



The Homogeneous Weight for R_k , Related Gray Map and New Binary Quasi-Cyclic Codes

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Abstract. Using theoretical results about the homogeneous weights for Frobenius rings, we describe the homogeneous weight for the ring family R_k , a recently introduced family of Frobenius rings which have been used extensively in coding theory. We find an associated Gray map for the homogeneous weight using first order Reed-Muller codes and we describe some of the general properties of the images of codes over R_k under this Gray map. We then discuss quasi-twisted codes over R_k and their binary images under the homogeneous Gray map. In this way, we find many optimal binary codes which are self-orthogonal and quasi-cyclic. In particular, we find a substantial number of optimal binary codes that are quasi-cyclic of index 8, 16 and 24, nearly all of which are new additions to the database of quasi-cyclic codes kept by Chen.

1. Introduction

Codes over Frobenius rings make up an important field of study in the literature of coding theory. An important aspect of Frobenius rings is that they possess a so-called generating character and it is possible to define a homogeneous weight on them. For different ways to describe the homogeneous weight and the related theoretical results we refer the reader to such works as [8], [12], [13] and [14].

Recently, a family of Frobenius rings denoted by R_k have been introduced and studied through different aspects with coding theory. These are finite commutative rings of characteristic 2 that are non-principal, non-chain when $k \geq 2$ and that generalize such rings as $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$. More on these rings can be found in [9], [10], [19], [20], etc.

In this work, contrary to the Lee weights used for the ring R_k up to now, we describe the homogeneous weight for R_k , using the theoretical results related to homogeneous weights for Frobenius rings, and in particular using a generating character for the ring R_k . We then find a Gray map, ψ_k , using the first order Reed-Muller codes, allowing us to map codes over R_k of length n to binary linear codes of length $2^{k-1}n$ in a distance-preserving way. We show that the ψ_k -images of all linear codes over R_k are self-orthogonal binary linear codes for $k \geq 2$. We also describe quasi-twisted codes over R_k , giving some theoretical results (such as all quasi-twisted codes over R_k of odd coindex are quasi-cyclic) and describe the binary images under the ψ_k -map. We also find many optimal binary codes as the ψ_k -images of quasi-twisted codes over R_k for

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suitable k . These binary images are at the same time self-orthogonal and quasi-cyclic of certain indices, almost all of which are new additions to the database of known quasi-cyclic codes kept in [7].

The rest of the work is organized as follows: In section 2 we give some of the preliminaries about the ring family R_k and the homogeneous weights on Frobenius rings. In section 3, we find the general form of the homogeneous weight for R_k and we also describe the related Gray map ψ_k . Section 4 includes the theoretical discussion on quasi-twisted codes over R_k . In section 5, we give the numerical results, where we tabulate the optimal binary codes we have obtained from the ψ_k -images of quasi-twisted codes over R_k . We then finish with section 6, where we mention the concluding remarks and possible directions of future research in related areas.

2. Preliminaries

2.1. The rings R_k

The family of rings denoted by R_k have been introduced in [9]. Leaving the details of these rings to the aforementioned work, we recall some of the basic properties, the proofs of which can be found in [9]. For $k \geq 1$, let

$$R_k = \mathbb{F}_2[u_1, u_2, \dots, u_k] / \langle u_i^2 = 0, u_i u_j = u_j u_i \rangle. \tag{1}$$

We actually take $R_0 = \mathbb{F}_2$, the binary field. The basis elements of R_k can be viewed, using subsets $A \subseteq \{1, 2, \dots, k\}$ by letting

$$u_A := \prod_{i \in A} u_i \tag{2}$$

with the convention that $u_\emptyset = 1$. Then any element of R_k can be represented as

$$\sum_{A \subseteq \{1, \dots, k\}} c_A u_A, \quad c_A \in \mathbb{F}_2. \tag{3}$$

The ring R_k is a local ring with maximal ideal $\langle u_1, u_2, \dots, u_k \rangle$ and $|R_k| = 2^{(2^k)}$. It is neither a principal ideal ring nor a chain ring when $k \geq 2$, but is a Frobenius ring for all $k \geq 0$.

An element of R_k is a unit if and only if the coefficient of u_\emptyset is 1 and each unit is also its own inverse. The following expresses this more accurately:

$$\forall a \in R_k \quad a^2 = \begin{cases} 1 & \text{if } a \text{ is a unit} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The following observations follow easily from the structure of these rings and will be needed later:

Lemma 2.1. (i) For any $a \in R_k$, we have

$$a \cdot (u_1 u_2 \cdots u_k) = \begin{cases} 0 & \text{if } a \text{ is a non-unit} \\ u_1 u_2 \cdots u_k & \text{if } a \text{ is a unit.} \end{cases}$$

(ii) For any unit $\alpha \in R_k$ and $x \in R_k$, we have

$$\alpha \cdot x = u_1 u_2 \cdots u_k \Leftrightarrow x = u_1 u_2 \cdots u_k.$$

We denote the set of units of R_k by $\mathcal{U}(R_k)$ and non-units by $\mathcal{D}(R_k)$. It is clear that

$$|\mathcal{U}(R_k)| = |\mathcal{D}(R_k)| = 2^{2^k - 1} \quad \text{and} \quad \mathcal{U}(R_k) = \mathcal{D}(R_k) + 1. \tag{5}$$

A linear code of length n over R_k is defined to be an R_k -submodule of R_k^n .

