm} (10)

i.e.,

$$\begin{align*}
(P + Q D) &= Q n \sum_{i=0}^{r-1} (P D)^{i+1} + \sum_{i=0}^{r-1} (Q D)^{i+1} P n P \sum_{i=0}^{r-1} (Q D)^{i+1} P n \\
&+ PQ n \sum_{i=0}^{r-1} (Q D)^{i+1} P n - PQ D^2 P D - PQ Q D (P D)^2.
\end{align*}$$  \hspace{1cm} (11)

Moreover, $\text{ind}(P + Q) \leq r + s + 3$.

**Proof.** Let $A = (P, Q) : X \oplus X \rightarrow X$ and $B = (P) : X \rightarrow X \oplus X$. Then $P + Q = AB$ and $BA = M$, where $M$ is defined as in Lemma 2.2. By Lemma 2.2, we obtain

$$R(\lambda, BA) = \begin{pmatrix} \lambda^{-2}(AI - Q)(AI + PQ)R(\lambda, Q)R(\lambda, P) & \lambda^{-2}(AI - Q)PQR(\lambda, Q) \\
\lambda^{-2}(AI + PQ)R(\lambda, Q)R(\lambda, P) & \lambda^{-2}(AI + PQ)R(\lambda, Q) \end{pmatrix}$$  \hspace{1cm} (12)

for $\lambda$ belonging to a punctured neighborhood of 0, which shows that $R(\lambda, BA)$ has a pole at $\lambda = 0$ of order at most $r + s + 2$. So, according to Lemma 1.1, $BA$ is Drazin invertible and $R(\lambda, BA)$ has the Laurent series

$$R(\lambda, BA) = \sum_{k=1}^{r+s+2} \lambda^{-k}(BA)^{k+1}(BA)^n - \sum_{k=0}^{\infty} \lambda k((BA)^D)^{k+1}$$

in a punctured neighborhood of 0. Thus, by Lemma 2.1, $AB$ is Drazin invertible, i.e., $P + Q$ is Drazin invertible. In addition, we have

$$(P + Q D) = (AB D)^2 B$$  \hspace{1cm} (13)

and $\text{ind}(P + Q) \leq \text{ind}(BA) + 1 \leq r + s + 3$.

According to Lemma 2.3 and the expression (12) for $R(\lambda, BA)$, $\alpha_0 - Q \alpha_0 + PQ \alpha_1 - PQ Q \alpha_2, \alpha_0 + P Q \alpha_2, -Q D - P D^2 (Q D)^2$ and $-Q D - P (Q D)^2$ are the coefficients at $\lambda^0$ of $\lambda^{-2}(AI - Q)(AI + PQ)R(\lambda, Q)R(\lambda, P), \lambda^{-2}(AI + PQ)$.
Therefore, from Eq. (13), we obtain that

\[
(\mathbf{BA})^D = -\frac{1}{2\pi i} \int \frac{1}{\lambda} R(\lambda, \mathbf{BA}) d\lambda
\]

This gives

\[
\begin{pmatrix}
\alpha_1 - \alpha Q_0 + P Q \alpha_1 - Q P Q \alpha_2 - P Q P Q - P^2 (Q^D)^2 \\
\alpha_0 + P Q \alpha_2
\end{pmatrix}
\]

Then

\[
((\mathbf{BA})^D)^2 = \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}
\]

where

\[
\begin{align*}
C_{11} &= (\alpha_1 - \alpha Q_0 + P Q \alpha_1 - Q P Q \alpha_2 - (P Q P Q + P^2 (Q^D)^2))(\alpha_0 + P Q \alpha_2), \\
C_{12} &= -(\alpha_1 - \alpha Q_0 + P Q \alpha_1 - Q P Q \alpha_2)(P Q P Q + P^2 (Q^D)^2) \\
&\quad + (P Q P Q + P^2 (Q^D)^2)(P Q P Q + P^2 (Q^D)^2), \\
C_{21} &= (\alpha_0 + P Q \alpha_2)(\alpha_1 - \alpha Q_0 + P Q \alpha_1 - Q P Q \alpha_2) - (P Q + P(Q^D)^2)(\alpha_0 + P Q \alpha_2), \\
C_{22} &= -(\alpha_0 + P Q \alpha_2)(P Q P Q + P^2 (Q^D)^2) + (P Q P Q + P^2 (Q^D)^2)^2.
\end{align*}
\]

