On Kuratowski $I$–Convergence of Sequences of Closed Sets

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Abstract. In this paper we extend the concepts of statistical inner and outer limits (as introduced by Talo, Sever and Başar) to $I$–inner and $I$–outer limits and give some $I$–analogue of properties of statistical inner and outer limits for sequences of closed sets in metric spaces, where $I$ is an ideal of subsets of the set $\mathbb{N}$ of positive integers. We extend the concept of Kuratowski statistical convergence to Kuratowski $I$–convergence for a sequence of closed sets and get some properties for Kuratowski $I$–convergent sequences. Also, we examine the relationship between Kuratowski $I$–convergence and Hausdorff $I$–convergence.

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [9] and Schoenberg [23]. The idea of $I$–convergence was introduced by Kostyrko et al. [11] as a generalization of statistical convergence which is based on the structure of the ideal $I$ of subsets of the set of positive integers. Nuray and Ruckle [18] independently introduced the same with another name generalized statistical convergence. Kostyrko et al. [12] gave some of basic properties of $I$–convergence and dealt with extremal $I$–limit points.


In set valued and variational analysis, limits of sequences of sets have the leading role. See [1, 8, 20]. The concepts of inner and outer limits for a sequence of sets are due to Painlevé, who introduced them in 1902 in his lectures on analysis at the University of Paris; set convergence was defined as the equality of these two limits. This convergence has been popularized by Kuratowski in his famous book Topologie [14] and thus, often called Kuratowski convergence of sequences of sets. For some properties of inner and outer limits we refer to [4, 5, 15, 20, 22, 24, 25, 28, 29]. Other convergence notions for sets are not equivalent to Kuratowski

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convergence but have significance for certain applications. One of them is Hausdorff convergence. We mention some references related to Hausdorff convergence: [3, 4, 14, 22, 25]. Nuray and Rhoades [19] first defined the statistical convergence for sequences of sets and studied Hausdorff and Wijsman statistical convergence.

In this paper our aim is to discuss two kinds of $I$–convergence for sequences of closed sets which are called Kuratowski $I$–convergence and Hausdorff $I$–convergence. For our purpose we give the definitions of $I$–outer and $I$–inner limits for a sequence of closed sets and investigate some properties of them.

2. Definitions and Notation

Let $K$ be a subset of positive integers $\mathbb{N}$ and $K(n) = ||k \leq n : k \in K||$, where $|A|$ denotes the number of elements in $A$. The natural density of $K$ is given by $\delta(K) = \lim_{n \to \infty} \frac{1}{n} K(n)$ if this limit exists.

A sequence $x = (x_n)$ is said to be statistically convergent to the number $L$ if the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero for every $\varepsilon > 0$. In this case we write $s\lim_{n \to \infty} x_n = L$.

Let $X \neq \emptyset$. A class $I$ of subsets of $X$ is said to be an ideal in $X$ provided:

(i) $\emptyset \in I$,
(ii) $A, B \in I$ implies $A \cup B \in I$,
(iii) $A \in I, B \subseteq A$ implies $B \in I$.

$I$ is called a nontrivial ideal if $X \notin I$. A nontrivial ideal $I$ in $X$ is called admissible ideal if $|x| \in I$ for each $x \in X$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of $X$ is said to be a filter in $X$ provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}, A \subseteq B$ implies $B \in \mathcal{F}$.

Lemma 2.1. [11] If $I$ is a nontrivial ideal in $X$, $X \neq \emptyset$, then the class $\mathcal{F}(I) = \{M \subseteq X : X \in I\}$

is a filter on $X$, called the filter associated with $I$.

Lemma 2.2. [21, Lemma 2.5] $K \subseteq F(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$ then $M \cap K \notin I$.

In what follows $(X, d)$ is a fixed metric space and $I$ denotes a non-trivial ideal of subsets of $\mathbb{N}$.

A sequence $[x_n]_{n \in \mathbb{N}}$ of elements of $X$ is said to be $I$–convergent to $\xi \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : d(x_n, \xi) \geq \varepsilon\}$ belongs to $I$. The element $\xi$ is called the $I$–limit of the sequence $x = [x_n]_{n \in \mathbb{N}}$. In this case we write $I – \lim_{n \to \infty} x_n = \xi$.

A sequence $[x_n]_{n \in \mathbb{N}}$ of elements of $X$ is said to be $I^*$–convergent to $\xi \in X$ if there exists a set $M \in \mathcal{F}(I)$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subseteq \mathbb{N}$ such that $\lim_{n \to \infty} d(x_{m_n}, \xi) = 0$. In this case we write $I^* – \lim_{n \to \infty} x_n = \xi$.

We say that an admissible ideal $I \subseteq 2^\mathbb{N}$ satisfies the property (AP), if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $I$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ of sets such that each symmetric difference $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^\infty B_j \in I$. (Hence $B_j \in I$ for each $j \in \mathbb{N}$).

Lemma 2.3. [11, Proposition 3.2] Let $I$ be an admissible ideal. If $I^* – \lim_{n \to \infty} x_n = \xi$, then $I^* – \lim_{n \to \infty} x_n = \xi$.

Lemma 2.4. [11, Theorem 3.2] Let $I \subseteq 2^\mathbb{N}$ be an admissible ideal. If the ideal $I$ has property (AP) and $(X, d)$ is an arbitrary metric space, then for arbitrary sequence $[x_n]_{n \in \mathbb{N}}$ of elements of $X$ we have $I – \lim_{n \to \infty} x_n = \xi$ implies $I^* – \lim_{n \to \infty} x_n = \xi$. 

An element \( \xi \in X \) is said to be an \( I \)-limit point of a sequence \( x = (x_k) \) if there is a set \( M = \{m_1 < m_2 < \cdots < m_k < \cdots \} \subset \mathbb{N} \) such that \( M \notin I \) and \( \lim_{k \to \infty} x_{m_k} = \xi \). The set of all \( I \)-limit points of a sequence \( x \) will be denoted by \( I(\Lambda_x) \).

An element \( \xi \in X \) is said to be an \( I \)-cluster point of a sequence \( x = (x_k) \) if for each \( \varepsilon > 0 \), we have \( \{k \in \mathbb{N} : d(x_k, \xi) < \varepsilon \} \notin I \). The set of all \( I \)-cluster points of \( x \) will be denoted by \( I(\Gamma_x) \).