2.2. The Homogeneous weight on Frobenius rings

Contrary to most of the work done on R_k , instead of the Lee weight and the related Gray map, we will be interested in the homogeneous weight and the related Gray map. Homogeneous weights were first introduced by Constantinescu and Heise [8]. We may consider these weights as a generalization of the Hamming weight for finite rings.

Definition 2.2 ([12, p. 19]). A real valued function w on the finite ring R is called a (left) homogeneous weight, if $w(0) = 0$ and the following are true.

(H1) For all $x, y \in R$, $Rx = Ry$ implies $w(x) = w(y)$.

(H2) There exists a real number γ such that

$$\sum_{y \in Rx} w(y) = \gamma |Rx| \text{ for all } x \in R \setminus \{0\}.$$

The number γ is the average value of w on R , and from condition (H2) we can deduce that the average value of w is constant on every non-zero principal ideal of R .

Homogeneous weights for Frobenius rings can be described by using the Möbius function. For a finite poset P , consider the function $\mu : P \times P \rightarrow \mathbb{C}$ implicitly defined by $\mu(x, x) = 1$ and $\sum_{y \leq t \leq x} \mu(t, x) = 0$ if $y < x$ and $\mu(y, x) = 0$ if $y \not\leq x$. It is called the Möbius function on P and induces for arbitrary pairs of real-valued functions f, g on P the following equivalences, referred to as Möbius inversion

$$g(x) = \sum_{y \leq x} f(y) \text{ for all } x \in P \Leftrightarrow f(x) = \sum_{y \leq x} g(y) \mu(y, x).$$

Let R be a finite ring and μ be the Möbius function on the set $\{Rx \mid x \in R\}$ of its principal left ideals (partially ordered by inclusion). Further let R^\times denote the set of units in R . The conditions for the existence and uniqueness of homogeneous weights on finite rings are given by the following theorem in [12].

Theorem 2.3 ([12, p. 19]). A real valued function w on the finite ring R is a homogeneous weight if and only if there exists a real number γ such that $w(x) = \gamma \left[1 - \frac{\mu(0, Rx)}{|R^\times x|} \right]$ for all $x \in R$.

Honold [14] described the homogeneous weights on Frobenius rings in terms of generating characters.

Proposition 2.4 ([14, p. 412]). Let R be a finite ring with generating character χ . Then every homogeneous weight on R is of the form

$$w : R \rightarrow \mathbb{R}, x \mapsto \gamma \left[1 - \frac{1}{|R^\times|} \sum_{u \in R^\times} \chi(xu) \right]. \tag{6}$$

By Property (H2), the average weight of a left principal ideal of R is γ . The following proposition shows that for any coset of either a left or a right ideal, the average weight is γ . This property is equivalent to R being Frobenius.

Proposition 2.5 ([13, p. 412]). Let I be either a left or a right ideal of a finite Frobenius ring R , and let $y \in R$. Then $\sum_{r \in I+y} w(r) = \gamma |I|$.

3. The Homogeneous Weight and the Related Gray Map for R_k

3.1. The Homogeneous weight for R_k

We will find the homogeneous weight for the ring family R_k using Proposition 2.4. We recall from [9] that the following is a generating character for the Frobenius rings R_k :

$$\chi\left(\sum_{A \subseteq \{1,2,\dots,k\}} c_A u_A\right) = (-1)^{wt(\bar{c})}, \tag{7}$$

where by $wt(\bar{c})$, we mean the Hamming weight of the \mathbb{F}_2 -coordinate vector of the element in the basis $\{u_A | A \subseteq \{1, 2, \dots, k\}\}$.