By Lemma 2.3 and Lemma 2.4, together with \(P Q + P Q Q = 0, Q^D = Q(Q^D)^2\) and \((Q^D)^2 P^2 (Q^D)^2 = -Q^D P(Q^D)^2\), we can deduce that

\[
\begin{align*}
C_{11} &= -\alpha_0 + Q \alpha_0 + Q^D P Q \alpha_2 - Q^D Q \alpha_0 - Q Q^D P Q \alpha_2 + P Q(Q^D)^2 \alpha_0 \\
&\quad + P Q P Q \alpha_2 + Q P Q(Q^D)^2 \alpha_0 - Q P Q(Q^D)^2 \alpha_0 - Q P Q P Q \alpha_2 + Q \alpha_1 \\
&\quad - P Q \alpha_2 - P Q^2 \alpha_0 - P Q P Q \alpha_2 + P^2 (Q^D)^2 \alpha_0 - P^2 (Q^D)^2 P Q \alpha_2 \\
&\quad = -\alpha_0 + Q \alpha_1 - P Q \alpha_2 + Q P Q \alpha_3, \\
C_{12} &= Q^D P^2(Q^D)^2 - Q^D(Q^D)^2 P^2(Q^D)^2 + P Q(Q^D)^3 P^2(Q^D)^2 - Q P Q(Q^D)^4 P^2(Q^D)^2 \\
&\quad + P(Q^D)^2 + P Q^2 P(Q^D)^2 + P^2(Q^D)^3 + P^2(Q^D)^2 P(Q^D)^2 \\
&\quad = P(Q^D)^2 + P^2(Q^D)^3, \\
C_{21} &= -\alpha_1 + Q \alpha_0 + Q P Q \alpha_2 - Q P \alpha_3 + P Q(Q^D)^3 Q \alpha_0 + P Q(Q^D)^3 P Q \alpha_2 \\
&\quad - Q^D \alpha_0 - Q^D P Q \alpha_2 - P(Q^D)^2 \alpha_0 - P(Q^D)^2 P Q \alpha_2 \\
&\quad = -\alpha_1 - P Q \alpha_3, \\
C_{22} &= (Q^D)^2 P^2(Q^D)^2 + P Q(Q^D)^4 P^2(Q^D)^2 + (Q^D)^2 + Q^D P(Q^D)^2 \\
&\quad + P(Q^D)^3 + P(Q^D)^2 P(Q^D)^2 \\
&\quad = (Q^D)^2 + P(Q^D)^3.
\end{align*}
\]

Thus,

\[
((\mathbf{BA})^D)^2 = \begin{pmatrix}
-\alpha_0 + Q \alpha_1 - P Q \alpha_2 + Q P Q \alpha_3 \\
-\alpha_1 - P Q \alpha_3
\end{pmatrix}
\]

Therefore, from Eq. (13), we obtain that

\[
(P + Q)^D = (I \ Q) \begin{pmatrix}
-\alpha_0 + Q \alpha_1 - P Q \alpha_2 + Q P Q \alpha_3 \\
-\alpha_1 - P Q \alpha_3
\end{pmatrix} P(Q^D)^2 + P^2(Q^D)^3 \\
(P(Q^D)^2 + P^2(Q^D)^3)
\]

\[
= -\alpha_0 P - P Q \alpha_2 P + P^2(Q^D)^3 + Q(Q^D)^2 + Q P(Q^D)^3 \\
= -\alpha_0 P - P Q \alpha_2 P + P^2(Q^D)^2 + Q^D.
\]
Instituting the expression (8) of \(a_0, a_2\) into Eq. (15), then we have

\[
(P + Q)^D = Q^n \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P_i P^n + P \sum_{i=0}^{r-1} (Q^D)^{i+2} P_i P^n 
+ PQ^n \sum_{i=0}^{s-2} Q^{i+1} (P^D)^{i+3} - PQ^D P^D - PQQ^D (P^D)^2
\]

from \(Q^n Q^s = 0, P^n P^m = 0, Q^D - Q^D P^D P = Q^D P^n\) and \(P(Q^D)^2 - P(Q^D)^2 P Q = P(Q^D)^2 P^n\). \(\square\)

**Remark 2.6.** In Theorem 2.5, we find that the representation (11) of \((P + Q)^D\) is the same when \(n \geq 2\).

If let \(A = (Q^D) : X \oplus X \rightarrow X\) and \(B = (P^D) : X \rightarrow X \oplus X\), then \(P + Q = AB\), and we have

**Theorem 2.7.** Let \(P, Q \in \mathcal{B}(X)\) be Drazin invertible, \(r = \text{ind}(P)\) and \(s = \text{ind}(Q)\). If \(PQ^2 + PQP = 0\) and \(PQ^n = 0\) for some integer \(n > 0\), then \(P + Q\) is Drazin invertible, and

\[
(P + Q)^D = Q^n \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P_i P^n + \sum_{i=0}^{r-2} (Q^D)^{i+3} P_i P^n Q 
+ Q^n \sum_{i=0}^{s-1} Q^i (P^D)^{i+2} Q - Q^D P^D Q - (Q^D)^2 PPPQ.
\]

The following corollary is the case when \(n = 1\) of Theorem 2.5.