Let \( L_x \) denote the set of all limit points \( \xi \) (accumulation points) of the sequence \( x \); i.e., \( \xi \in L_x \) if there exists an infinite set \( K = \{k_1 < k_2 < k_3 < \cdots \} \) such that \( x_{k_n} \to \xi \) as \( n \to \infty \).

Obviously, for an admissible ideal \( I \) we have \( I(\Lambda_x) \subseteq I(\Gamma_x) \subseteq L_x \).

**Lemma 2.5.** [6, Lemma 3.1] \( K \) be a compact subset of \( X \). Then we have \( K \cap I(\Gamma_x) = \emptyset \) for every \( x = (x_n) \) with \( \{n \in \mathbb{N} : x_n \in K \} \notin I \).

The concepts of \( I \)-limit superior and inferior were introduced by Demirci [7] as follows: Let \( I \) be an admissible ideal and \( x = (x_k) \) be a real number sequence.

\[
I - \lim sup_{k \to \infty} x_k := \left\{ \begin{array}{c}
sup B_x, \quad B_x \neq \emptyset, \\
-\infty, \quad B_x = \emptyset,
\end{array} \right.
\]

\[
I - \lim inf_{k \to \infty} x_k := \left\{ \begin{array}{c}
\inf A_x, \quad A_x \neq \emptyset, \\
\infty, \quad A_x = \emptyset,
\end{array} \right.
\]

where \( A_x := \{a \in \mathbb{R} : \{k \in \mathbb{N} : x_k < a \} \notin I \} \) and \( B_x := \{b \in \mathbb{R} : \{k \in \mathbb{N} : x_k > b \} \notin I \} \).

**Lemma 2.6.** [7, Theorem 1] If \( \beta = I - \lim sup_{k \to \infty} x_k \) is finite, then for every \( \varepsilon > 0 \),

\[
\{k \in \mathbb{N} : x_k > \beta - \varepsilon \} \notin I \quad \text{and} \quad \{k \in \mathbb{N} : x_k > \beta + \varepsilon \} \in I.
\]

Conversely, if (1) holds for every \( \varepsilon > 0 \) then \( \beta = I - \lim sup_{k \to \infty} x_k \).

The dual statement for \( I - \lim inf \) is as follows:

**Lemma 2.7.** [7, Theorem 2] If \( \alpha = I - \lim inf_{k \to \infty} x_k \) is finite, then for every \( \varepsilon > 0 \),

\[
\{k \in \mathbb{N} : x_k < \alpha + \varepsilon \} \notin I \quad \text{and} \quad \{k \in \mathbb{N} : x_k < \alpha - \varepsilon \} \in I.
\]

Conversely, if (2) holds for every \( \varepsilon > 0 \) then \( \alpha = I - \lim inf_{k \to \infty} x_k \).

Let \((X,d)\) be a metric space. The distance between a subset \( A \) of \( X \) and \( x \in X \) is given by \( d(x,A) = \inf\{d(x,y) : y \in A\} \), where it is understood that the infimum of \( d(x,\cdot) \) is \( \infty \) if \( A = \emptyset \). For each closed subset \( A \) of \( X \), the function \( x \to d(\cdot,A) \) is Lipschitz continuous, i.e. for each \( x, y \in X \)

\[
\left| d(x,A) - d(y,A) \right| \leq d(x,y).
\]

The open ball with center \( x \) and radius \( \varepsilon > 0 \) in \( X \) is denoted by \( B(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\} \). Also, for any set \( A \) and \( \varepsilon > 0 \), we write \( B(A,\varepsilon) = \{x \in X \mid d(x,A) < \varepsilon\} \).

Now we recall some basic properties of Kuratowski convergence. We use the following notation:

\[
\mathcal{N} := \{N \subseteq \mathbb{N} : \mathbb{N}\setminus N \text{ finite}\}
\]

\[
\mathcal{N}^\# := \{\text{subsequences of } \mathcal{N} \text{ containing all } n \text{ beyond some } n_0\}
\]

\[
\mathcal{N}^\infty := \{N \subseteq \mathbb{N} : \text{finite}\} = \{\text{all subsequences of } \mathbb{N} \}.
\]

We write \( \lim_{n \to \infty} \) when \( n \to \infty \) as usual in \( \mathbb{N} \), but \( \lim_{n \in \mathcal{N}^\infty} \) in the case of convergence of a subsequence designated by an index set \( N \) in \( \mathcal{N}^\infty \).
Definition 2.8. For a sequence \((A_n)\) of closed subsets of \(X\), the outer limit is the set

\[
\limsup_{n \to \infty} A_n := \left\{ x \mid \forall \varepsilon > 0, \exists N \in \mathcal{N}^0, \forall n \in N : A_n \cap B(x, \varepsilon) \neq \emptyset \right\}
\]

while the inner limit is the set

\[
\liminf_{n \to \infty} A_n := \left\{ x \mid \exists N \in \mathcal{N}, \forall n \in N, \exists x_n \in A_n : \lim_{n \to N} x_n = x \right\}.
\]

The limit of a sequence \((A_n)\) of closed subsets of \(X\) exists if the outer and inner limit sets are equal, that is, \(\lim_{n \to \infty} A_n = \liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n\).

Talo et al. [27] introduced Kuratowski statistical convergence of sequences of closed sets. The statistical outer limit and statistical inner limit of a sequence \((A_n)\) of closed subsets of \(X\) are defined by

\[
\text{st} - \limsup_{n \to \infty} A_n := \left\{ x \mid \forall \varepsilon > 0, \exists N \in \mathcal{S}, \forall n \in N : A_n \cap B(x, \varepsilon) \neq \emptyset \right\},
\]

\[
\text{st} - \liminf_{n \to \infty} A_n := \left\{ x \mid \forall \varepsilon > 0, \exists N \in \mathcal{S}, \forall n \in N : A_n \cap B(x, \varepsilon) \neq \emptyset \right\},
\]

where

\[
\mathcal{S} := \{ N \subseteq \mathbb{N} : \delta(N) = 1 \} \quad \text{and} \quad \mathcal{S}^0 := \{ N \subseteq \mathbb{N} : \delta(N) \neq 0 \}.
\]

The statistical limit of a sequence \((A_n)\) exists if its statistical outer and statistical inner limits coincide; i.e., \(\text{st} - \lim_{n \to \infty} A_n = \text{st} - \limsup_{n \to \infty} A_n = \text{st} - \liminf_{n \to \infty} A_n\).

