Filomat 31:9 (2017), 2891–2900
https://doi.org/10.2298/FIL1709891A

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Weyl Type Theorems for Complex Symmetric Operator Matrices

Il Ju An[a], Eungil Ko[b], Ji Eun Lee[c]

[a] *Hanyang University, Department of Mathematics*
[b] *Ewha Womans University, Department of Mathematics*
[c] *Sejong University, Department of Mathematics-Applied Statistics*

Abstract. In this paper, we study Weyl type theorems for complex symmetric operator matrices. In particular, we give a necessary and sufficient condition for complex symmetric operator matrices to satisfy a-Weyl's theorem. Moreover, we also provide the conditions for such operator matrices to satisfy generalized a-Weyl's theorem and generalized a-Browder's theorem, respectively. As some applications, we give various examples of such operator matrices which satisfy Weyl type theorems.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators acting on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_s(T)$, and $\sigma_a(T)$ for the spectrum, the point spectrum, the surjective spectrum, and the approximate point spectrum of T, respectively.

If $T \in \mathcal{L}(\mathcal{H})$, we shall write $N(T)$ and $R(T)$ for the null space and the range of T, respectively. Also, let $\alpha(T) := dimN(T)$ and $\beta(T) := dimN(T^*)$, respectively. For $T \in \mathcal{L}(\mathcal{H})$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the *descent* of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. For any conjugation C, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [7] for more details). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that T is complex symmetric with conjugation C. This concept is due to the fact that T is a complex symmetric operator if and only if it is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension (see [7]). All normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators,

2010 *Mathematics Subject Classification*. Primary 47A10, 47A53, 47A55
Keywords. Weyl type theorems; Complex symmetric operator matrices
Received: 28 November 2015; Accepted: 19 August 2016
Communicated by Dragan S. Djordjević
The first author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT and future Planning(2015027497). This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2009-0093827). This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A3006841).
Email addresses: dlfwn79@khu.ac.kr (Il Ju An), eiko@ewha.ac.kr (Eungil Ko), jieun7@ewhain.net: jieunlee7@sejong.ac.kr (Ji Eun Lee)

and some Volterra integration operators are included in the class of complex symmetric operators. We refer the reader to [7]-[9] for more details.

The Weyl type theorems for upper triangular operator matrices have been studied by many authors. In general, even though Weyl type theorems hold for entry operators T_1 and T_2, neither $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ nor $\begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}$ satisfies Weyl type theorems (see [10], [11], [13], [14], [3], and ect.). So many authors have been studied the relation between a diagonal matrix and an upper triangular operator matrix of Weyl type theorems. Recently, in [17], they provide several forms of complex symmetric operator matrices $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ and have studied a-Weyl's theorem and a-Browder's theorem for complex symmetric operator matrices $\begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix}$. We now consider how Weyl type theorems hold for upper triangular operator matrices when some entry operators are complex symmetric.

In this paper, we focus on the operator matrix $\begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ when B is complex symmetric with the conjugation C. In this case, we are interested in which the operator matrix $\begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix}$ satisfies Weyl type theorems under what behavior of the entry operator A. In particular, we give a necessary and sufficient condition for this complex symmetric operator matrices to satisfy a-Weyl's theorem. Moreover, we also provide the conditions for such operator matrices to satisfy generalized a-Weyl's theorem and generalized a-Browder's theorem, respectively. As some applications, we give various examples of such operator matrices which satisfy Weyl type theorems.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in \mathcal{L}(\mathcal{H})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and *index of a semi-Fredholm operator* $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm of finite ascent and descent, respectively. The left essential spectrum $\sigma_{SF+}(T)$, the right essential spectrum $\sigma_{SF-}(T)$, the essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined as follows;

$$\sigma_{SF+}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm}\},$$

$$\sigma_{SF-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not lower semi-Fredholm}\},$$

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_{SF+}(T) \cup \sigma_{SF-}(T) = \sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write acc Δ for the accumulation points of $\Delta \subseteq \mathbb{C}$. If we write iso $\Delta = \Delta \setminus \operatorname{acc} \Delta$, then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty \},$$

and $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$. We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and that *Browder's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$. We recall the definitions of some spectra;

$$\sigma_{ea}(T) := \cap\{\sigma_a(T + K) : K \in \mathcal{K}(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in \mathcal{K}(\mathcal{H})\}$$

is the Browder essential approximate point spectrum. We put

$$\pi^a_{00}(T) := \{\lambda \in \text{iso}\, \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}$$

and $p^a_{00}(T) = \sigma_a(T) \setminus \sigma_{ab}(T)$.

Let $T \in \mathcal{L}(\mathcal{H})$. We say that *a-Browder's theorem holds for* T if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = p^a_{00}(T),$$

and *a-Weyl's theorem holds for* T if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi^a_{00}(T).$$

It is known that

$$a\text{-Weyl's theorem} \Longrightarrow a\text{-Browder's theorem} \Longrightarrow \text{Browder's theorem},$$

$$a\text{-Weyl's theorem} \Longrightarrow \text{Weyl's theorem} \Longrightarrow \text{Browder's theorem}.$$