**Corollary 2.8.** [9, 12] Let \(P, Q \in \mathcal{B}(X)\) is Drazin invertible, \(r = \text{ind}(P)\) and \(s = \text{ind}(Q)\). If \(PQ = 0\). Then \(P + Q\) is Drazin invertible, and

\[
(P + Q)^D = Q^n \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P_i P^n.
\]

**Remark 2.9.** When \(n = 2\) in Theorem 2.5 and Theorem 2.7, we obtain the results of [19, Theorem 2.1, Theorem 2.2] and [7, Lemma 4]. When \(n = 3\) in Theorem 2.5, we get the result of [16, Theorem 5].

In fact, the condition \(PQ^2 = 0\) in [16, Theorem 5] can be obtained from \(P^2 Q + PQQ = 0\) and \(PQ = 0\). On the other hand, since \(\text{ind}(P^2) = \left\lceil \frac{\text{ind}(P^2)}{2} \right\rceil = \text{ind}(P)\) and \(P^k P^n = 0\) \((k \geq \text{ind}(P))\), \(X\) in [16, Theorem 5] can be simplified as \(X = \sum_{i=0}^{r-1} (Q^D)^{i+3} P_i P^n + \sum_{i=0}^{r-1} Q^r Q (P^D)^{i+3} - (Q^D)^2 P^D - Q^D (P^D)^2\), where \(r = \text{ind}(P), s = \text{ind}(Q)\). Thus, the representation of \((P + Q)^D\) in [16, Theorem 5] is reduced to the formula of (11).

### 3. Application to Bounded Operator Matrices

Let \(Y, Z\) be Banach spaces, and let \(A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\) be a bounded linear operator matrix on \(Y \times Z\). In the following, we illustrate an application of our results to establish representations for \(A^D\) under some conditions.
Lemma 3.1. [8] Let \( M_1 = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \), \( M_2 = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \) be operator matrices. If \( \operatorname{ind}(A) = a \), \( \operatorname{ind}(D) = d \), then \( M_1 \) and \( M_2 \) are Drazin invertible, and

\[
M_1^D = \begin{pmatrix} A^D & 0 \\ X_1 & D^D \end{pmatrix}, \quad M_2^D = \begin{pmatrix} A^D & X_2 \\ 0 & D^D \end{pmatrix},
\]

where \( X_1 = D^n \sum_{i=0}^{d-1} D^i C(A^D)^{i+2} + \sum_{i=0}^{a-1} (D^iA)^{i+2} CA^D - D^i CA^D \),

\[
X_2 = A^n \sum_{i=0}^{d-1} A^i B(D^D)^{i+2} + \sum_{i=0}^{a-1} (A^D)^{i+2} BD^i D^\pi - A^D BD^D.
\]

The case \( BC = 0 \), \( BDC = 0 \) and \( BD^2 = 0 \) has been studied in [10] and the case \( ABC = 0 \), \( BDC = 0 \), \( CBC = 0 \) and \( D^2C = 0 \) in [16] for matrices. We focus our attention in the generalization of the mentioned results.

Theorem 3.2. Let \( A \in \mathcal{B}(\mathcal{Y}), D \in \mathcal{B}(\mathcal{Z}) \) be Drazin invertible, \( a = \operatorname{ind}(A) \), \( d = \operatorname{ind}(D) \). Assume that one of the following holds:

1. \( ABC + BDC = 0 \), \( CBC + D^2C = 0 \) and \( D^nC = 0 \) for some integer \( n > 0 \). Further, \( BD^{n-1}C = 0 \) if \( n \) is odd;
2. \( CAB + CBD = 0 \), \( CBC + CA^2 = 0 \) and \( CA^n = 0 \) for some integer \( n > 0 \). Further, \( CA^{n-1}B = 0 \) if \( n \) is odd.