For example, for the case of $k = 2$, we have $\chi(0) = 1$

$$\begin{aligned} \chi(1) &= \chi(u) = \chi(v) = \chi(uv) = -1 \\ \chi(1 + u) &= \chi(1 + v) = \chi(1 + uv) = \chi(u + v) = \chi(u + uv) = \chi(v + uv) = 1 \\ \chi(1 + u + v) &= \chi(1 + u + uv) = \chi(1 + v + uv) = \chi(u + v + uv) = -1 \\ \chi(1 + u + v + uv) &= 1. \end{aligned}$$

The following lemma will be a key in proving the main theorem about the homogeneous weight on R_k :

Lemma 3.1. *Let x be any element in R_k such that $x \neq 0$ and $x \neq u_1 u_2 \dots u_k$. Then*

$$\sum_{\alpha \in \mathcal{U}(R_k)} \chi(\alpha x) = 0.$$

Proof. Since χ is a generating character, it is non-trivial when restricted to any non-zero ideal, and thus we have,

$$\sum_{\alpha \in R_k} \chi(\alpha x) = 0. \tag{8}$$

But, by (5), since $\alpha \in \mathcal{U}(R_k)$ if and only if $1 + \alpha \in \mathcal{D}(R_k)$, and

$$\chi((\alpha + 1)x) = \chi(\alpha x + x) = \chi(\alpha x) \cdot \chi(x),$$

the above sum becomes

$$0 = \sum_{\alpha \in \mathcal{U}(R_k)} \chi(\alpha x) + \sum_{\alpha \in \mathcal{U}(R_k)} \chi(\alpha x)\chi(x) = (1 + \chi(x)) \sum_{\alpha \in \mathcal{U}(R_k)} \chi(\alpha x). \tag{9}$$

The proof is done if $\chi(x) = 1$. Now, assume $\chi(x) = -1$. Let us label the sum:

$$F(x) = \sum_{\alpha \in \mathcal{U}(R_k)} \chi(\alpha x).$$

As α runs through all the units of R_k , we can easily observe that $F(x) = F(\beta x)$ for all $\beta \in \mathcal{U}(R_k)$. Thus (9) can be written as

$$(1 + \chi(\alpha x))F(x) = 0, \quad \forall \alpha \in \mathcal{U}(R_k). \tag{10}$$

So, the proof will be complete if we prove that $\chi(\alpha x) = 1$ for at least one value of $\alpha \in \mathcal{U}(R_k)$. Assume to the contrary that $\chi(\alpha x) = -1$ for all $\alpha \in \mathcal{U}(R_k)$. But, then this means we must have $\chi(\beta x) = 1$ for all $\beta \in \mathcal{D}(R_k)$. Now, the ideal generated by x must contain $u_1 u_2 \dots u_k$. Since $x \neq u_1 u_2 \dots u_k$, $\alpha x \neq u_1 u_2 \dots u_k$ for any $\alpha \in \mathcal{U}(R_k)$ by Lemma 2.1. Thus we must have $\beta x = u_1 u_2 \dots u_k$ for some $\beta \in \mathcal{D}(R_k)$. But this is a contradiction since $\chi(u_1 u_2 \dots u_k) = -1$. \square

We are now ready to describe the homogeneous weight for R_k :

Theorem 3.2. *The homogeneous weight on R_k is found to be:*

$$w_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2\gamma & \text{if } x = u_1u_2 \dots u_k \\ \gamma & \text{otherwise.} \end{cases}$$

Proof. Suppose $x = u_1u_2 \dots u_k$. Then by Lemma 2.1, $\alpha x = x$ for all $\alpha \in \mathcal{U}(R_k)$. Thus $\chi(\alpha x) = -1$ for all $\alpha \in \mathcal{U}(R_k)$. Hence, by Proposition 2.4, we have

$$w_{\text{hom}}(x) = \gamma \left[1 - \frac{1}{|\mathcal{U}(R_k)|} \sum_{\alpha \in \mathcal{U}(R_k)} (-1) \right] = 2\gamma.$$

If $x \neq 0$ and $x \neq u_1u_2 \dots u_k$, then by Lemma 3.1, we have $\sum_{\alpha \in R_k} \chi(\alpha x) = 0$. Thus we obtain

$$w_{\text{hom}}(x) = \gamma \left[1 - \frac{1}{|\mathcal{U}(R_k)|} 0 \right] = \gamma.$$

□

3.2. The Gray map for the homogeneous weights

We will assign a numerical value to γ so that it is possible to define a distance-preserving Gray map. In [16], hyperplanes in projective geometries were used to define a Gray map for the homogeneous weight on R_k . Consequently, the choice of γ that was imposed by the combinatorial structure was found to be 2^{2^k-2} . A corresponding Gray map from R_k to $\mathbb{F}_2^{2^k-1}$ was described using the hyperplanes in $PG_{2^k-1}(\mathbb{F}_2)$.

We will adopt the same choice of γ , but instead of the combinatorial description of the Gray map we will give an algebraic description which uses a well-known family of binary codes, namely first order Reed-Muller codes.