Let $T_n = T|_{R(T^n)}$ for each nonnegative integer n; in particular, $T_0 = T$. If T_n is upper semi-Fredholm for some nonnegative integer n, then T is called a *upper semi-B-Fredholm operator*. In this case, by [4], T_m is a upper semi-Fredholm operator and $ind(T_m) = ind(T_n)$ for each $m \geq n$. Thus, we can consider the *index* of T as the index of the semi-Fredholm operator T_n. Similarly, we define *lower semi-B-Fredholm operators*. We say that $T \in \mathcal{L}(\mathcal{H})$ is *B-Fredholm* if it is both upper and lower semi-B-Fredholm. Let $SBF^-_+(\mathcal{H})$ be the class of all upper semi-B-Fredholm operators such that $ind(T) \leq 0$, and let

$$\sigma_{SBF^-_+}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF^-_+(\mathcal{H})\}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *B-Weyl* if it is B-Fredholm of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a } B\text{-Weyl operator }\}.$$

In addition, we state two spectra as follows;

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} | \, T - \lambda \notin LD(\mathcal{H})\},$$

$$\sigma_{RD}(T) = \{\lambda \in \mathbb{C} | \, T - \lambda \notin RD(\mathcal{H})\},$$

where $LD(\mathcal{H}) = \{T \in \mathcal{H} | \, p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed}\}$, and $RD(\mathcal{H}) = \{T \in \mathcal{H} | \, q(T) < \infty \text{ and } R(T^{q(T)}) \text{ is closed}\}$. The notation $p_0(T)$ (respectively, $p^a_0(T)$) denotes the set of all poles (respectively, left poles) of T, while $\pi_0(T)$ (respectively, $\pi^a_0(T)$) is the set of all eigenvalues of T which is an isolated point in $\sigma(T)$ (respectively, $\sigma_a(T)$).

Let $T \in \mathcal{L}(\mathcal{H})$. We say that

(i) T satisfies *generalized Browder's theorem* if $\sigma(T) \setminus \sigma_{BW}(T) = p_0(T)$;

(ii) T satisfies *generalized a-Browder's theorem* if $\sigma_a(T) \setminus \sigma_{SBF^-_+}(T) = p^a_0(T)$;

(iii) T satisfies *generalized Weyl's theorem* if $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$;

(iv) T satisfies *generalized a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SBF^-_+}(T) = \pi^a_0(T)$.

It is known that

$$\begin{array}{ccc} \text{generalized } a\text{-Weyl's theorem} & \Longrightarrow & \text{generalized Weyl's theorem} \\ \Downarrow & & \Downarrow \\ \text{generalized } a\text{-Browder's theorem} & \Longrightarrow & \text{generalized Browder's theorem.} \end{array}$$

An operator $T \in \mathcal{L}(\mathcal{H})$ has the *single-valued extension property at* $\lambda_0 \in \mathbb{C}$ if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation $(T - \lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$ on U. The operator T is said to have the *single-valued extension property* if T has the single-valued extension property at every $\lambda_0 \in \mathbb{C}$.

3. Wyel Type Theorem

In this section, we study Weyl type theorems for complex symmetric operator matrices. In [17], they provide several forms of complex symmetric operator matrices $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$. Indeed, if C is a conjugation on $\mathcal{H}$, then $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ is complex symmetric with $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ if and only if T_2 and T_3 are complex symmetric with a conjugation C and $T_4 = C{T_1}^*C$. For example, the complex symmetric operator matrix $\begin{pmatrix} S^* & 0 \\ 0 & S \end{pmatrix}$ does not satisfy Weyl's theorem where S is the unilateral shift on $\mathcal{H}$. They also have studied a-Weyl's theorem and a-Browder's theorem for complex symmetric operator matrices $\begin{pmatrix} T_1 & T_2 \\ 0 & C{T_1}^*C \end{pmatrix}$. In this paper, we study generalized Weyl theorem and generalized a-Weyl theorem for complex symmetric operator matrices $\begin{pmatrix} T_1 & T_2 \\ T_3 & C{T_1}^*C \end{pmatrix}$ where C is a conjugation on $\mathcal{H}$. Put $\Delta^* := \{\overline{z} : z \in \Delta\}$ for any set Δ in $\mathbb{C}$. For our study, we start with the following lemmas.

Lemma 3.1. ([17]) *If C is a conjugation on $\mathcal{H}$ and $A \in \mathcal{L}(\mathcal{H})$, then the following identities hold:*
(i) $\sigma(A)^* = \sigma(CAC)$, $\sigma_p(A)^* = \sigma_p(CAC)$, $\sigma_a(A)^* = \sigma_a(CAC)$, *and* $\sigma_s(A) = \sigma_s(CAC)^*$.
(ii) $\sigma_e(A)^* = \sigma_e(CAC)$, *and* $\sigma_w(A)^* = \sigma_w(CAC)$.

Remark that if S is a complex symmetric operator with the conjugation C, then it is known from [16, Lemma 3.5] that S has the single-valued extension property if and only if S^* has. With the similar proof of [16], we have the following lemma.

Lemma 3.2. *Let C be a conjugation on $\mathcal{H}$ and $S \in \mathcal{L}(\mathcal{H})$. Then S has the single-valued extension property if and only if CSC has.*

Lemma 3.3. *If C is a conjugation on $\mathcal{H}$ and $A \in \mathcal{L}(\mathcal{H})$, then the following identities hold:*
(i) $\sigma_b(A)^* = \sigma_b(CAC)$ *and* $\sigma_D(A)^* = \sigma_D(CAC)$.
(ii) $\sigma_{LD}(A)^* = \sigma_{LD}(CAC)$ *and* $\sigma_{RD}(A) = \sigma_{RD}(CAC)^*$.
(iii) $\sigma_{BF}(A)^* = \sigma_{BF}(CAC)$ *and* $\sigma_{BW}(A)^* = \sigma_{BW}(CAC)$.