Then the operator matrix \( \mathcal{A} \) is Drazin invertible, and

\[
\mathcal{A}^D = \begin{pmatrix} A^D + BC(A^D)^{i+1} & X + BC(A^D)^{i+2}X + A^i XD^\pi + BCX(D^\pi)^2 \\
C(A^D)^{i+2} + DC(A^D)^{i+1} & D^\pi + CA^i XD^\pi + DC(A^D)^{i+2}X(D^\pi)^2 
\end{pmatrix}.
\]

where \( X = A^n \sum_{i=0}^{d-1} A^i B(D^D)^{i+2} + \sum_{i=0}^{a-1} (A^D)^{i+2} BD^i D^\pi - A^D BD^D \).

Proof. We consider the splitting \( \mathcal{A} = P + Q \), where \( P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \), \( Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \). Then

\[
P^nQ = \begin{pmatrix} \sum_{k=0}^{n-1} A^k BD^{n-1-k}C & 0 \\ D^nC & 0 \end{pmatrix}.
\]

If (1) holds, then \( D^{2k}C = C(-BC)^k \) by \( CBC + D^2C = 0 \). Thus, using \( ABC + BDC = 0 \), we have

\[
\sum_{k=0}^{n-1} A^k BD^{n-1-k}C = \sum_{k=0}^{n-1} (A^{2k+1} BD^{n-2-2k}C + A^{2k} BD^{n-1-2k}C)
\]

\[
= \sum_{k=0}^{n-1} (A^{2k+1} BC(-BC)^{\frac{n-2-2k}{2}} + A^{2k} BDC(-BC)^{\frac{n-1-2k}{2}})
\]

\[
= \sum_{k=0}^{n-1} A^{2k}(ABC + BDC)(-BC)^{\frac{n-2-2k}{2}}
\]

\[
= 0
\]
when \( n \) is even, and
\[
\sum_{k=0}^{n-1} A^k BD^{n-1-k} C = BD^{n-1} C + \sum_{k=1}^{n-1} A^k BD^{n-1-k} C
\]
\[
= \sum_{k=1}^{n-1} \left( A^{2k} BD^{n-1-2k} C + A^{2k-1} BD^{n-2k} C \right)
\]
\[
= \sum_{k=1}^{n-1} \left( A^{2k} BC(BC)^{\frac{n-2k}{2}} + A^{2k-1} BDC(BC)^{\frac{n-2k}{2}} \right)
\]
\[
= \sum_{k=1}^{n-1} A^{2k-1}(ABC + BDC)(BC)^{\frac{n-2k}{2}}
\]
\[
= 0
\]
when \( n \) is odd. So, \( P^n Q = 0 \) according to \( D^n C = 0 \). On the other hand, a straightforward calculation shows that \( P^n Q + QPQ = 0 \). The desired result follows from Theorem 2.5 and Lemma 3.1.

Similarly, if (2) holds, then we conclude that \( QP^2 + QPQ = 0 \) and \( QP^n = 0 \). Therefore, the claim follows from Theorem 2.7. \( \square \)

If we consider the splitting \( M = P + Q \), where \( P = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \) then we obtain the following result.

**Theorem 3.3.** Let \( A \in \mathcal{B}(Y), D \in \mathcal{B}(Z) \) be Drazin invertible, \( a = \text{ind}(A), d = \text{ind}(D) \). Assume that one of the following holds:

1. \( CAB + DCB = 0, BCB + A^2B = 0 \) and \( A^nB = 0 \) for some integer \( n > 0 \), further, \( CA^{n-1}B = 0 \) if \( n \) is odd;
2. \( BCA + BDC = 0, BCB + BD^2 = 0 \) and \( BD^n = 0 \) for some integer \( n > 0 \), further, \( BD^{n-1}C = 0 \) if \( n \) is odd.

Then the operator matrix \( \mathcal{A} \) is Drazin invertible, and
\[
\mathcal{A}^D = \left( \begin{array}{ccc} A^{n} + BXA^{n} + BD^{n}X + AB(D^n)^{X} & X^{+}CBX(A^{n})^{Y} + CBX(A^{n})^{Y} & BD^{n}X + AB(D^n)^{Y} \\ X^{+}CBX(A^{n})^{Y} + CBX(A^{n})^{Y} & D^{n}X + CBX(D^n)^{Y} & BD^{n}X + AB(D^n)^{Y} \end{array} \right)
\]

where \( X = D^{\frac{n}{2}} \sum_{i=0}^{d-1} D^{i}C(A^{D})^{i+2} + \sum_{i=0}^{a-1} (D^{i})^{\frac{a+2}{2}}CA^{i}A^{n} - D^{D}CA^{D} \).

**Acknowledgments**

The author would like to thank the anonymous referees for their very detailed comments and many constructive suggestions which helped to improve the paper.

**References**