3. Kuratowski \(I\)–Convergence

In this section, we introduce Kuratowski \(I\)–convergence of sequences of closed sets. We use the analogous idea employed by Kuratowski [14] and Talo et al. [27] for convergence and statistical convergence of sequences of closed sets. Let us consider

\[
\mathcal{N}_I := \{ N \subseteq \mathbb{N} : N \notin I \} = \mathcal{F}(I) \quad \text{and} \quad \mathcal{N}_I^0 := \{ N \subseteq \mathbb{N} : N \in I \}.
\]

Firstly, we define the \(I\) analogues for outer and inner limits of a sequence of closed sets.

Definition 3.1. The \(I\)–outer limit and \(I\)–inner limit of a sequence \((A_n)\) of closed subsets of \(X\) are defined as follows:

\[
I - \limsup_{n \to \infty} A_n := \left\{ x \mid \forall \varepsilon > 0, \exists N \in \mathcal{N}_I^0, \forall n \in N : A_n \cap B(x, \varepsilon) \neq \emptyset \right\},
\]

and

\[
I - \liminf_{n \to \infty} A_n := \left\{ x \mid \forall \varepsilon > 0, \exists N \in \mathcal{N}_I, \forall n \in N : A_n \cap B(x, \varepsilon) \neq \emptyset \right\}.
\]

The \(I\)–limit of a sequence \((A_n)\) exists if its \(I\)–outer and \(I\)–inner limits coincide. In this situation we say that the sequence of sets is Kuratowski \(I\)–convergent and we write

\[
I - \liminf_{n \to \infty} A_n = I - \limsup_{n \to \infty} A_n = I - \lim A_n.
\]
Moreover, it’s clear from the inclusion $N_I \subset N^e_I$ that
\[ I - \liminf_{n \to \infty} A_n \subseteq I - \limsup_{n \to \infty} A_n \]
so that in fact, $I - \limsup_{n \to \infty} A_n = A$ if and only if
\[ I - \limsup_{n \to \infty} A_n \subseteq A \subseteq I - \liminf_{n \to \infty} A_n. \]

**Remark 3.2.** $I - \liminf_{n \to \infty} A_n = A$ if and only if the following conditions are satisfied:
(i) for every $x \in A$ and for every $\varepsilon > 0$ we have $\{k \in \mathbb{N} : B(x, \varepsilon) \cap A_k \neq \emptyset\} \in \mathcal{F}(I)$;
(ii) for every $x \in X \setminus A$ there exists $\varepsilon > 0$ such that $\{k \in \mathbb{N} : B(x, \varepsilon) \cap A_k = \emptyset\} \in \mathcal{F}(I)$.

We give some examples of ideals and corresponding $I$–convergence.

(I) Put $I_0 = \{\emptyset\}$. $I_0$ is the minimal ideal in $\mathbb{N}$. Then for a sequence $(A_n)$ of closed sets we have
\[ I_0 - \liminf_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n \quad \text{and} \quad I_0 - \limsup_{n \to \infty} A_n = \cl \bigcup_{n=1}^{\infty} A_n, \]
where $\cl(A)$ denotes the closure of the set $A$ in the metric space $(X, d)$. A sequence $(A_n)$ is Kuratowski $I_0$–convergent if and only if it is constant set.

(II) Let $M \subseteq \mathbb{N}$, $M \neq \emptyset$. Put $I_M = 2^M$. Then $I_M$ is a nontrivial ideal in $\mathbb{N}$. Then for a sequence $(A_n)$ of closed sets we have
\[ I_M - \liminf_{n \to \infty} A_n = \bigcap_{n \in \mathbb{N} \setminus M} A_n \quad \text{and} \quad I_M - \limsup_{n \to \infty} A_n = \cl \bigcup_{n \in \mathbb{N} \setminus M} A_n. \]
A sequence $(A_n)$ is Kuratowski $I_M$–convergent if and only if it is constant set on $\mathbb{N} \setminus M$, i.e. there is a closed set $A$ such that $A_n = A$ for each $n \in \mathbb{N} \setminus M$.

(III) Take for $I$ the class $I_f$ of all finite subsets of $\mathbb{N}$. Then $I_f$ is a non-trivial admissible ideal and Kuratowski $I_f$–convergence coincides with the usual Kuratowski convergence.

(IV) Denote by $I_\delta$ the class of all $A \subseteq \mathbb{N}$ with $\delta(A) = 0$. Then $I_\delta$ is non-trivial admissible ideal and Kuratowski $I_\delta$–convergence coincides with the Kuratowski statistical convergence.

Note that if $I$ is an admissible, then $I_f \subseteq I$. It is clear that
\[ \liminf_{n \to \infty} A_n \subseteq I - \liminf_{n \to \infty} A_n \subseteq I - \limsup_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n. \]
Hence every Kuratowski convergent sequence is Kuratowski $I$–convergent, i.e.,
\[ \lim_{n \to \infty} A_n = A \text{ implies } I - \lim_{n \to \infty} A_n = A. \]

But, the converse of this claim does not hold in general.

**Example 3.3.** Let $X = \mathbb{R}^2$ (with the usual Euclidean metric). We decompose the set $\mathbb{N}$ into countably many disjoint sets
\[ N_j = \{2^{j+1}(2s - 1) : s \in \mathbb{N}\}, \quad (j = 1, 2, 3, \ldots). \]
It is obvious that $\mathbb{N} = \bigcup_{j=1}^{\infty} N_j$ and $N_i \cap N_j = \emptyset$ for $i \neq j$. Denote by $I$ the class of all $A \subseteq \mathbb{N}$ such that $A$ intersects only a finite number of $N_j$. It is easy to see that $I$ is an admissible ideal. Define $(A_n)$ as follows: for $n \in N_j$ we put
\[ A_n = \left\{ (x, y) \in \mathbb{R}^2 : \frac{1}{(j + 1)^2} \leq x^2 + y^2 \leq \frac{1}{j} \right\} \quad (j = 1, 2, 3, \ldots). \]
Let $\varepsilon > 0$. Choose $p \in \mathbb{N}$ such that $\frac{1}{p} < \varepsilon$. Then
\[
\left\{ n \in \mathbb{N} : A_n \cap B(0, \varepsilon) = \emptyset \right\} \subseteq N_1 \cup N_2 \cup \cdots \cup N_p.
\]
Thus $\left\{ n \in \mathbb{N} : A_n \cap B(0, \varepsilon) = \emptyset \right\} \in \mathcal{I}$ i.e., $\left\{ n \in \mathbb{N} : A_n \cap B(0, \varepsilon) \neq \emptyset \right\} \in \mathcal{F}(\mathcal{I})$. So $I - \lim_{n \to \infty} A_n = \{0\}$. However
\[
\liminf_{n \to \infty} A_n = \emptyset \quad \text{and} \quad \limsup_{n \to \infty} A_n = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \right\}.
\]
Therefore $(A_n)$ is not Kuratowski convergent.

