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# Teichmüller Space of a Countable Set of Points on the Riemann Sphere

**Masahiko Taniguchi**[a]

[a]*Department of Mathematics, Nara Women's University*

**Abstract.** We introduce the Teichmüller space $T(E)$ of an ordered countable set $E$ of infinite number of distinct points on the Riemann sphere. We discuss the relation between the Teichmüller distance on $T(E)$ and a natural one on the configuration space for $E$. Also we give a system of global holomorphic coordinates for $T(E)$ when $E$ is determined from a finitely generated semigroup consisting of Möbius transformations with the totally disconnected forward limit set.

## 1. Introduction

Let $E = \{z_k\}_{k=1}^{\infty}$ be an ordered countable set of an infinite number of distinct points on $\widehat{\mathbb{C}}$. We define a natural kind of the deformation space of $E$ as follows.

**Definition 1.1.** Let $QC(E)$ be the set of all ordered countable sets $E' = \{z'_k\}_{k=1}^{\infty}$ of an infinite number of distinct points on $\widehat{\mathbb{C}}$ such that there are quasiconformal self-homeomorphisms $f$ of $\widehat{\mathbb{C}}$ which are *order-preserving from $E$ onto $E'$* in a sense that $f(z_k) = z'_k$ for every $k$.

We say that two points $E_1$ and $E_2$ of $QC(E)$ are *equivalent* if there is a conformal self-homeomorphism, or equivalently a Möbius transformation, $\phi$ of $\widehat{\mathbb{C}}$ which is order-preserving from $E_1$ onto $E_2$.

The *Teichmüller space* $T(E)$ of $E$ consists of all equivalence classes $[E]$ of $E \in QC(E)$.

The Teichmüller metric on $T(E)$ is defined as usual.

**Definition 1.2.** The *Teichmüller distance* between $[E_1]$ and $[E_2]$ in $T(E)$ is defined by setting

$$d_T([E_1], [E_2]) = \inf_g \log K_g,$$

where $g$ moves all quasiconformal self-homeomorphisms $g$ of $\widehat{\mathbb{C}}$ order-preserving from $E_1$ onto $E_2$.

It is clear that the Teichmüller distance $d_T$ is actually a distance, and hence $T(E)$ equipped with $d_T$ is a metric space.

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*Email address:* tanig@cc,nara-wu.ac.jp (Masahiko Taniguchi)

**Remark 1.3.** We can define the Teichmüller space $T(E)$ of an ordered countable set of infinite number of points on a general Riemann surface $R$, and can equip $T(E)$ with a natural complex Banach manifold structure under some conditions. We will discuss it in a forthcoming paper. In this paper, we give some interesting examples in the next section instead.

Without loss of generality, we may assume in the sequel that an ordered countable set $E = \{z_k\}_{k=1}^{\infty}$ is *normalized*, i.e.

$$z_1 = \infty, \quad z_2 = 0, \quad z_3 = 1.$$

Then, every point $[E']$ of $T(E)$ contains a single normalized ordered countable set, say $E'$, and hence in the sequel, we identify $T(E)$ with the set $NQC(E)$ of all normalized ones in $QC(E)$, and write $[E']$ simply as $E'$.

In particular, a simple canonical parametrization of $T(E)$ is defined as follows.

**Definition 1.4.** The *configuration space* of normalized ordered countable sets of infinite number of distinct points on $\mathbb{C} - \{0, 1\}$ is the subset

$$\Sigma = \{\{w_k\}_{k=1}^{\infty} \in \Omega = (\mathbb{C} - \{0, 1\})^{\infty} \mid w_m \neq w_n \text{ if } m \neq n\}$$

of the product space $\mathbb{C}^{\infty}$.

The topology of $\mathbb{C}^{\infty}$ is usually induced from component-wise convergence. In this paper, we assume that the subset $\Omega$, and hence also $\Sigma$, is equipped with the hyperbolic $\ell^{\infty}$ distance defined by

$$d\left(\{w_k\}, \{w'_k\}\right) = \sup_k d_h(w_k, w'_k)$$

for every $\{w_k\}, \{w'_k\} \in \Omega$, where $d_h$ is the hyperbolic distance on $\mathbb{C} - \{0, 1\}$. In general, $\Sigma$ is not necessarily open in $\Omega$.