Thus for us the homogeneous weight on R_k is now defined as follows:

$$w_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2^{2^k-1} & \text{if } x = u_1u_2 \dots u_k \\ 2^{2^k-2} & \text{otherwise.} \end{cases} \tag{11}$$

Let us recall that first order Reed-Muller codes, denoted by $RM(1, m)$ have the following well-known properties:

1. $RM(1, m)$ is a binary linear code of length 2^m and of dimension $m + 1$. It contains the all 1-vector of length 2^m .
2. The minimum weight of $RM(1, m)$ is 2^{m-1} .
3. Every nonzero codeword other than the all 1-vector has weight 2^{m-1} . In other words the weight enumerator of $RM(1, m)$ is given by $1 + (2^{m+1} - 2)z^{2^{m-1}} + z^{2^m}$.

Notice that R_k can be viewed as an \mathbb{F}_2 -vector space with a basis consisting of $\{u_A | A \subseteq \{1, 2, \dots, k\}\}$. The basis has 2^k elements. Then if we take $RM(1, 2^k - 1)$, and define an \mathbb{F}_2 -linear map $\psi_k : R_k \rightarrow RM(1, 2^k - 1)$ in such a way that the basis elements are mapped to the basic generating vectors of $RM(1, 2^k - 1)$, the map ψ will satisfy the following property:

Theorem 3.3. *The map ψ_k defined above is a distance preserving isometry from $(R_k, \text{homogeneous distance})$ to $(\mathbb{F}_2^{2^k-1}, \text{Hamming distance})$.*

Example 3.4. *For $k = 1$, we get the ring to be $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$ and in this case the homogeneous weight coincides with the usual Lee weight that was defined in [9]. The map ψ is the usual Gray map defined in the same work, namely $\psi_1(0) = (0, 0)$, $\psi_1(1) = (0, 1)$, $\psi_1(u) = (1, 1)$ and $\psi_1(1 + u) = (1, 0)$.*

Example 3.5. For the case when $k = 2$, we describe R_2 as $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, to go by the original notation used in [20]. The homogeneous weight is given by

$$\omega_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 8 & \text{if } x = uv \\ 4 & \text{otherwise.} \end{cases} \tag{12}$$

The map ψ_2 in this case can be described by assigning the basis elements as follows:

$$\psi_2(uv) = (1, 1, 1, 1, 1, 1, 1, 1), \quad \psi_2(u) = (1, 1, 1, 1, 0, 0, 0, 0)$$

$$\psi_2(v) = (1, 1, 0, 0, 1, 1, 0, 0), \quad \psi_2(1) = (1, 0, 1, 0, 1, 0, 1, 0).$$

The maps ψ_k are naturally extended (component-wise) to R_k^n as well. This allows us to consider the ψ_k -images of codes over R_k as well. Thus we have the following theorem:

Theorem 3.6. Let C be a linear code over R_k of length n . Then $\psi_k(C)$ is a binary linear code of length $2^{2^k-1}n$. Moreover the homogeneous weight distribution of C is the same as the Hamming weight distribution of $\psi_k(C)$.

Note that when $k \geq 2$, all the homogeneous weights are divisible by 4. Considering the binary images we get the following observation:

Theorem 3.7. Let C be any linear code over R_k of length n with $k \geq 2$. Then $\psi_k(C)$ is a self-orthogonal binary linear code of length $2^{2^k-1}n$.

Remark 3.8. The size of a linear code over R_k of length n is at most $2^{2^k n}$. An inductive argument shows that $k + 1 < 2^k - 1$ for all $k \geq 3$. Thus, for $k \geq 3$, if C is a linear code over R_k of length n , then

$$\begin{aligned} |\psi_k(C)| &\leq 2^{2^k n} = 2^{2^{k+1} \cdot n/2} \\ &< 2^{2^{2^k-1} \cdot n/2} = \sqrt{|\mathbb{F}_2^{2^{2^k-1} \cdot n}|}. \end{aligned}$$

This shows that the ψ_k -image of a code over R_k cannot be self-dual for any $k \geq 3$. On the other hand, since

$$|R_2^n| = 2^{4n} = \sqrt{2^{8n}}$$

and by Theorem 3.7, $\psi_2(R_2^n)$ is self-orthogonal, we see that $\psi_2(R_2^n)$ is a Type II (i.e. all weights are divisible by 4) binary self-dual code of length n and minimum distance 4 for all $n \geq 1$. When $n = 1$, we get a $[16, 8, 4]$ extremal Type II binary self-dual code.

4. Quasi-Twisted Codes and their Images

We describe a special class of codes over R_k with respect to the homogeneous distance. Cyclic codes are a special class of codes, which possess an algebraic structure that allows them to be encoded and decoded easily and it also provides more information about the code. Cyclic codes over R_2 were studied in [19] and later in more generality in [10] with respect to the Lee metric.