Proof. (i) Let $\lambda \notin \sigma_b(A)^*$. Then $A - \overline{\lambda}$ is Fredholm and we can let $p(A - \overline{\lambda}) = q(A - \overline{\lambda}) = n < \infty$. By Lemma 3.1(ii), we know that $CAC - \lambda$ is Fredholm. Now we will prove that $N((CAC - \lambda)^n) = N((CAC - \lambda)^{n+1})$. Since $N((CAC - \lambda)^n) \subseteq N((CAC - \lambda)^{n+1})$, it suffices to show that $N((CAC - \lambda)^{n+1}) \subseteq N((CAC - \lambda)^n)$. If $x \in N((CAC - \lambda)^{n+1})$, then $(CAC - \lambda)^{n+1}x = 0$ yields $(A - \overline{\lambda})^{n+1}Cx = 0$. This means that $Cx \in N((A - \overline{\lambda})^{n+1}) = N((A - \overline{\lambda})^n)$. Thus $(A - \overline{\lambda})^nCx = 0$ and so $(CAC - \lambda)^nx = 0$. Therefore, $x \in N((CAC - \lambda)^n)$. Hence

$N((CAC-\lambda)^n) = N((CAC-\lambda)^{n+1})$. So $CAC-\lambda$ has finite ascent. On the other hand, we will show that $R(CAC-\lambda)^n \subset R(CAC-\lambda)^{n+1}$. If $y \in R(CAC-\lambda)^n$, set $y = (CAC-\lambda)^n x$ for some $x \in \mathcal{H}$. Since

$$Cy = C(CAC-\lambda)^n x = (A-\overline{\lambda})^n Cx \in R(A-\overline{\lambda})^n = R(A-\overline{\lambda})^{n+1},$$

there is $z \in \mathcal{H}$ such that $Cy = (A-\overline{\lambda})^{n+1}z$. Thus

$$y = C(A-\overline{\lambda})^{n+1}z = (CAC-\lambda)^{n+1}Cz \in R(CAC-\lambda)^{n+1},$$

and $R(CAC-\lambda)^n \subset R(CAC-\lambda)^{n+1}$. Since the opposite inclusion obviously satisfies, $CAC-\lambda$ has finite descent. The converse holds using a similar way. Hence $\sigma_b(A)^* = \sigma_b(CAC)$. From this, we also know that $\sigma_D(A)^* = \sigma_D(CAC)$.

(ii) Since $\sigma_{RD}(T) = \sigma_{LD}(T^*)^*$ for any $T \in \mathcal{L}(\mathcal{H})$, we only consider $\sigma_{LD}(A)^* = \sigma_{LD}(CAC)$. From the proof of (i), $p(A-\overline{\lambda}) < \infty$ if and only if $p(CAC-\lambda) < \infty$. Set $k = p(A-\overline{\lambda}) + 1$. Then $p(CAC-\lambda) + 1 = k$. Assume that $R((CAC)^k - \lambda))$ is closed. If $y \in \overline{R(A^k - \overline{\lambda})}$, then choose a sequence $\{x_n\} \subset \mathcal{H}$ such that $\lim_{n\to\infty}(A^k - \overline{\lambda})x_n = y$ in norm. This gives that

$$Cy = \lim_{n\to\infty} C(A^k - \overline{\lambda})x_n = \lim_{n\to\infty}((CAC)^k - \lambda)Cx_n.$$

Thus $Cy \in \overline{R((CAC)^k - \lambda)} = R((CAC)^k - \lambda)$ and so $y \in R(A^k - \overline{\lambda})$. Hence $R(A^k - \overline{\lambda})$ is closed. The reverse implication follows in a similar way. Therefore, $\mathrm{R}(A^{p(A-\overline{\lambda})+1} - \overline{\lambda})$ is closed if and only if $\mathrm{R}((CAC)^{p(CAC-\lambda)+1} - \lambda)$ is closed. Hence we conclude that $\sigma_{LD}(A)^* = \sigma_{LD}(CAC)$.

(iii) Let $\lambda \notin \sigma_{BF}(CAC)$. Then, from [4, Theorem 2.7], $CAC-\lambda = \begin{pmatrix} S & 0 \\ 0 & N \end{pmatrix}$ where S is Fredholm and N is a nilpotent. Put $C = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ where J is a conjugation. Then it follows that $\overline{\lambda} \notin \sigma_{BF}(A)$. The reverse implication follows in a similar method. Hence $\sigma_{BF}(A)^* = \sigma_{BF}(CAC)$.

Let $\lambda \notin \sigma_{BW}(CAC)$. Then, from [4], $CAC-\lambda = \begin{pmatrix} S & 0 \\ 0 & N \end{pmatrix}$ where S is Weyl and N is a nilpotent. Put $C = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ where J is a conjugation. Then $A-\overline{\lambda} = C(CAC-\lambda)C = \begin{pmatrix} JSJ & 0 \\ 0 & JNJ \end{pmatrix}$. Since S is Weyl and N is a nilpotent, it follows that JSJ is Weyl and JNJ is a nilpotent. Therefore, $\overline{\lambda} \notin \sigma_{BW}(A)$ from [4]. The reverse implication follows in a similar method. Hence $\sigma_{BW}(A)^* = \sigma_{BW}(CAC)$. So, this completes the proof. □

Throughout this paper, for operators $A, B \in \mathcal{L}(\mathcal{H})$ and a conjugation C on $\mathcal{H}$, put $M(A,B) = \left\{ \begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) : B \text{ is complex symmetric with the conjugation } C \right\}$. We study a-Weyl theorem and generalized a-Weyl theorem for complex symmetric operator matrices in $M(A,B)$.