In what follows $\mathcal{I}$ denotes a non-trivial admissible ideal of subsets of $\mathbb{N}$.

**Proposition 3.4.** Let $(A_n)$ be a sequence of closed subsets of $X$. Then
\[
I - \liminf_{n \to \infty} A_n = \bigcap_{N \in \mathcal{N}_I} \text{cl} \bigcup_{n \in N} A_n \quad \text{and} \quad I - \limsup_{n \to \infty} A_n = \bigcap_{N \in \mathcal{N}_I} \text{cl} \bigcup_{n \in N} A_n.
\]

**Proof.** We prove only the first equality because the proof of the second one is similar to the first one. Let $x \in I - \liminf_{n \to \infty} A_n$ be arbitrary and $N \in \mathcal{N}_I$ be arbitrary. For every $\varepsilon > 0$ there exists $N_1 \in \mathcal{N}_I$ such that for every $n \in N_1$
\[
A_n \cap B(x, \varepsilon) \neq \emptyset.
\]
From Lemma 2.2 we have $N \cap N_1 \notin \mathcal{I}$. So there exists $n_0 \in N \cap N_1$ such that $A_{n_0} \cap B(x, \varepsilon) \neq \emptyset$. Therefore,
\[
\left( \bigcup_{n \in N} A_n \right) \cap B(x, \varepsilon) \neq \emptyset.
\]
This means that $x \in \text{cl} \bigcup_{n \in N} A_n$. This holds for any $N \in \mathcal{N}_I$. Consequently,
\[
x \in \bigcap_{N \in \mathcal{N}_I} \text{cl} \bigcup_{n \in N} A_n.
\]
For the reverse inclusion, suppose that $x \notin I - \liminf_{n \to \infty} A_n$. Then, there exists $\varepsilon > 0$ such that
\[
N = \left\{ n \in \mathbb{N} : A_n \cap B(x, \varepsilon) = \emptyset \right\} \notin \mathcal{I},
\]
i.e., $N \in \mathcal{N}_I$. Thus
\[
\left( \bigcup_{n \in N} A_n \right) \cap B(x, \varepsilon) = \emptyset.
\]
This means that $x \notin \text{cl} \bigcup_{n \in N} A_n$. This completes the proof. \qed

As a consequence of Proposition 3.4, for any given sequence $(A_n)$ the sets $I - \liminf_{n \to \infty} A_n$ and $I - \limsup_{n \to \infty} A_n$ are closed.

**Proposition 3.5.** Let $(A_n)$ be a sequence of closed subsets of $X$. Then
\[
I - \liminf_{n \to \infty} A_n = \left\{ x \mid I - \lim_{n \to \infty} d(x, A_n) = 0 \right\},
\]
\[
I - \limsup_{n \to \infty} A_n = \left\{ x \mid I - \liminf_{n \to \infty} d(x, A_n) = 0 \right\}.
\]
Proof. For any closed set $A$ we have

$$d(x, A) \geq \varepsilon \iff A \cap B(x, \varepsilon) = \emptyset. \quad (3)$$

Suppose that $I - \lim_{n \to \infty} d(x, A_n) = 0$. Then for every $\varepsilon > 0$

$$\{ n \in \mathbb{N} : d(x, A_n) \geq \varepsilon \} \in I.$$  

By (3), for every $\varepsilon > 0$ we obtain

$$\{ n \in \mathbb{N} : A_n \cap B(x, \varepsilon) = \emptyset \} \in I.$$

This means that

$$\{ n \in \mathbb{N} : A_n \cap B(x, \varepsilon) \neq \emptyset \} \in \mathcal{F}(I).$$

That is, $x \in I - \liminf_{n \to \infty} A_n$.

Now, we show the reverse inclusion. Let $x \in I - \liminf_{n \to \infty} A_n$. Then for every $\varepsilon > 0$ there exists $N \in \mathcal{N}_I$ such that $A_n \cap B(x, \varepsilon) \neq \emptyset$ for every $n \in N$. Since

$$\{ n \in \mathbb{N} : A_n \cap B(x, \varepsilon) = \emptyset \} \subseteq \mathbb{N} \setminus N,$$

we have

$$\{ n \in \mathbb{N} : A_n \cap B(x, \varepsilon) = \emptyset \} \in I.$$

By (3)

$$\{ n \in \mathbb{N} : d(x, A_n) \geq \varepsilon \} \in I.$$

That is, $I - \lim_{n \to \infty} d(x, A_n) = 0$.

Similarly, for any closed set $A$ we have

$$d(x, A) < \varepsilon \iff A \cap B(x, \varepsilon) \neq \emptyset. \quad (4)$$

Suppose that $I - \liminf_{n \to \infty} d(x, A_n) = 0$. Then for every $\varepsilon > 0$

$$\{ n \in \mathbb{N} : d(x, A_n) < \varepsilon \} \notin I.$$

By (4), for every $\varepsilon > 0$ we obtain

$$\{ n \in \mathbb{N} : A_n \cap B(x, \varepsilon) \neq \emptyset \} \notin I.$$

This means that $x \in I - \limsup_{n \to \infty} A_n$.

Now, we show the reverse inclusion. Let $x \in I - \limsup_{n \to \infty} A_n$. Then for every $\varepsilon > 0$

$$\{ n \in \mathbb{N} : A_n \cap B(x, \varepsilon) \neq \emptyset \} \notin I.$$

By (4) and Lemma 2.7, we have $I - \liminf_{n \to \infty} d(x, A_n) = 0$. \hfill \Box

**Proposition 3.6.** Let $(A_n)$ be a sequence of closed subsets of $X$. Then

$$I - \liminf_{n \to \infty} A_n = \left\{ x \mid \forall n \in \mathbb{N}, \exists y_n \in A_n : I - \lim_{n \to \infty} y_n = x \right\}. \quad (5)$$
Proof. Let $x \in I - \liminf_{n \to \infty} A_n$ be arbitrary. By Proposition 3.5,

$$I - \lim_{n \to \infty} d(x, A_n) = 0.$$ 

For every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : d(x, A_n) \geq \frac{\varepsilon}{2} \right\} \in I.$$

Since $A_n$ is closed, for $n \in \mathbb{N}$, there exists $y_n \in A_n$ such that $d(x, y_n) \leq 2d(x, A_n)$. Now, we define the sequence $\{y_n | y_n \in A_n, n \in \mathbb{N}\}$. Then $I - \lim_{n \to \infty} y_n = x$.