Cyclic codes have two possible generalizations. One is obtained by replacing the ordinary shift with a λ -shift, which results in λ -constacyclic codes. The other possible generalization is achieved through replacing the shift with a composition of the shift, thus giving rise to the so-called quasi-cyclic codes. Constacyclic and quasi-cyclic codes over R_2 were studied in [15] and [5] respectively with respect to the Lee metric. Recently, the concepts of constacyclic and quasi-cyclic codes have been combined to give rise to a new generalization that generalizes all these concepts, and is called quasi-twisted. Quasi-twisted codes have been studied for their help in finding many good codes over different alphabets. For the rest of the paper, quasi-twisted codes will shortly be denoted as QT codes while quasi-cyclic codes will be denoted by QC codes.

Definition 4.1. Let R be a commutative ring with identity and suppose that $\lambda \in R$ is a unit. A λ -shift on R^n is defined to be the map T_λ with the property:

$$T_\lambda(a_0, a_1, \dots, a_{n-1}) = (\lambda a_{n-1}, a_0, a_1, \dots, a_{n-2}).$$

When $\lambda = 1$ we simply denote it by T and we mean it to be the cyclic shift. This leads to the following definition on codes over R :

Definition 4.2. Let C be a linear code over R of length n . We say, C is a cyclic code if $T(C) = C$, a λ -constacyclic code if $T_\lambda(C) = C$. For $\ell|n$, we say C is an ℓ -QC code if $T^\ell(C) = C$, and it is a (λ, ℓ) -QT code if $T_\lambda^\ell(C) = C$. ℓ is called the index of QT codes while n/ℓ is referred to as the coindex of the QT codes.

Structural properties of QT codes over finite fields have been given in [3], [4] and [6]. We recall that to any vector $(a_0, a_1, \dots, a_{n-1}) \in R^n$, we can assign a polynomial $a_0 + a_1x + \dots + a_{n-1}x^{n-1} \in R[x]$. This correspondence leads to the following well-known theorems:

Theorem 4.3. (i) C is a λ -constacyclic code over R of length n if and only if the polynomial correspondence of C is an ideal in $R[x]/(x^n - \lambda)$.

(ii) Suppose $n = \ell \cdot m$. Then a (λ, ℓ) -QT code over R of length n algebraically is an $(R[x]/(x^m - \lambda))$ -submodule of $(R[x]/(x^m - \lambda))^\ell$.

The following theorem, a special case of which was proved in [15], can easily be proved:

Theorem 4.4. Let R be a finite commutative ring with identity and let λ be a unit in R with $\lambda^2 = 1$. If n is odd, the map

$$\mu : R[x]/(x^n - 1) \rightarrow R[x]/(x^n - \lambda)$$

given by $\mu(f(x)) = f(\lambda x)$ is a ring isomorphism.

Since in R_k , all units λ , satisfy $\lambda^2 = 1$, we get the following corollary:

Corollary 4.5. Let n be odd and $\lambda \in R_k$ be any unit. Then C is a λ -constacyclic code over R_k of length n if and only if C is a cyclic code over R_k of length n .

Considering the algebraic structure of the QT codes and the above ring isomorphism, we also get the following result about QT codes over R_k :

Corollary 4.6. Let $n = \ell \cdot m$, where m is odd and $\lambda \in R_k$ is any unit. Then C is a (λ, ℓ) -QT code over R if and only if C is an ℓ -QC code.

The above corollaries imply that we do not need to consider constacyclic codes over R_k of odd lengths and QT-codes of odd coindex as they are cyclic and QC respectively.

4.1. The Binary Images of QT-codes over R_k

We start by observing that

$$\psi_k(a_0, a_1, \dots, a_{n-1}) = (\psi_k(a_0), \psi_k(a_1), \dots, \psi_k(a_{n-1}))$$

for all $a_i \in R_k$. Note that $\psi(a_i)$ is a binary vector of size 2^{2^k-1} . Thus we have

$$\begin{aligned} \psi_k \circ T(a_0, a_1, \dots, a_{n-1}) &= \psi_k(a_{n-1}, a_0, a_1, \dots, a_{n-2}) \\ &= (\psi_k(a_{n-1}), \psi_k(a_0), \dots, \psi_k(a_{n-2})) \\ &= T^{2^{2^k-1}} \circ \psi_k(a_0, a_1, \dots, a_{n-1}). \end{aligned}$$

In other words, we have

$$\psi_k \circ T = T^{2^{2^k-1}} \circ \psi_k. \tag{13}$$

This leads to the following theorem:

Theorem 4.7. *If C is a cyclic code over R_k of length n , then $\psi_k(C)$ is a binary 2^{2^k-1} -QC code of length $2^{2^k-1}n$. If C is an ℓ -QC code over R_k of length n , then $\psi_k(C)$ is a binary $(2^{2^k-1} \cdot \ell)$ -QC code of length $2^{2^k-1}n$.*