Theorem 3.4. *Let $T \in M(A,B)$. Suppose that A is complex symmetric which has the single-valued extension property.*

(a) *Then the following statements are equivalent;*
(i) *A satisfies Weyl's theorem.*
(ii) *A satisfies a-Weyl's theorem.*
(iii) *T satisfies Weyl's theorem.*
(iv) *T satisfies a-Weyl's theorem.*

(b) *Then the following statements are equivalent;*
(i) *A satisfies generalized Weyl's theorem.*
(ii) *A satisfies generalized a-Weyl's theorem.*
(iii) *T satisfies generalized Weyl theorem.*
(vi) *T satisfies generalized a-Weyl theorem.*

Proof. (a) (i)$\Longleftrightarrow$(ii): Since A is complex symmetric, it follows from [17, Lemma 3.22] that $\sigma(A) = \sigma_a(A)$ and $\sigma_w(A) = \sigma_{ea}(A)$. So (i)$\Longleftrightarrow$(ii) is obvious.

(iii)$\Longleftrightarrow$(iv): Since B is complex symmetric with the conjugation C, it follows that T is also complex symmetric with the conjugation $\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$. Again by [17], $\sigma(T) = \sigma_a(T)$ and $\sigma_w(T) = \sigma_{ea}(T)$. So (iii)$\Longleftrightarrow$(iv) clearly holds.

(i)$\Longleftrightarrow$(iii) Since A is complex symmetric and has the single-valued extension property, it follows from [17] that A^* has the single-valued extension property. Therefore CA^*C has the single-valued extension property from Lemma 3.2. Hence we get that

$$\sigma_a(T) = \sigma_a(A) \cup \sigma_a(CA^*C) \text{ and } \sigma_e(T) = \sigma_e(A) \cup \sigma_e(CA^*C) \tag{1}$$

from [22, Page 4-5]. Combining Lemma 3.1 with (1) and [17, Lemma 3.22], we obtain that

$$\begin{cases} \sigma(T) = \sigma_a(T) = \sigma_a(A) \cup \sigma_a(A^*)^* = \sigma(A) \\ \sigma_w(T) = \sigma_e(T) = \sigma_e(A) \cup \sigma_e(A^*)^* = \sigma_e(A) = \sigma_w(A). \end{cases}$$

Now, we show that $\pi_{00}(T) = \pi_{00}(A)$. Since $\sigma(T) = \sigma(A)$, we only need to show that

$$0 < \alpha(T - \lambda) < \infty \Longleftrightarrow 0 < \alpha(A - \lambda) < \infty,$$

for every $\lambda \in \mathrm{iso}\sigma(T) = \mathrm{iso}\sigma(A)$. Note that from [12] that

$$\begin{aligned} N(A-\lambda) \oplus \{0\} &\subseteq N(T-\lambda) \\ &\subseteq (A-\lambda)^{-1}(B(N(CA^*C-\lambda))) \oplus N(CA^*C-\lambda). \end{aligned} \tag{2}$$

From the first inclusion in (2), we know that $0 < \alpha(T - \lambda) < \infty \Longrightarrow 0 < \alpha(A - \lambda) < \infty$. For the reverse, let $0 < \alpha(A - \lambda) < \infty$ for $\lambda \in \mathrm{iso}\sigma(A)$. Since A is complex symmetric, it follows from [15, Lemma 4.3] that $0 < \alpha(A^* - \overline{\lambda}) < \infty$. Now, we will show that $\alpha(CA^*C - \lambda) < \infty$. If $\alpha(A^* - \overline{\lambda}) = k < \infty$, then we can choose a linearly independent set $\{e_1, e_2, \cdots, e_k\}$ in $N(A^* - \overline{\lambda})$. If $\sum_{i=1}^k a_iCe_i = 0$ for $a_1, a_2, \cdots, a_k \in \mathbb{C}$, then $0 = C\Big(\sum_{i=1}^k a_iCe_i\Big) = \sum_{i=1}^k \overline{a_i}e_i$, and so $a_i = 0$ for all $i = 1, 2, \cdots, k$. Thus $\{Ce_1, Ce_2, \cdots, Ce_k\}$ are linearly independent set in $CN(A^* - \overline{\lambda}) = N(CA^*C - \lambda)$. Hence $\alpha(CA^*C - \lambda) = k = \alpha(A^* - \overline{\lambda})$. Moreover, if $CA^*C - \lambda$ is injective, then it follows from (2) that $A - \lambda$ is also injective. This is a contradiction, so that $0 < \alpha(CA^*C - \lambda)$. This means that $0 < \alpha(T - \lambda) < \infty$. Hence $\pi_{00}(T) = \pi_{00}(A)$. Therefore this completes the proof.

(b) (i)$\Longleftrightarrow$(ii) Since A is complex symmetric, it follows from [18, Theorem 4.4] that $\sigma(A) = \sigma_a(A)$ and $\sigma_{SBF_+^-}(A) = \sigma_{BW}(A)$. So this implication is obvious.

(iii)$\Longleftrightarrow$(iv) Since B is complex symmetric with the conjugation C, it follows that T is also complex symmetric with the conjugation $\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$. Again by [18], $\sigma(T) = \sigma_a(T)$ and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. So this relations is clear.