On the contrary, assume that $x$ belongs to the right-hand side set of the equality (5). Then, there exist $\{y_n | y_n \in A_n, n \in \mathbb{N}\}$ such that $I - \lim_{n \to \infty} y_n = x$. Then for every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : d(x, y_n) \geq \varepsilon \right\} \in I.$$

The inequality $d(x, y_n) \geq d(x, A_n)$ yields the inclusion

$$\left\{ n \in \mathbb{N} : d(x, A_n) \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : d(x, y_n) \geq \varepsilon \right\}.$$

So,

$$\left\{ n \in \mathbb{N} : d(x, A_n) \geq \varepsilon \right\} \in I.$$

This means that $I - \lim_{n \to \infty} d(x, A_n) = 0$. By Proposition 3.5 we have $x \in I - \liminf_{n \to \infty} A_n$. \qed

The following result is well known in the theory of Kuratowski convergence. $x \in \liminf_{n \to \infty} A_n$ if and only if there exist $N \in \mathcal{N} = \mathcal{N}_I$, and $x_n \in A_n$ for all $n \in N$ such that $\lim_{n \in N} x_n = x$. For Kuratowski $I$–convergence, if $I$ has property (AP), then this fact holds.

Corollary 3.7. Let $I$ be an admissible ideal. If the ideal $I$ has property (AP) then

$$I - \liminf_{n \to \infty} A_n = \left\{ x | \exists N \in \mathcal{N}_I, \forall n \in N, \exists y_n \in A_n : \lim_{n \in N} y_n = x \right\}. \tag{6}$$

Proof. Suppose that $I$ satisfies condition (AP). Let $x \in I - \liminf_{n \to \infty} A_n$. Then $I - \lim_{n \to \infty} d(x, A_n) = 0$. By condition (AP) we have $I^* - \lim_{n \to \infty} d(x, A_n) = 0$. Then there is a set $M \in \mathcal{F}(I)$ such that

$$\lim_{m \in M} d(x, A_m) = 0.$$ 

Since $A_m$ is closed, for $m \in M$, there exists $y_m \in A_m$ such that $d(x, y_m) \leq 2d(x, A_m)$. Now, we define the sequence $\{y_m | y_m \in A_m, m \in M\}$. Then $\lim_{m \in M} y_m = x$.

On the contrary, assume that $x$ belongs to the right-hand side set of the equality (6). Let us define

$$z_n = \begin{cases} y_n, & \text{if } n \in N, \\ \text{arbitrary element of } A_n, & \text{if } n \notin N. \end{cases}$$

Then $I^* - \lim_{n \to \infty} z_n = x$. So $I - \lim_{n \to \infty} z_n = x$. By Proposition 3.6, we have $x \in I - \liminf_{n \to \infty} A_n$. \qed

Remark 3.8. In Corollary 3.7 the property (AP) can not be dropped. Let $X = \mathbb{R}$ (with the usual Euclidean metric) and $I$ be the ideal introduced in Example 3.3. Define $(A_n)$ as follows: for $n \in N_j$ we put $A_n = \{\frac{1}{j}\}$ ($j = 1, 2, 3, ...$). Then the sequence $\{y_n | y_n \in A_n, n \in \mathbb{N}\}$ can be defined as follows: for $n \in N_j$ we put $y_n = \frac{1}{j}$ ($j = 1, 2, 3, ...$). Clearly, $I - \lim_{n \to \infty} y_n = 0$. So $I - \liminf_{n \to \infty} A_n = \{0\}$.

Suppose in contrary that $0$ belongs to the right-hand side set of the equality (6). Then there is a set $M \in \mathcal{F}(I)$ such that for $m \in M$, there exists $y_m \in A_m$ and

$$\lim_{m \in M} y_m = 0. \tag{7}$$
By the definition of $F(I)$ we have $M = N \setminus H$, where $H \in I$. By the definition of $I$ there is a $p \in N$ such that $H \subseteq N_1 \cup N_2 \cup \ldots \cup N_p$.

But then $M$ contains the set $N_{p+1}$ and so $y_m = \frac{1}{p+1}$ for infinitely many $m$'s from $M$. This contradicts (7).

**Corollary 3.9.** Let $X$ be a normed linear space and $(A_n)$ be a sequence of subsets of $X$. If the ideal $I$ has property (AP) and there is a set $K \in F(I)$ such that $A_n$ is convex for each $n \in K$, then $I - \lim \inf_{n \to \infty} A_n$ is convex and so, when it exists, is $I - \lim \inf_{n \to \infty} A_n$.

**Proof.** Let $I - \lim \inf_{n \to \infty} A_n = A$. If $x_1$ and $x_2$ belong to $A$, by Corollary 3.7, we can find for all $n \in N$ in some set $N \in F(I)$ points $y^{s}_{n}$ and $y^{e}_{n}$ in $A_n$ such that

$$\lim_{n \to N} y^{s}_{n} = x_1 \quad \text{and} \quad \lim_{n \to N} y^{e}_{n} = x_2.$$ 

Since $K \in F(I)$, we have $M \in F(I)$ with $M = N \cap K$. Then for arbitrary $\lambda \in [0, 1]$ and $n \in M$, let us define

$$y^{\lambda}_{n} := (1 - \lambda)y^{s}_{n} + \lambda y^{e}_{n} \quad \text{and} \quad x_{\lambda} := (1 - \lambda)x_1 + \lambda x_2.$$ 

Then

$$\lim_{n \to M} y^{\lambda}_{n} = x_{\lambda}.$$ 

By Corollary 3.7, we obtain $x_{\lambda} \in A$. This means that $A$ is convex. \qed

**Proposition 3.10.** Let $(A_n)$ be a sequence of closed subsets of $X$. Then

$$I - \lim \sup_{n \to \infty} A_n = \left\{ x \mid \exists N \in N^{\#}_I, \forall n \in N, \exists y_n \in A_n : x \in I(\Gamma_y) \right\}. \tag{8}$$

**Proof.** Let $x \in I - \lim \inf_{n \to \infty} A_n$ be arbitrary. By Proposition 3.5,

$$I - \lim \inf_{n \to \infty} d(x, A_n) = 0.$$ 

By Lemma 2.7, for every $\varepsilon > 0$ we have

$$\left\{ n \in N : d(x, A_n) < \frac{\varepsilon}{2} \right\} \notin I.$$

Since $A_n$ is closed, for $n \in N$, there exists $y_n \in A_n$ such that $d(x, y_n) \leq 2d(x, A_n)$. Now, we define the sequence $\{y_n \mid y_n \in A_n, n \in N\}$. Then

$$\left\{ n \in N : d(x, y_n) < \varepsilon \right\} \notin I.$$ 

Therefore $x \in I(\Gamma_y)$.