Now, let λ be any unit in R_k . Since $\lambda(u_1u_2 \cdots u_k) = u_1u_2 \cdots u_k$ by Lemma 2.1, we have $w_{\text{hom}}(a) = w_{\text{hom}}(\lambda \cdot a)$ for all units λ and elements a in R_k . But this means that for any unit $\lambda \in R_k$ and any element $a \in R_k$, $\psi_k(\lambda \cdot a)$ is permutation equivalent to $\psi_k(a)$. This means that (13) will have the following form

$$\psi_k \circ T_\lambda(a_0, a_1, \dots, a_{n-1}) \simeq T^{2^{2^k-1}} \circ \psi_k(a_0, a_1, \dots, a_{n-1}), \tag{14}$$

where \simeq denotes permutation-equivalence. Thus we have the following version of Theorem 4.7:

Theorem 4.8. *If C is a λ -constacyclic code over R_k of length n , then $\psi_k(C)$ is equivalent to a binary 2^{2^k-1} -QC code of length $2^{2^k-1}n$. If C is a (λ, ℓ) -QT code over R_k of length n , then $\psi_k(C)$ is equivalent to a binary $(2^{2^k-1} \cdot \ell)$ -QC code of length $2^{2^k-1}n$.*

In the case of R_2 , which is the most common case we will use in our constructions, we get the following corollary:

Corollary 4.9. *Let C be an ℓ -QC code of length n over R_2 . Then $\psi_2(C)$ is a binary 8ℓ -QC code of length $8n$. If C is a (λ, ℓ) -QT code of length n over R_2 , then $\psi_2(C)$ is permutation-equivalent to a binary 8ℓ -QC code of length $8n$.*

4.2. One-Generator QT codes

QT codes are structurally complex codes and as such, in their literature the most common types of such codes that have been considered are the so-called one-generator QT codes. The QC, cyclic and constacyclic analogues can easily be considered. Assume that $g(x) \in R[x]/(x^m - \lambda)$ is a polynomial, with λ a unit. Then the one-generator λ -constacyclic code generated by $g(x)$ is simply defined to be the principal ideal $\langle g(x) \rangle$ in the ring $R[x]/(x^m - \lambda)$. It is clear that such a code will be generated by the following matrix:

$$G = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_{m-1} \\ \lambda g_{m-1} & g_0 & g_1 & \cdots & g_{m-2} \\ \lambda g_{m-2} & \lambda g_{m-1} & g_0 & \cdots & g_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda g_1 & \lambda g_2 & \lambda g_3 & \cdots & g_0 \end{bmatrix},$$

where $g(x) = g_0 + g_1x + \cdots + g_{m-1}x^{m-1}$. In some contexts such a matrix is called a λ -twistulant matrix or λ -circulant matrix. When $\lambda = 1$, we simply get a circulant matrix as the generator matrix of a one-generator cyclic code.

Definition 4.10. *A one-generator (λ, ℓ) -QT code over R is a linear code over R generated by a matrix of the form*

$$[G_1|G_2|\cdots|G_\ell],$$

where each G_i is an $m \times m$ λ -twistulant matrix.

The following theorem, whose cyclic analogue is proved in [5], is easily obtained.

Theorem 4.11. *Let $C = \langle g(x) \rangle$ be a λ -constacyclic code over R_k of length m where $g(x)$ is a monic polynomial with $\deg(g(x)) = m - k$. Then C is a free module of rank k if and only if $g(x)|x^m - \lambda$.*

This theorem can then be used to obtain the following result for a special type of one-generator QT codes:

Theorem 4.12. *Suppose C is a (λ, ℓ) -QT code of length $n = m\ell$ generated by $(f_1(x)g(x), f_2(x)g(x), \dots, f_\ell(x)g(x))$, where $x^m - \lambda = g(x)h(x)$ with $g(x)$ and $h(x)$ monic polynomials in $R_k[x]/(x^m - \lambda)$ and $f_i(x)$ is relatively prime to $h(x)$ for all $i = 1, 2, \dots, \ell$. Then C is a free module with rank $m - \deg(g(x))$. In other words, $\psi_k(C)$ is of dimension $2^k(m - \deg(g(x)))$.*

There is a natural projection from R_k to its residue field, namely, \mathbb{F}_2 , which we denote by μ_k . μ_k essentially works as reduction modulo the maximal ideal, that is

$$\mu_k\left(\sum_{A \subseteq \{1, \dots, k\}} c_A u_A\right) = c_\emptyset. \tag{15}$$

Non-units are sent to 0, while units are mapped to 1.

The following lemma provides a natural interval for the minimum homogeneous weight of a code over R_k :

Lemma 4.13. *Let C be a linear code over R_k and suppose the minimum Hamming weight of $\mu_k(C)$ is d . Then*

$$2^{2^k-2}d \leq d_{\text{hom}}(C) \leq 2^{2^k-1}d.$$

Proof. For any codeword $\bar{c} = (c_1, c_2, \dots, c_m) \in C$, the projection under μ_k has at least d non-zero coordinates, which means \bar{c} has at least d unit coordinates, all non-zero. Thus the homogeneous weight of \bar{c} is at least $d \cdot 2^{2^k-2}$, giving us the left hand inequality.