(i)$\Longleftrightarrow$(iii) Since A is complex symmetric and has the single-valued extension property, it follows that CA^*C has the single-valued extension property from Lemma 3.2 and [17]. Moreover, T also has the single-valued extension property. By the proof of Theorem 3.4, we know that $\sigma(T) = \sigma(A)$. Now, it suffices to show that $\sigma_{BW}(T) = \sigma_{BW}(A)$ and $\pi_0(T) = \pi_0(A)$. For the first equality, without loss of generality, we let $0 \notin \sigma_{BW}(T)$. Then T is B-Weyl. Since T has the single-valued extension property at 0, it follows that T is Drazin invertible by [2]. Therefore, T has finite ascent and descent.

Claim If $T = \begin{pmatrix} A & B \\ 0 & CA^*C \end{pmatrix}$ has finite descent, then A has finite descent.

Let $q(T) := k$ for any positive integer k. We now claim that $q(CA^*C) = k$, which we only need to prove that $R(CA^{*k}C) \subseteq R(CA^{*k+1}C)$. Let $z \in R(CA^{*k}C)$. Then $z = CA^{*k}Cy$ for some $y \in \mathcal{H}$. Since $T^k(0 \oplus y) \in R(T^{k+1})$, there exists $x_0 \oplus y_0 \in \mathcal{H} \oplus \mathcal{H}$ such that

$$\begin{pmatrix} A^k & A^{k-1}B + \cdots + BCA^{*k-1}C \\ 0 & CA^{*k}C \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} A^{k+1} & A^kB + \cdots + BCA^{*k}C \\ 0 & CA^{*k+1}C \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

It follows that

$$\begin{aligned} & (A^{k-1}By + A^{k-2}BCA^*Cy + \cdots + BCA^{*k-1}Cy) \oplus CA^{*k}Cy \\ = \; & (A^{k+1}x_0 + A^kBy_0 + A^{k-1}BCA^*Cy_0 + \cdots + BCA^{*k}Cy_0) \oplus CA^{*k+1}Cy_0. \end{aligned}$$

Then $z = CA^{*k+1}Cy_0 \in R(CA^{*k+1}C)$. Hence $R(CA^{*k}C) \subseteq R(CA^{*k+1}C)$ and this implies that $q(CA^*C) = k < \infty$. Then $q(A^*) = k < \infty$ by Lemma 3.3 (i). Since A is complex symmetric, it follows from [15, Lemma 4.2] that $q(A) = k < \infty$. This completes the proof of this lemma.

Since T and A have the single-valued extension property, $\sigma_{BW}(T) = \sigma_D(T)$ and $\sigma_{BW}(A) = \sigma_D(A)$. Moreover, since $\sigma_D(T) \subseteq \sigma_D(A) \cup \sigma_D(CA^*C)$, it follows that $\sigma_{BW}(T) =\subset \sigma_{BW}(A)$. The reverse inclusion is trivial. Hence $\sigma_{BW}(T) = \sigma_{BW}(A)$.

For the second equality, it suffices to show that $\alpha(T-\lambda) > 0$ if and only if $\alpha(A-\lambda) > 0$. Since $\alpha(A-\lambda) > 0$ implies $\alpha(T-\lambda) > 0$, we consider the reverse implication. If $\alpha(T-\lambda) > 0$, then $\alpha(A-\lambda) > 0$ or $\alpha(CA^*C-\lambda) > 0$. But, since A is complex symmetric, we know that $A - \lambda$ is one-to-one if and only if $A^* - \overline{\lambda}$ is one-to-one if and only if $CA^*C - \lambda$ is one-to-one. Hence $\alpha(A-\lambda) > 0$ and, therefore, $\pi_0(T) = \pi_0(A)$. So this completes the proof. □

Let us recall that the Hilbert Hardy space, denoted by H^2, consists of all analytic functions f on the open unit disk $\mathbb{D}$ with the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

It is clear that $H^2 = \overline{\text{span}\{z^n : n = 0, 1, 2, 3, \cdots\}}$.

For any $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by the formula

$$T_\varphi f = P(\varphi f)$$

for $f \in H^2$ where P denotes the orthogonal projection of L^2 onto H^2. Let C_1 and C_2 be the conjugations on H^2 given by

$$(C_1 f)(z) = \overline{f(\overline{z})} \text{ and } (C_2 f)(z) = \overline{f(-\overline{z})}$$

for all $f \in H^2$, respectively.

Corollary 3.5. *Let C_1 and C_2 be the conjugations on H^2 given by $(C_1 f)(z) = \overline{f(\overline{z})}$ and $(C_2 f)(z) = \overline{f(-\overline{z})}$ for all $f \in H^2$. Suppose that*

$$T = \begin{pmatrix} T_\varphi & T_\psi \\ 0 & C_1 {T_\varphi}^* C_1 \end{pmatrix} \text{ or } T = \begin{pmatrix} T_\psi & T_\varphi \\ 0 & C_2 {T_\psi}^* C_2 \end{pmatrix}$$

are in $\mathcal{L}(H^2 \oplus H^2)$ where

$$\begin{cases} \varphi(z) = \varphi_0 + 2\sum_{k=1}^{\infty} \hat{\varphi}(2k) Re\{z^{2k}\} + 2i \sum_{k=1}^{\infty} \hat{\varphi}(2k-1) Im\{z^{2k-1}\} \\ \psi(z) = \psi_0 + 2\sum_{n=1}^{\infty} \hat{\psi}(n) Re\{z^n\}. \end{cases} \tag{3}$$

If T_φ or T_ψ have the single-valued extension property, then T satisfies a-Weyl's theorem.