**Definition 1.5.** Let $E$ be an ordered countable set as above. Then, there is a natural injection

$$\iota : T(E) = NQC(E) \to \Sigma,$$

by sending $E' = \{z'_k\}_{k=1}^{\infty} \in NQC(E)$ to $\{z'_{k+3}\}_{k=1}^{\infty} \in \Sigma$.

We have equipped $T(E)$ with the Teichmüller distance $d_T$, while the image $\iota(T(E))$ with the hyperbolic $\ell^{\infty}$ one. Hence, we need to clarify the relation between these two distances.

Here, we consider the following condition.

**Definition 1.6.** We say that a normalized ordered countable set $E = \{z_k\}$ is *uniformly discrete* if

$$\inf_{m,n \geq 4, m \neq n} d_h(z_m, z_n) > 0.$$

**Example 1.7.** The density of the hyperbolic metric $d_h$ on $\mathbb{C} - \{0, 1\}$ near $\infty$ is comparable with

$$\frac{1}{|z| \log |z|},$$

and hence $z_k = a^{2^k}$ with $a > 1, k \geq 4$ give a uniformly discrete normalized ordered countable set.

Let $UD$ be the subset of $T(E)$ consisting of those which are uniformly discrete.

**Lemma 1.8.** *The subset $UD$ is open in $T(E)$.*

*Proof.* Suppose that $E' = \{z'_k\} \in T(E)$ is uniformly discrete, and set

$$a = \inf_{m,n\geq 4, m\neq n} d_h(z'_m, z'_n) > 0.$$

Then a classical theorem due to Teichmüller (cf. [1]) states that there is an $\epsilon > 0$ such that every $K$-qc self-homeomprphism $f$ of $\mathbb{C}$ fixing $0, 1$ with $K < 1 + \epsilon$ satisfies

$$d_h(z'_m, f(z'_m)) \leq a/3$$

for every $m \geq 4$. Actually, $\epsilon = e^{a/3} - 1$ is available. Thus we conclude the assertion. □

**Theorem 1.9.** *The injection $\iota : T(E) \to \Sigma$ is continuous and non-expanding. Moreover, $\iota$ is a locally bi-Lipschitz homeomorphism of UD onto its image, i.e., for every $E' \in UD$, there are a neighborhood $V$ of $E'$ and an $M > 0$ such that*

$$\frac{1}{M} d_T(E_1, E_2) \leq d(\iota(E_1), \iota(E_2)) \leq M\, d_T(E_1, E_2),$$

*for every $E_1, E_2 \in V$.*

*Proof.* Again by the classical theorem due to Teichmüller stated in the above proof, we see that

$$d(\iota(E_1), \iota(E_2)) \leq d_T(E_1, E_2)$$

for every $E_1, E_2$ in $T(E)$, which means that $\iota$ is continuous and non-expanding.

Next, fix $\{w_k\}_{k=1}^{\infty}$ in $\iota(UD)$. Then the assumption implies that

$$a = \inf_{m\neq n} d_h(w_m, w_n) > 0.$$

In particular, the $(a/3)$-neighborhoods $U_k$ of $w_k$ with respect to the hyperbolic metric $d_h$ are mutually disjoint in $\mathbb{C} - \{0, 1\}$. Here replacing $a/3$ by a smaller positive constant if necessary, we may assume that $U_k$ is either a topological disk or an annulus for every $k$. In the latter case, we can find a positive $a'$ $(< a/3)$, depending only on $a$, such that either the $a'$-neighborhood of $w_k$ is a topological disk or the hyperbolic distance between $w_k$ and the boundary of $U_k$ is greater than $a/4$ for every $k$.