For the upper bound, suppose $(a_1, a_2, \dots, a_m) \in \mu_k(C)$ is a binary codeword in $\mu_k(C)$ whose Hamming weight is d . Since μ_k maps units to 1 and non-units to 0, this means there exists $(c_1, c_2, \dots, c_m) \in C$ such that exactly d of the c_i s are units. But then $u_1 u_2 \dots u_k(c_1, c_2, \dots, c_m) \in C$ as well, since C is linear over R_k and this last codeword has exactly d coordinates that are equal to $u_1 u_2 \dots u_k$ and the rest equal to 0. Since the homogeneous weight of $u_1 u_2 \dots u_k$ is equal to 2^{2^k-1} , we see that

$$w_{\text{hom}}(u_1 u_2 \dots u_k(c_1, c_2, \dots, c_m)) = 2^{2^k-1}d,$$

giving us the necessary upper bound. \square

Corollary 4.14. *Let $C = (g_1(x), g_2(x), \dots, g_\ell(x))$ be a one generator (λ, ℓ) -QT code over R_k of length $n = m\ell$, and suppose the number of unit coefficients of $g_i(x)$ is d_i for $i = 1, 2, \dots, \ell$. Then*

$$d_{\text{hom}}(C) \leq 2^{2^k-1}(d_1 + d_2 + \dots + d_\ell).$$

5. Examples of Optimal Binary Codes from ψ_k -Images of QT Codes over R_k

In this section we will be giving some examples of optimal codes that we have obtained from the ψ_k -images of QT-codes over R_k . The optimality of these codes have been established by theoretical upper bounds and specific constructions in [11]. It turns out that many of our constructions serve as alternative (and usually much simpler) constructions for the optimal codes. Because the images under ψ of our codes are binary QC codes of high index, we also compare our results with those collected in [7], the database of known QT codes. It turns out that we have found many new additions to this database through our constructions.

Before we proceed, we would like to observe that, ψ_k is from R_k to $\mathbb{F}_2^{2^{2^k-1}}$. Thus for example for $k = 3$, ψ_3 is from R_3 to \mathbb{F}_2^{128} while for $k = 4$, ψ_4 is from R_4 to $\mathbb{F}_2^{2^{15}}$. So, for practical purposes, it is not feasible to look beyond $k = 2$. That is why, in what follows we will first write down some of the general examples, but then devote separate subsections to the feasible cases of $k = 1$ and $k = 2$.

5.1. The ψ_k -images of the repetition code

Let C be the code of length n over R_k generated by $(111 \dots 1)$. It is clear that C is a cyclic and QC code for any suitable index. C is a free code of free rank 1. Thus $|C| = 2^{2^k}$. Considering the homogeneous weights and the ψ_k -image we get the following family of binary codes:

Theorem 5.1. *Let C be the code of length n generated by $(11 \dots 1)$ over R_k . Then $\psi_k(C)$ is a binary linear code of parameters $[n \cdot 2^{2^k-1}, 2^k, n \cdot 2^{2^k-2}]$. Moreover, when $k \geq 2$, these codes are all self-orthogonal binary codes.*

Example 5.2. *Putting $k = 2$ into Theorem 5.1, we get self-orthogonal binary linear codes of parameters $[8n, 4, 4n]$ from the ψ_2 -images of the repetition code over R_2 . For $n = 1$ up to 6, we get the self-orthogonal binary codes of parameters $[8, 4, 4]$, $[16, 4, 8]$, $[24, 4, 12]$, $[32, 4, 16]$, $[40, 4, 20]$ and $[48, 4, 24]$, all of which are optimal linear codes according to [11]. When $n = 7$ and 8 we get self-orthogonal binary linear codes of parameters $[56, 4, 28]$ and $[64, 4, 32]$. The optimal codes of these lengths have parameters $[56, 4, 29]$ and $[64, 4, 33]$. But since these latter codes cannot be self-orthogonal, the codes we obtain, have the best possible minimum distance among all the self-orthogonal codes of those lengths and dimensions.*

Example 5.3. *Putting $k = 3$ into Theorem 5.1, we get self-orthogonal binary linear codes of parameters $[128n, 8, 64n]$ from the ψ_3 -images of the repetition code over R_3 . The only cases for which we can make comparisons are the cases when $n = 1$ and $n = 2$. With these values, we obtain self-orthogonal binary linear codes of parameters $[128, 8, 64]$ and $[256, 8, 128]$, both of which are optimal as linear codes.*