Proof. Suppose that φ and ψ have the forms in (3). Then, by [19, Corollary 2.6], T_φ and T_ψ are complex symmetric with conjugations C_2 and C_1, respectively. Since T_φ satisfies Weyl's theorem by Coburn's theorem, it follows that T satisfies a-Weyl's theorem from Theorem 3.4. □

Example 3.6. Let C be a conjugation on $l^2(\mathbb{Z})$ given by $Cx = \overline{x}$ for all x and let U_1 and U_2 be bilateral shifts on $l^2(\mathbb{Z})$. Then $\begin{pmatrix} U_1 & U_2 \\ 0 & CU_1^*C \end{pmatrix} \in \mathcal{L}(l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}))$ satisfies a-Weyl's theorem from Theorem 3.4.

Corollary 3.7. *Let $T \in M(N, B)$ where N is normal and $B = CB^*C$ for a conjugation C. Then T satisfies generalized a-Weyl theorem.*

Proof. Since N is normal, it follows that N is complex symmetric and has the single-valued extension property. Thus N satisfies generalized Weyl's theorem. Hence T satisfies generalized a-Weyl theorem from Theorem 3.4. □

From the similar way with the proof of Theorem 3.4 and [18, Theorem 4.6], we get the following corollary.

Corollary 3.8. *Let $T \in M(A, B)$. If A is complex symmetric which has the single-valued extension property, then the following statements are equivalent;*
(i) *A satisfies Browder's theorem.*
(ii) *A satisfies a-Browder's theorem.*
(iii) *A satisfies generalized Browder's theorem.*
(iv) *A satisfies generalized a-Browder's theorem.*
(v) *T satisfies Browder's theorem.*
(vi) *T satisfies a-Browder's theorem.*
(vii) *T satisfies generalized Browder's theorem.*
(viii) *T satisfies generalized a-Browder's theorem.*

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if every $\lambda \in \text{iso}\sigma(T)$ is an eigenvalue of T. In [17], they proved that if $T \in M(A, B)$ where A and A^* are isoloid operators with the single-valued extension property and if Weyl's theorem holds for both A and A^*, then a-Weyl's theorem holds for T. Finally, we consider complex symmetric operator matrices where main diagonal operators are not complex symmetric.

Theorem 3.9. *Let $T \in M(A, B)$ where A and A^* have the single-valued extension property. Then the following statements hold:*
(a) *If A satisfies generalized Weyl theorem, then T satisfies generalized a-Weyl theorem.*
(b) *If A is isoloid, then the following statements are equivalent;*
(i) *A satisfies generalized Weyl theorem.*
(ii) *A satisfies generalized a-Weyl theorem.*
(iii) *T satisfies generalized Weyl theorem.*
(iv) *T satisfies generalized a-Weyl theorem.*
(c) *If A is isoloid, then the following statements are equivalent;*
(i) *A and A^* satisfies Weyl's theorem.*
(ii) *T satisfies Weyl's theorem.*
(iii) *T satisfies a-Weyl theorem.*

Proof. (a) Suppose A satisfies generalized Weyl theorem. Since A and A^* have the single-valued extension property, T has the single-valued extension property from 3.3. Since T is complex symmetric, T satisfies generalized a-Browder's theorem. Then

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \pi_0^a(T).$$

Now we will show the converse inclusion. If $\lambda \in \pi_0^a(T)$, then $\lambda \in \text{iso}\sigma_a(T)$ and $\alpha(T - \lambda) > 0$. Since $\sigma_a(A) \subset \sigma_a(T)$, it follows that $\lambda \in \text{iso}\sigma_a(A)$ and $\alpha(A-\lambda) > 0$. Moreover, since A^* has the single-valued extension property, $\lambda \in \text{iso}\sigma(A)$. Thus $\lambda \in \pi_0(A)$. Since A satisfies generalized Weyl's theorem, $\lambda \in \sigma(A) \setminus \sigma_{BW}(A)$.

But, since A has the single-valued extension property, $\sigma_{BW}(A) = \sigma_D(A)$ from [2]. So, $\lambda \notin \sigma_D(A) = \sigma_D(A^*)^* = \sigma_D(CA^*C)$ from Lemma 3.3. But, $\sigma_D(T) \subset \sigma_D(A) \cup \sigma_D(CA^*C)$, so that $\lambda \notin \sigma_D(T)$. Since T has the single-valued extension property, $\lambda \notin \sigma_{BW}(T)$. Hence $\lambda \notin \sigma_{SBF_+^-}(T)$. So T satisfies generalized a-Weyl theorem.

(b) Since A^* has the single-valued extension property, it follows that $\sigma(A) = \sigma_a(A)$. The implication (i)$\Longleftrightarrow$(ii) holds clearly. Since T is complex symmetric, then (iii)$\Longleftrightarrow$(iv) holds from [18].