On the contrary, assume that $x$ belongs to the right-hand side set of the equality (8). Then there exist $N \in N^{\#}_I$ and a sequence $\{y_n \mid y_n \in A_n, n \in N\}$ such that $x \in I(\Gamma_y)$. That is, for every $\varepsilon > 0$

$$\left\{ n \in N : d(x, y_n) < \varepsilon \right\} \notin I.$$ 

The inequality $d(x, y_n) \geq d(x, A_n)$ yields the inclusion

$$\left\{ n \in N : d(x, y_n) < \varepsilon \right\} \subseteq \left\{ n \in N : d(x, A_n) < \varepsilon \right\}.$$ 

So, the set

$$N' = \left\{ n \in N : d(x, A_n) < \varepsilon \right\} \notin I.$$ 

That is, $N' \in N^{\#}_I$. By (4), for every $n \in N'$ we obtain $A_n \cap B(x, \varepsilon) \neq \emptyset$. This means that $x \in I - \lim \sup_{n \to \infty} A_n$. \qed
Remark 3.11. In Proposition 3.10 the set of \( I \)-cluster points can not be replaced by the set of \( I \)-limit points. Let
\((A_n)\) and \((y_n)\) be the sequences introduced in Remark 3.8. Let us take \( I = \Gamma_g \). It can be easily shown that \( \delta(N) = 1/2 \).
From Example 2.1 of [6] we have \( 0 \in I_0(\Gamma_g) \) but \( 0 \notin I_0(A_g) \). So, \( 0 \in I_0 - \limsup_{n \to \infty} A_n \). However
\[
0 \notin \left\{ x \mid \forall n \in \mathbb{N}, \exists y_n \in A_n : \lim_{n \to \infty} y_n = x \right\}.
\]

By Proposition 3.6 and Proposition 3.10, note that \( I - \lim inf_{n \to \infty} A_n \) is the set of \( I \)-limits of sequence \((y_n)_{n \in \mathbb{N}}\) with \( y_n \in A_n \) and \( I - \lim sup_{n \to \infty} A_n \) is the set of \( I \)-cluster points of sequence \((y_n)_{n \in \mathbb{N}}\) with \( y_n \in A_n \).

Lemma 3.12. Let \((A_n)\) and \((B_n)\) be two sequences of closed subsets of \( X \). If there is a set \( K \in \mathcal{N}_I \) such that \( A_n \subseteq B_n \) for each \( n \in K \), then the inclusions
\[
I - \lim inf_{n \to \infty} A_n \subseteq I - \lim inf_{n \to \infty} B_n \quad \text{and} \quad I - \lim sup_{n \to \infty} A_n \subseteq I - \lim sup_{n \to \infty} B_n
\]
hold.

Proof. To prove the first inclusion suppose that there exists \( K \in \mathcal{N}_I \) such that for each \( n \in K \) the inclusion \( A_n \subseteq B_n \) holds. In this case for each \( x \in I - \lim inf_{n \to \infty} A_n \), we obtain
\[
d(x, B_n) \leq d(x, A_n).
\]
By Proposition 3.5, we have
\[
I - \lim_{n \to \infty} d(x, A_n) = 0.
\]
Consequently, combining (9) and (10), we have \( I - \lim_{n \to \infty} d(x, B_n) = 0 \). Namely \( x \in I - \lim inf_{n \to \infty} B_n \).

The proof of second inclusion is analogous to that of the first one and so we omit the details. \( \square \)

Corollary 3.13. Let \((A_n)\) and \((B_n)\) be two sequences of closed subsets of \( X \). Then, the following statements hold:

1. \( I - \lim sup_{n \to \infty} (A_n \cap B_n) \subseteq I - \lim sup_{n \to \infty} A_n \cap I - \lim sup_{n \to \infty} B_n \).
2. \( I - \lim inf_{n \to \infty} (A_n \cap B_n) \subseteq I - \lim inf_{n \to \infty} A_n \cap I - \lim inf_{n \to \infty} B_n \).
3. \( I - \lim sup_{n \to \infty} (A_n \cup B_n) = I - \lim sup_{n \to \infty} A_n \cup I - \lim sup_{n \to \infty} B_n \).
4. \( I - \lim inf_{n \to \infty} (A_n \cup B_n) = I - \lim inf_{n \to \infty} A_n \cup I - \lim inf_{n \to \infty} B_n \).

Proof. For each \( n \in \mathbb{N} \), the inclusions \( A_n \cap B_n \subseteq A_n, A_n \cap B_n \subseteq B_n, A_n \subseteq A_n \cup B_n \) and \( B_n \subseteq A_n \cup B_n \) hold. Now, the proof is immediate by Lemma 3.12. \( \square \)

Definition 3.14. A sequence \((A_k)\) is said to be \( I \)-monotonic increasing, if there exists a subset \( K = \{k_1 < k_2 < k_3 < \cdots \} \in F(I) \) such that \( A_{k_n} \subseteq A_{k_{n+1}} \) for every \( n \in \mathbb{N} \). Similarly, sequence \((A_k)\) is said to be \( I \)-monotonic decreasing, if there exists a subset \( K = \{k_1 < k_2 < k_3 < \cdots \} \in F(I) \) such that \( A_{k_n} \supseteq A_{k_{n+1}} \) for every \( n \in \mathbb{N} \).