In any case, there exists a positive $\eta$, sufficiently smaller than $a'$, satisfying the following condition: For every $\{w'_k\}$ in

$$V = \{\{w'_k\} \in \Sigma \mid d\left(\{w'_k\}, \{w_k\}\right) < \eta\},$$

we can construct explicitly a normalized quasiconformal self-homeomorphism $f$ of $\mathbb{C}$, order-preserving from $\{w_k\}$ onto $\{w'_k\}$, such that $f$ is the identical map on $\mathbb{C} - \bigcup_k U_k$ and the maximal dilatation of $f$ is bounded by

$$\exp\left(Md(\{w'_k\}, \{w_k\})\right),$$

where $M$ is a positive constant depending only on $a, a'$, and $\eta$.

Thus we conclude that $V \subset \iota(UD)$ and $\iota$ is a locally bi-Lipschitz homeomorphism of $UD$ to its image. □

**Remark 1.10.** The continuous and non-expanding injection $\iota$ seems not always to be locally bi-Lipschitz on $T(E)$. Also see Remark 2.11 below.

On the other hand, the standard complex analytic structure of $T(E)$, if exists, is given by using Beltrami coefficients, and the corresponding normalized quasiconformal homeomorphisms gives holomorphic families (cf. [4]). In particular, components $w_k$ of $\Sigma$ always induce some complex Banach manifold structure on $T(E)$ if $\iota(T(E))$ is open in $\Omega$.

## 2. Countable Sets Ordered by Möbius Action

Let $S$ be a semigroup generated by Möbius transformations of the Riemann sphere $\widehat{\mathbb{C}}$. We give a canonical order to $S$ by using the word length of the reduced word expressions and in addition by giving elements with same length the lexicographic order with respect to the ordered generators. So, we consider $S$ as an ordered set $\{\sigma_k\}_{k=1}^{\infty}$. Here we always assume that the identity $e$ is added to $S$ and $\sigma_1 = e$.

For the sake of simplicity, we restrict ourselves to the case that *$S$ is finitely generated* in the sequel.

**Definition 2.1.** Let $S = [g_1, \cdots, g_n]$ be a semigroup generated by a finite number of ordered non-identical Möbius transformations $g_1, \cdots, g_m$. Assume that $\infty$ is a fixed point of $g_1$ and that $g_k$ have mutually distinct fixed points in $\mathbb{C}$.

We say that an ordered countable set $E$ of an infinite number of distinct points on $\widehat{\mathbb{C}}$ including $\infty$ is *$S$-invariant* if

$$g(E) \subset E \text{ for every } g \in S$$

and

$$\bigcup_{k=1}^{n} g_k(E) \supset E.$$

We say that the *order of $E$ is induced from the $S$-action* if there is an order-preserving injection

$$\tau : E \to S.$$

**Remark 2.2.** The second condition of $S$-invariance is equivalent to

$$E = \bigcup_{k=1}^{n} g_k(E).$$

If $S$ is a group, then the first condition implies that $g(E) = E$ for every $g \in S$, and hence the second condition is unnecessary.

For a countable set $E$ with an order induced from the $S$-action, we consider the Teichüller space $T(E; S)$ of $E$ with $S$-action as follows.

**Definition 2.3.** Let $E, S = [g_1, \cdots, g_n], \tau : E \to S$ be as above and $QC(E; S)$ the set of all triples $(E'; S', \tau')$ of ordered countable sets $E' = \{w_k\}_{k=1}^{\infty}$ of infinite number of distinct points on $\widehat{\mathbb{C}}$, semigroups $S' = [g_1', \cdots, g_n']$ isomorphic to $S$ by the isomorphisms $\sigma_{S'} : S \to S'$ which send $g_k$ to $g_k'$ for every $k$, and order-preserving injections $\tau' : E' \to S'$ such that $\tau'(E') = \sigma_{S'} \circ \tau(E)$ and there are quasiconformal self-homeomorphisms $f$ of $\widehat{\mathbb{C}}$ which equal $\tau'^{-1} \circ \sigma_{S'} \circ \tau$ on $E$.

We say that two points $(E_1; S_1, \tau_1)$ and $(E_2; S_2, \tau_2)$ of $QC(E; S)$ are *equivalent* if there is a conformal self-homeomorphism $\phi$ of $\widehat{\mathbb{C}}$ such that $\phi(E_1) = E_2$ and

$$\tau_2 \circ \phi = (\sigma_{S_2} \circ \sigma_{S_1}^{-1}) \circ \tau_1$$

on $E_1$.