(i)$\Longleftrightarrow$(iii): We will show that if T satisfies generalized Weyl theorem, then A satisfies generalized Weyl theorem. Let $\lambda \in \sigma(A) \setminus \sigma_{BW}(A)$. Since A has the single-valued extension property, $\lambda \notin \sigma_D(A)$. But, $\sigma_D(A) = \sigma_D(A^*)^* = \sigma_D(CA^*C)$ from Lemma 3.3 so that $\lambda \notin \sigma_D(CA^*C)$. Since $\sigma_D(T) \subset \sigma_D(A) \cup \sigma_D(CA^*C)$, it follows that $\lambda \notin \sigma_D(T) = \sigma_{BW}(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$. Hence $\lambda \in \mathrm{iso}\sigma_a(T)$ and $\alpha(T - \lambda) > 0$. Since $\sigma_a(A) \subset \sigma_a(T) = \sigma(T)$, $\lambda \in \mathrm{iso}\sigma_a(A)$. But, A^* has the single-valued extension property, $\lambda \in \mathrm{iso}\sigma(A)$. Since A is isoloid, $\alpha(A - \lambda) > 0$ and so $\lambda \in \pi_0(A)$.

For the converse inclusion, let $\lambda \in \pi_0(A)$. Then $\lambda \in \mathrm{iso}\sigma(A)$ and $\alpha(A - \lambda) > 0$. But, Since $\sigma(A) = \sigma(A^*)^* = \sigma(CA^*C)$ from Lemma 3.1, $\lambda \in \mathrm{iso}\sigma(CA^*C)$. Since $\sigma(T) \subset \sigma(A) \cup \sigma(CA^*C)$, $\lambda \in \mathrm{iso}\sigma(T)$. Moreover, we know $\alpha(T - \lambda) > 0$ implies $\lambda \in \pi_0(T)$. Since T has generalized Weyl's theorem, $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Moreover, since T has the single-valued extension property, $\lambda \notin \sigma_D(T)$. Thus $A - \lambda$ is left Drazin invertible. But A^* has the single-valued extension property, hence $A - \lambda$ is Drazin invertible. Therefore, $\lambda \in \sigma(A) \setminus \sigma_{BW}(A)$. That is, $\pi_0(A) \subseteq \sigma(A) \setminus \sigma_{BW}(A)$. Hence A has generalized Weyl theorem.

(c) We only consider (ii)$\Longrightarrow$(i). We show that Weyl's theorem holds for T if and only if Weyl's theorem holds for A. Since A^* has the single-value extension property, it follows from Lemma 3.1 that $\sigma(T) = \sigma(A)$. On the other hand, we have $\sigma_w(T) \subset \sigma_w(A) \cup \sigma_w(CA^*C) = \sigma_w(A)$. Since the converse inclusion holds, $\sigma_w(T) = \sigma_w(A)$. Now, we will show that $0 < \alpha(T - \lambda) < \infty$ iff $0 < \alpha(A - \lambda) < \infty$. Using (2), if $0 < \alpha(T - \lambda) < \infty$, then $0 < \alpha(A - \lambda) < \infty$. But, since T is complex symmetric, $0 < \alpha(T^* - \overline{\lambda}) < \infty$. Therefore, $0 < \alpha(A^* - \overline{\lambda}) < \infty$. So, $\pi_0(T) = \pi_0(A)$. Hence Weyl's theorem holds for T^* if and only if Weyl's theorem holds for A^* by similar arguments. □

Corollary 3.10. *Let $T \in M(A, N)$ where A is decomposable and N is normal or nilpotent of order 2 with $N = CN^*C$. If A satisfies generalized Weyl's theorem, then T satisfies generalized a-Weyl's theorem.*

Proof. The proof follows from Theorem 3.9. □

For $u \in H^2$ with power series representation $u(z) = \sum_{n=0}^{\infty} a_n z^n$, it is well known that $\lim_{r\to 1^-} u(rz)$ exists for almost every $z \in \partial\mathbb{D}$, and so one defines $\widetilde{u}(e^{i\theta}) := \sum_{n=0}^{\infty} a_n e^{in\theta}$ for almost every $\theta \in [0, 2\pi)$. A function $u \in H^2$ is called *inner* if $|\widetilde{u}(e^{i\theta})| = 1$ for almost every $\theta \in [0, 2\pi)$. For a nonconstant inner function u, the *model space* is given by $\mathcal{K}_u{}^2 := H^2 \ominus uH^2$ (see [7] and [21] for more details). For an inner function u and $\varphi \in L^2$, the *truncated Toeplitz operator* $A_\varphi^u : \mathcal{K}_u{}^2 \to \mathcal{K}_u{}^2$ is the compressed operator of T_φ to the space $\mathcal{K}_u{}^2$, that is,

$$A_\varphi^u := P_u T_\varphi P_u$$

where P_u denotes the orthogonal projection of L^2 onto $\mathcal{K}_u{}^2$. It is evident that A_φ^u is bounded on $\mathcal{K}_u{}^2$ whenever $\varphi \in L^\infty$. Define an antilinear operator C on $\mathcal{K}_u{}^2$ by $Cf = \overline{zf}u$. It is known from [7] that $\overline{zf}u \in \mathcal{K}_u{}^2$ for all $f \in \mathcal{K}_u{}^2$ and C is a conjugation operator on $\mathcal{K}_u{}^2$.