Theorem 3.15. Suppose that \((A_k)\) is \( I \)-monotonic increasing sequence of closed subsets of \( X \). Then \( I - \lim_{k \to \infty} A_k \) exists and
\[
I - \lim_{k \to \infty} A_k = cl \bigcup_{n \in \mathbb{N}} A_{k_n}.
\]

Proof. Let \((A_k)\) is a \( I \)-monotonic increasing sequence of closed subsets of \( X \) and \( A = cl \bigcup_{n \in \mathbb{N}} A_{k_n} \). Then, \( A_{k_n} \subseteq A \) for every \( n \in \mathbb{N} \). If \( A = \emptyset \), then \( A_{k_n} = \emptyset \) for every \( n \in \mathbb{N} \). So, \( I - \lim A_k = \emptyset \). Let \( A \neq \emptyset \) and \( x \in cl \bigcup_{n \in \mathbb{N}} A_{k_n} \). In this case, for every \( \varepsilon > 0 \)
\[
B(x, \varepsilon) \cap \bigcup_{n \in \mathbb{N}} A_{k_n} \neq \emptyset.
\]
Then there exists \( n_0 \in \mathbb{N} \) such that \( B(x, \varepsilon) \cap A_{k_{n_0}} \neq \emptyset \). Since \((A_k)\) is an increasing sequence, \( A_{k_{n_0}} \subseteq A_k \) for all \( n \geq n_0 \). Define the set \( M \) as
\[
M = \{m \mid m = k_n, \ n \geq n_0, \ n \in \mathbb{N}\}.
\]
Then \( M \in F(I) \) and \( B(x, \varepsilon) \cap A_m \neq \emptyset \) for all \( m \in M \). Consequently, we obtain \( x \in I - \lim_{k \to \infty} A_k \).

Now we show that \( I - \lim_{k \to \infty} A_k \subseteq A \). Let \( x \in I - \lim_{k \to \infty} A_k \) be arbitrary. Then for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for every \( k \in N \) we have \( A_k \cap B(x, \varepsilon) \neq \emptyset \). By Lemma 2.2, since \( K \in F(I) \) and \( N \notin I \), we have \( K \cap N \notin I \). So, there exists \( k_{n_0} \in K \cap N \) such that \( B(x, \varepsilon) \cap A_{k_{n_0}} \neq \emptyset \).

Therefore we obtain
\[
B(x, \varepsilon) \cap \bigcup_{n \in \mathbb{N}} A_{k_n} \neq \emptyset.
\]
This means that \( x \in cI \cup \bigcap_{n \in \mathbb{N}} A_{k_n} \). This step concludes the proof. \( \square \)

**Theorem 3.16.** Suppose that \((A_k)\) is an \( I \)–monotonic decreasing sequence of closed subsets of \( X \). Then \( I - \lim_{k \to \infty} A_k \) exists and
\[
I - \lim_{k \to \infty} A_k = \bigcap_{n \in \mathbb{N}} A_{k_n}.
\]

**Proof.** Let \( A = \bigcap_{n \in \mathbb{N}} A_{k_n} \). Clearly if \( x \in A \), then \( x \in A_{k_n} \) for every \( n \in \mathbb{N} \). Define \( M = \{m \mid m = k_n, n \in \mathbb{N}\} \). Then \( M \in F(I) \). Also for all \( \varepsilon > 0 \) and \( m \in M \) we have \( B(x, \varepsilon) \cap A_m \neq \emptyset \). This means that \( x \in I - \lim_{k \to \infty} A_k \).

Now we show that \( I - \lim_{k \to \infty} A_k \subseteq A \). Let \( x \in I - \lim_{k \to \infty} A_k \) be arbitrary. Then, for every \( \varepsilon > 0 \) there exists \( N \notin I \) such that for every \( m \in N \), \( A_m \cap B(x, \varepsilon) \neq \emptyset \). Since \( I \) is an admissible, \( N \) is infinite. So for every \( n \in \mathbb{N} \) there exists \( m \in N \) such that \( k_n \leq m \). Since the sequence \((A_k)\) is decreasing, the inclusion \( A_{k_n} \supseteq A_m \) holds and consequently \( B(x, \varepsilon) \cap A_{k_n} \neq \emptyset \). This means that \( x \in cI_{A_{k_n}} \). Since \( A_{k_n} \) is closed, \( x \in A_{k_n} \).

Therefore \( x \in \bigcap_{n \in \mathbb{N}} A_{k_n} \). This step concludes the proof. \( \square \)

In the next section we introduce Hausdorff \( I \)–convergence of closed sets. Then, we compare Hausdorff \( I \)–convergence and Kuratowski \( I \)–convergence of the sequence of closed sets.

### 4. Hausdorff \( I \)–Convergence

The Hausdorff distance \( h(E, F) \) between the subsets \( E \) and \( F \) of \( X \) is defined as follows:
\[
h(E, F) = \max \{D(E, F), D(F, E)\},
\]
where
\[
D(E, F) = \sup_{x \in E} d(x, F) = \inf \{\varepsilon > 0 : E \subseteq B(F, \varepsilon)\}
\]
unless both \( E \) and \( F \) are empty in which case \( h(E, F) = 0 \). Note that if only one of the two sets is empty then \( h(E, F) = \infty \).

It is known, for a long time (see [3, 14]), that
\[
h(E, F) = \sup_{x \in X} |d(x, E) - d(x, F)|.
\]

**Definition 4.1.** Let \((A_n)\) be a sequence of closed subsets of \( X \). We say that the sequence \((A_n)\) is Hausdorff \( I \)–convergent to a closed subset \( A \) of \( X \) if
\[
I - \lim_{n \to \infty} h(A_n, A) = 0.
\]
In this case, we write \( A = H_I - \lim_{n \to \infty} A_n \).
**Lemma 4.2.** Suppose that \( \{A; A_n, n \in \mathbb{N}\} \) is a family of closed subsets of \( X \). Then \( A = H_I - \lim_{n \to \infty} A_n \) if and only if either there exists \( M \in F(I) \) such that \( A \) and \( A_n \) are empty for all \( n \in M \) or for any \( \varepsilon > 0 \) the sets
\[
\{ n \in \mathbb{N} : A \not\subseteq B(A_n, \varepsilon) \} \quad \text{and} \quad \{ n \in \mathbb{N} : A_n \not\subseteq B(A, \varepsilon) \}
\] (12)
belong to \( I \).

**Proof.** If \( A = \emptyset \), then for every \( \varepsilon > 0 \)
\[
\{ n \in \mathbb{N} : h(A_n, A) \geq \varepsilon \} = \{ n \in \mathbb{N} : A_n \neq \emptyset \}.
\]
Thus \( \{ n \in \mathbb{N} : A_n \neq \emptyset \} \in I \). Namely, \( \{ n \in \mathbb{N} : A_n = \emptyset \} \notin F(I) \).