The *Teichmüller space $T(E; S)$ of $E$ with $S$-action* consists of all equivalence classes $[E; S, \tau]$ of $(E'; S', \tau') \in QC(E; S)$, which we write simply by $[E]$ whenever $S'$ and $\tau'$ are clear.

The Teichmüller distance $d_T$ on $T(E; S)$ is defined similarly as before.

Now, we give two typical examples where the closures of all fixed points of elements in $S - \{e\}$ are totally disconnected. In each examples, we can give a system of global coordinates for the standard complex Banach manifold structure on $T(E; S)$.

**Example 2.4.** We consider a semigroup $S = [g_1, \cdots, g_n]$ generated of contractive similarities

$$g_1(z) = \lambda_1 z, \quad g_2(z) = \lambda_2(z-1)+1,$$
$$g_3(z) = \lambda_3(z-\alpha_3)+\alpha_3, \ \cdots, \ g_n(z) = \lambda_n(z-\alpha_n)+\alpha_n.$$

Here, $n \geq 3$, $0 < |\lambda_k| < 1$, and $0, 1, \alpha_3, \cdots, \alpha_n$ are distinct.

Assume that the attractor of the IFS (iterative function system) given by $S$ is dust-like. In other words, the forward limit set $\Lambda(S)$ is totally disconnected. For the backgrounds on IFSs, see [2] and the references of it.

The standard order of $S$ starts with $e, g_1, g_2, \cdots, g_n$, and so on. Let $E = \{z_k\}_{k=1}^{\infty}$ be the set consisting of $z_1 = \infty$ and all other fixed points of elements in $S$. Here, we have already normalized $S$ so that the fixed points $z_2$ and $z_3$ of $g_1$ and $g_2$ are 0 and 1, respectively. Also we set $z_{k+1} = \alpha_k$ for every $k = 3, \cdots, n$.

Define the injection $\tau : E \to S$ as in Definition 2.1 by sending $\infty, 0, 1, \cdots, \alpha_n$, and every other fixed point $z_k \in E \cap \mathbb{C}$ of some $g_{(k)}$ to $e, g_1, \cdots, g_n$, and $g_{(k)}$ for every $k \geq n+2$, respectively, where $g_{(k)}$ is assumed to have the smallest order among all $g \in S - \{e\}$ having $z_k$ as the fixed point, and assume that the order of $E$ is induced from the $S$-action by $\tau$. Then we can define $T(E; S)$.

By the normalization, we have a canonical continuous injection

$$j : T(E; S) \to ND(n)$$

by sending $[E', S' = [g'_1, \cdots, g'_n], \tau']$ to $(g'_1, \cdots, g'_n)$. Here, $ND(n)$ is the subspace of $(\mathrm{CS})^n$ consisting of all $(g_1, \cdots, g_n)$ normalized as above, where CS is the space of all contractive similarities.

**Theorem 2.5 ([2]).** *The Teichmüller space $T(E; S)$ is identified with the dust-likeness locus $DL(n)$ consisting of all $S' = [g'_1, \cdots, g'_n] \in ND(n)$ with totally disconnected $\Lambda(S')$.*

*In particular, $T(E; S)$ is a domain in $ND(n)$, and the $(2n-2)$ fixed points, say $z_4, \cdots, z_{2n+1}$, of*

$$g_3, \cdots, g_n, g_1 g_2, \cdots, g_1 g_n, g_2 g_1,$$

*in $E$ gives a system of global coordinates for $T(E; S)$.*

Actually, the first assertion has been shown as Theorem 1.2 in [2].