Corollary 3.11. *Let u be a finite Blaschke product with zeros $a_1, a_2, \cdots, a_n$, i.e., $u(z) := \left(\Pi_{j=1}^{n} \frac{a_j - z}{1 - \overline{a}_j z}\right)$ for $a_j \in \mathbb{D}$. If $T = \begin{pmatrix} A_\varphi^u & A_\psi^u \\ 0 & A_\varphi^u \end{pmatrix}$ is in $\mathcal{L}(\mathcal{K}_u{}^2 \oplus \mathcal{K}_u{}^2)$ where A_φ^u is isoloid, then A_φ^u satisfies generalized Weyl theorem if and only if T satisfies generalized a-Weyl theorem.*

Proof. Suppose that u be a finite Blaschke product. Then u is inner function and $\mathcal{K}_u^2$ is a finite model space. Then $\sigma(A_\varphi^u)$ is finite and so A_φ^u has the single-valued extension property by [1]. From [21, Lemma 2.1] or [7, Proposition 3], the truncated Toeplitz operators A_φ^u and A_ψ^u are complex symmetric with the conjugation

$Cf = \overline{zfu}$ on $\mathcal{K}_u^2$. Moreover, the operator matrix $\begin{pmatrix} A_\varphi^u & A_\psi^u \\ 0 & A_\varphi^u \end{pmatrix}$ is complex symmetric with the conjugation $\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$. Hence the results hold from Theorem 3.9. □

Example 3.12. For $x \in \mathbb{C}^n$, define $C^j(\sum_{i=1}^n \alpha_i e_i) = \sum_{i=1}^n \overline{\alpha_i} e_{n-i+1}$. Put $C = \oplus C^j$. Then C is a conjugation on $\mathcal{H}$ where $dim\mathcal{H} = \aleph_0$. Suppose that S is written as $S = \oplus_{j=1}^\infty S_j$ where

$$S_j = \begin{pmatrix} 0 & \lambda_1^{(j)} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2^{(j)} & \cdots & 0 \\ \cdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_{n_j-1}^{(j)} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to an orthonormal basis of S_j with $|\lambda_k^{(j)}| = |\lambda_{n_j-k}^{(j)}|$ for all $1 \le k \le n_j - 1$. Then S is complex symmetric with C from [23, Theorem 3.1]. Let W be a weighted shift on $\mathcal{H}$ defined by

$$W = (x_1, x_2, x_3, \cdots) := (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots).$$

If $T = \begin{pmatrix} W^* & S \\ 0 & CWC \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. Then T satisfies generalized a-Weyl's theorem. Indeed, since $\sigma(W^*) = \sigma_{BW}(W^*) = \{0\}$ and $\pi_0(W^*) = \emptyset$, it follows that W^* satisfies generalized Weyl's theorem. Moreover, in this case, W and W^* have the single-valued extension property. Hence T satisfies the generalized a-Weyl's theorem from Theorem 3.9.

References

[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Pub. 2004.
[2] P. Aiena, M.T. Biondi, C. Carpintero, *On Drazin invertibility*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2839–2848.
[3] I. J. An, *Weyl type theorems for* 2×2 *operator matrices*, Kyung Hee Univ., Ph.D. Thesis. 2013.
[4] M. Berkani, *On a class of quasi-Fredholm operators*, Int. Eq. Op. Th. **34**(1999), 244-249.
[5] S. R. Garcia, *Aluthge transforms of complex symmetric operators*, Int. Eq. Op. Th. **60**(2008), 357-367.
[6] S. R. Garcia, *Means of unitaries, conjugations, and the Friedrichs operator*, J. Math. Anal. Appl. **335**(2007), 941-947.
[7] S. R. Garcia, M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358**(2006), 1285-1315.
[8] S. R. Garcia, M. Putinar, *Complex symmetric operators and applications* II, Trans. Amer. Math. Soc. **359**(2007), 3913-3931.
[9] S. R. Garcia, W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362**(2010), 6065-6077.
[10] S. V. Djordjevic, Y. M. Han, *A note on Weyl's theorem for operator matrices*, Proc. Amer. Math. Soc. **130**(2003), 2543-2547.
[11] J. K. Han, H.Y. Lee, W.Y. Lee, *Invertible completions of* 2×2 *upper triangular operator matrices,* Proc. Amer. Math. Soc. **128** (1999), 119-123.
[12] I. S. Hwang and W.Y. Lee, *The boundedness below of* 2×2 *upper triangular operator matrices,* Int. Eq. Op. Th. **39** (2001), 267-276.
[13] W.Y. Lee, *Weyl spectra of operator matrices*, Proc. Amer. Math. Soc. **129**(2001), 131-138.
[14] W. Y. Lee, *Weyl's theorem for operator matrices*, Int. Eq. Op. Th. **32** (1998), 319-331.
[15] S. Jung, E. Ko, M. Lee, J. Lee, *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl. **379**(2011), 325-333.
[16] S. Jung, E. Ko, J. Lee, *On scalar extensions and spectral decompositions of complex symmetric operators*, J. Math. Anal. Appl. **382**(2011), 252-260.
[17] S. Jung , E. Ko, J. Lee, *On complex symmetric operator matrices*, J. Math. Anal. Appl. **406**(2013), 373-385.
[18] S. Jung, E. Ko, J. Lee, *Properties of complex symmetric operators*, Operators and matrices. **8**(4)(2014), 957-974.
[19] E. Ko, J. Lee, *On complex symmetric Toeplitz operators*, J. Math. Anal. Appl. **434**(2016), 20-34.
[20] K. Laursen, M. Neumann, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.
[21] D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices, **1**(2007), 419-526.
[22] S. Zhang, H. Zhang, J. Wu, *Spectra of upper-triangular operator matrix*, preprint.
[23] S. Zhu, C. G. Li, *Complex symmetric weigthed shifts*, Trans. Amer. Math. Soc., **365**(1)(2013), 511-530.