Conversely, there exists \( M \in F(I) \) such that \( A_n \) is empty for all \( n \in M \). Then, for every \( \varepsilon > 0 \)
\[
\{ n \in \mathbb{N} : h(A_n, \emptyset) \geq \varepsilon \} \in I.
\]
So \( A = \emptyset \).

On the other hand if \( A \neq \emptyset \), then (11) holds if and only if for every \( \varepsilon > 0 \)
\[
\{ n \in \mathbb{N} : h(A_n, A) > \varepsilon \} \in I
\]
or equivalently,
\[
\{ n \in \mathbb{N} : h(A_n, A) < \varepsilon \} \in F(I).
\]
By the definition of Hausdorff metric,
\[
\{ n \in \mathbb{N} : A \not\subseteq B(A_n, \varepsilon) \quad \text{and} \quad A_n \not\subseteq B(A, \varepsilon) \} \in F(I).
\]
Consequently,
\[
\{ n \in \mathbb{N} : A \not\subseteq B(A_n, \varepsilon) \} \cup \{ n \in \mathbb{N} : A_n \not\subseteq B(A, \varepsilon) \} \in I.
\]
This completes the proof. \( \square \)

The next theorem answers a natural question about relationships between Hausdorff \( I \)–convergence and Kuratowski \( I \)–convergence.

**Theorem 4.3.** Suppose that \( \{A; A_n, n \in \mathbb{N}\} \) is a family of closed subsets of \( X \) with \( A \neq \emptyset \). Then Hausdorff \( I \)–convergence implies Kuratowski \( I \)–convergence, i.e.,
\[
H_I - \lim_{n \to \infty} A_n = A \text{ implies } I - \lim_{n \to \infty} A_n = A.
\]

**Proof.** Take \( x \in A \). By (12), for any \( \varepsilon > 0 \)
\[
M = \{ n \in \mathbb{N} : A \subseteq B(A_n, \varepsilon) \} \in F(I).
\]
Then, for \( n \in M \) we have \( B(x, \varepsilon) \cap A_n \neq \emptyset \). So condition (i) in Remark 3.2 is provided.

Conversely, \( x \notin A \). Then, there exists \( \varepsilon > 0 \) such that \( x \notin B(A, \varepsilon) \), i.e., \( d(x, A) > \varepsilon \). By (12)
\[
K = \{ n \in \mathbb{N} : A_n \subseteq B(A, \varepsilon) \} \in F(I).
\]
Take \( \delta = d(x, A) - \varepsilon \). Then, for \( n \in K \) we obtain \( B(x, \delta) \cap A_n = \emptyset \). So condition (ii) in Remark 3.2 is provided.

From conditions (i) and (ii) in Remark 3.2 we have \( I - \lim_{n \to \infty} A_n = A \). \( \square \)

**Definition 4.4.** The sequence \( \{A_n\} \) is said to be \( I \)–bounded if there exists a compact set \( K \) such that
\[
\{ n \in \mathbb{N} : A_n \not\subseteq K \} \in I.
\]
Now, our aim is to show that, for a $I$–bounded closed set, Kuratowski $I$–convergence is equivalent to Hausdorff $I$–convergence.

**Theorem 4.5.** Let $(A_n)$ be a $I$–bounded sequence of closed subsets of $X$. If $\lim_{n \to \infty} A_n = A$ with $A \neq \emptyset$, then $H_I - \lim_{n \to \infty} A_n = A$.

**Proof.** Let $(A_n)$ be a $I$–bounded sequence of closed subsets of $X$. Then there is a compact subset $K$ of $X$ such that

$$M = \{ n \in \mathbb{N} : A_n \subseteq K \} \in F(I).$$

By Lemma 3.12, $I - \lim_{n \to \infty} A_n = A \subseteq K$. So, the closed set $A$ is compact. Then given $\epsilon > 0$, $A$ has a finite cover with open balls of radius $\epsilon$; i.e., there exists \{x_1, x_2, x_3, \ldots, x_n\} with $x_i \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{\epsilon}{2}\right).$$

Since $I - \lim_{n \to \infty} A_n = A$ and $x_i \in A$ for $i \in \{1, 2, \ldots, n\}$, we obtain $I - \lim_{n \to \infty} d(x_i, A_n) = 0$. Therefore, for each $i$

$$\{|n \in \mathbb{N} : d(x_i, A_n) < \epsilon / 2\} \in F(I).$$

Let us define

$$N = \bigcap_{i=1}^{n} \{|n \in \mathbb{N} : d(x_i, A_n) < \epsilon / 2\}.$$  

Then $N \in F(I)$. Thus, we obtain

$$d(y, A_n) \leq d(y, x_i) + d(x_i, A_n) < \epsilon$$

for any $y \in A$ and $n \in N$. So, $A \subseteq B(A_n, \epsilon)$ for every $n \in N$. This means that \{n \in \mathbb{N} : A \nsubseteq B(A_n, \epsilon)\} $\in I$.

Now, suppose that $C = \{n \in \mathbb{N} : A_n \nsubseteq B(A, \epsilon)\} \notin I$ for some $\epsilon > 0$. Then, there exists a sequence \{y_k | y_k \in A_n \cap B(A, \epsilon), k \in C\}. By Lemma 2.2, $M \cap C \notin I$. Hence, \{k \mid y_k \in K\} $\notin I$. By Lemma 2.5, the sequence \{y_n\} has at least $I$–cluster point that belongs to $I - \lim \sup_{n \to \infty} A_n = A$ but does not belong to $B(A, \epsilon) \supseteq A$, which leads to a contradiction. So we have shown that \{n \in \mathbb{N} : A_n \nsubseteq B(A, \epsilon)\} $\in I$. This completes the proof. \(\square\)

**5. Conclusion**

In this paper we give the definitions and some properties of $I$–outer and $I$–inner limits for a sequence of closed sets. We have also introduced two kinds of $I$–convergence for sequences of closed sets which are called Kuratowski $I$–convergence and Hausdorff $I$–convergence. We prove that Hausdorff $I$–convergence implies Kuratowski $I$–convergence. Additionally, for a $I$–bounded sequence of closed sets, we show that these convergences are equivalent.

Continuity properties of a set-valued mapping can be defined on the basis of Kuratowski convergence or Hausdorff convergence (see Chapter 1 in [1], Chapter 3 in [8] and Chapter 5 in [20]). In the light of the main results of our paper, one can define $I$–continuity for a set-valued mapping and get $I$–analogue of continuity properties.

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