Next, by definition, $z_{k+1} = \alpha_k$ for every $k = 3, \cdots, n$, and the fixed points $z_{2n+1}$ of $g_2 g_1$ and $z_{n+k}$ of $g_1 g_k$ with $k \geq 2$ are

$$\frac{\lambda_2 - 1}{\lambda_1 \lambda_2 - 1} \quad \text{and} \quad \frac{\lambda_1 \alpha_k (\lambda_k - 1)}{\lambda_1 \lambda_k - 1},$$

respectively. In particular,

$$\lambda_1 = \frac{z_{n+2}}{z_{2n+1}}, \quad \lambda_2 = \frac{z_{2n+1} - 1}{z_{n+2} - 1},$$

and hence we can conclude that

$$\lambda_k = \frac{z_{2n+1} z_{n+k} - z_{n+2} z_{k+1}}{z_{n+2}(z_{n+k} - z_{k+1})}$$

for $k = 3, \cdots, n$. It is clear that these relations gives a bi-rational homeomorphism of $T(E; S)$ in $\mathbb{C}^{2n-2}$ with coordinates $z_4, \cdots, z_{2n+1}$ onto the domain $DL(n)$ in $\mathbb{C}^{2n-2}$ with coordinates

$$\lambda_1, \cdots, \lambda_n, \alpha_3, \cdots, \alpha_n.$$

**Example 2.6 (cf.[8]).** Let $G = \langle g_1, \cdots, g_n\rangle$ be a Schottky group, i.e., a finitely generated purely loxodromic free discrete group with totally disconnected limit set, generated by ordered $n$ ($\geq 2$) Möbius transformations $g_1, \cdots, g_n$. For the backgrounds on Kleinian groups, see for instance [6].

The standard order of $G$ starts with

$$e, g_1, g_1^{-1}, g_2, g_2^{-1}, \cdots, g_n, g_n^{-1},$$

and so on. We consider the set $E = \{z_k\}_{k=1}^{\infty}$ consisting of all fixed points of elements in $G - \{e\}$. We normalize $E$ so that $z_1 = \infty$ and $z_2 = 0$ are the repelling and the attracting fixed point of $g_1$, and $z_3 = 1$ is the repelling fixed point of $g_2$. Also we assume that $z_4$ is the attracting fixed point of $g_2$, and that $z_{2k-1}$ and $z_{2k}$ are the repelling and the attracting fixed points of $g_k$, respectively, for $k = 3, \cdots, n$. Recall that the sets of the fixed points of different $g \in G - \{e\}$ are mutually distinct and the attracting fixed point of $g$ is the repelling one of $g^{-1}$.

Define the injection $\tau : E \to G$ by sending the repelling fixed point $z_k$ of $g_{(k)}$ to $g_{(k)}$ for every $k \geq 2n + 1$, where $g_{(k)}$ is assumed to have the smallest order among all $g \in G - \{e\}$ having $z_k$ as the repelling fixed points, and assume that the order of $E$ is induced from the $G$-action by $\tau$. Then we can define $T(E; G)$.

By the normalization, we have a canonical continuous injection

$$j : T(E; G) \to \mathrm{NDef}(n)$$

by sending $[E'; G' = \langle g'_1, \cdots, g'_n\rangle, \tau']$ to $(g'_1, \cdots, g'_n)$. Here, $\mathrm{NDef}(n)$ is the subspace of $(PSL(2, \mathbb{C}))^n$ consisting of all $(g_1, \cdots, g_n)$ normalized as above, where $PSL(2, \mathbb{C})$ is the space of all Möbius transformations.

**Theorem 2.7.** *The Teichmüller space $T(E; G)$ is identified with the normalized Schottky locus $NS(n)$ in* $\mathrm{NDef}(n)$ *consisting of all normalized system of ordered generators of Schottky groups canonically isomorphic to G.*

*In particular, $T(E; G)$ is a domain in* $\mathrm{NDef}(n)$ *and $3n - 3$ fixed points*

$$z_4, \cdots z_{2n}, z_{1\cdot 2}, z_{2\cdot 1}, \cdots, z_{n\cdot 1}$$

*in E give a system of global coordinates for $T(E; G)$. Here $z_{k\cdot j}$ is the repelling fixed points of $g_k g_j g_k^{-1}$ for every j and k with $j \neq k$.*

*Proof.* First, recall that $NS(n)$ is a domain in $\mathrm{NDef}(n)$ and it is clear that $NS(n) \subset j(T(E; G))$. On the other hand, take a point $[E'; G', \tau']$ in $j(T(E; G))$. Then the definition of $T(E; G)$ implies that $G'$ is a finitely generated purely loxodromic free subgroup of $PSL(2, \mathbb{C})$ and the set of all fixed points of elements in $G' - \{e\}$ are mutually disjoint and is contained in a totally disconnected closed subset of $\widehat{\mathbb{C}}$. Hence by Lemma 2.8 below, $G'$ is discrete, which means that $j([E'; G', \tau']) \in NS(n)$.

Next, the ordered fix points and the multiplier determine the Möbius transformation. Hence those of $g_1, \cdots, g_n$ give a system of global coordinates for $NS(n)$. On the other hand, the multiplier of $g_1$ is $g_1(1)$, which is the repelling fixed point $z_{1\cdot 2}$ of $g_1 g_2 g_1^{-1}$, and the multipliers of other $g_k$ are given by

$$\frac{z_{k\cdot 1} - z_{2k}}{z_{k\cdot 1} - z_{2k-1}}.$$

In particular, these relations gives a bi-rational homeomorphism of $T(E; G)$ in $\mathbb{C}^{3n-3}$ with coordinates

$$z_4, \cdots z_{2n}, z_{1\cdot 2}, z_{2\cdot 1}, \cdots, z_{n\cdot 1}$$

onto the domain $NS(n)$ in $\mathbb{C}^{3n-3}$ with coordinates consisting of the ordered fixed points and the multipliers of all $g_k$. □

**Lemma 2.8.** *Let G be a subgroup of $PSL(2, \mathbb{C})$ which contains no elliptic elements and fixes no points in $\widehat{\mathbb{C}}$. If the set of all fixed points of elements in $G - \{e\}$ is contained in a proper closed subset of $\widehat{\mathbb{C}}$, then G is discrete.*

*Proof.* By the classification theorem (Proposition of [7]), the assumptions implies that $G$ is either discrete or a non-elementary subgroup of $PSL(2,\mathbb{R})$ without elliptic elements. But it is elementary to show that the latter is also discrete. □

**Remark 2.9.** We can consider an infinitely generated normalized Schottky group $G$ as in [5]. Even in this case, we can consider an ordered set $E$ consisting of all fixed points of elements in $G - \{e\}$ and the corresponding Teichmüller space $T(E;G)$, where some ordered subset of $E$ gives a system of global coordinates of $T(E;G)$.

Note that in the previous proofs of theorems, we could use the following classical variants of global coordinates.

**Definition 2.10.** The *cross-ratio* $\chi(a,b,c,d)$ of distinct 4 points $a,b,c,d$ in $\mathbb{C}$ is defined by

$$\chi(a,b,c,d) = \frac{a-b}{a-c}\frac{d-c}{d-b}$$

(and by taking the limit if one of them is $\infty$).

When $a,b,c,d$ are in $E$, then $\chi(a,b,c,d)$ can be considered as a function on $T(E)$, which we call the *cross-ratio coordinate* for $a,b,c,d \in E$.

Let $CR(E)$ be the set of all cross-ratio coordinates corresponding to 4 distinct ordered points of $E$ with the induced order, which are clearly countable. Without using the normalization, $CR(E)$ gives an continuous injection $CR : T(E) \to \Omega$.

**Remark 2.11.** We can define a metric on $T(E)$ by using $CR(E)$, which we called the CR-metric in [2]. Again, we know that the CR metric subordinates to the Teichmüller metric.

Finally, recall that, if $T(E)$ is finite-dimensional, then $T(E)$ is locally compact. Hence we can consider a natural kind of compactification of it.

**Definition 2.12.** Let $X$ be a locally compact, but non-compact, Hausdorff space, and $\mathcal{F}$ be a set of continuous maps of $X$ to $\widehat{\mathbb{C}}$.

Then, a compactification $X^*$ of $X$ such that every element in $\mathcal{F}$ can be extended to a continuous map of $X^*$ and that the family of all extended maps separates points of $X^* - X$ is called an *$\mathcal{F}$-compactification* of $X$.

**Proposition 2.13.** *There exists an $CR(E)$-compactification $T(E)^*$ of $T(E)$, which is unique up to homeomorphisms fixing $T(E)$ point-wise.*

For the reference and the proof, see [3].

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