5.2. Optimal binary codes from $(1 + u, 3)$ -QT codes over R_1

Note that the ring $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$, the first example of the rings we study, has been studied already in the literature for cyclic, quasi-cyclic and constacyclic codes. We may refer the reader to [1], [2], [17] and [18]. Now, the Gray map ψ_1 on R_1 that we have defined is the same map used in these works that we have mentioned and the homogeneous weight coincides with the Lee weight used. That is why, we will focus on $(1 + u)$ -QT codes here. We will fix the index at 3. The results that we have found are quite different than the ones found in the above-mentioned works. We tabulate our results in the following table. To save space, we will replace $1 + u$ by 3. A typical generator of the QT-code will be given in the form $(g_1g_2 \dots g_m|h_1h_2 \dots h_m|r_1r_2 \dots r_m)$. Thus for example $(1u|30|u3)$ will denote the one generator QT-code generated $(g(x), h(x), r(x))$ where $g(x) = 1 + ux$, $h(x) = 1 + u$ and $r(x) = u + (1 + u)x$.

Table 1: Optimal binary codes from $(1 + u, 3)$ -QT codes over R_1 of length $3m$

m	Generator of the code	Binary Image under ψ_1
2	$(0u 0u uu)$	$[12, 2, 8]$
2	$(10 11 3u)$	$[12, 4, 6]$
2	$(0u 33 13)$	$[12, 3, 6]$
3	$(00u 011 u33)$	$[18, 5, 8]$
3	$(00u 111 111)$	$[18, 4, 8]$
3	$(001 113 1u1)$	$[18, 6, 8]$
3	$(0uu 0uu uu0)$	$[18, 2, 12]$
4	$(0011 001u 00u1)$	$[24, 8, 8]$
4	$(0011 0013 1u1u)$	$[24, 7, 10]$
4	$(000u 00uu 0uuu)$	$[24, 4, 12]$
4	$(0u0u 0u0u uuuu)$	$[24, 2, 16]$
5	$(0011u 001u3 00u33)$	$[30, 8, 12]$
5	$(001u1 0013u 01111)$	$[30, 9, 12]$
5	$(13131 uuuuu 13131)$	$[30, 2, 20]$
6	$(uuuu11 uuu103 u1u311)$	$[36, 11, 12]^b$
6	$(u1u103 u10101 113133)$	$[36, 6, 16]$
6	$(u1u3u1 010301 133113)$	$[36, 4, 18]$
6	$(0u0u0u 0u0u0u uuuuuu)$	$[36, 2, 24]$
7	$(uuu1013 uu01033 uu11101)$	$[42, 11, 16]$
7	$(uu10333 u1330u1 u03u331)$	$[42, 6, 20]$
7	$(1313131 uuuuuuu 1313131)$	$[42, 2, 28]$

Remark 5.4. The $[36, 11, 12]$ -code that is marked with ^b is the best known code of these parameters. The theoretical upper bound for the minimum distance of the code of length 36 and dimension 11 is 13, which has not been attained yet. All the rest of the codes in the table are optimal, meaning that they attain the theoretical upper bounds. We also note that all the binary codes given in the above table are equivalent to 6-QC codes.

A pattern in the table has led us to observe the following: Suppose m is even. Then, since $(1 + u) \cdot u = u$, the $(1 + u, 3)$ -QT code C generated by $(0u0u \dots 0u|0u0u \dots 0u|uu \dots u)$ contains just four codewords given by

$$(00 \dots 0|00 \dots 0|00 \dots 0), (0u0u \dots 0u|0u0u \dots 0u|uu \dots u),$$

$$(u0u0 \dots u0|u0u0 \dots u0|uu \dots u), (uu \dots u|uu \dots u|00 \dots 0).$$

The homogeneous weight distribution of this code is given by $1 + 3z^{4m}$. Thus in the ψ_1 image we get a binary $[6m, 2, 4m]$ -code.

When m is odd, we take the generator to be of the form $(1313 \dots 1|1313 \dots 1|uu \dots u)$. Remembering that 3 stands for $1 + u$ and that $(1 + u)(1 + u) = 1$ and $1 + (1 + u) = u$, again C has 4 codewords in this case, given by

$$(00 \dots 0|00 \dots 0|00 \dots 0), (1313 \dots 1|1313 \dots 1|uu \dots u)$$

$$(3131 \dots 3|3131 \dots 3|uu \dots u), (uu \dots u|uu \dots u|00 \dots 0).$$

This code also has minimum homogeneous weight $4m$.

Thus we have found the following result:

Theorem 5.5. For any $m \geq 1$, we can obtain a binary code of parameters $[6m, 2, 4m]$ as the ψ_1 -image of a $(1 + u, 3)$ -QT code over R_1 . Note that all such codes attain the Griesmer bound and thus are optimal.

5.3. Optimal binary codes from QT codes over R_2

We give some examples of optimal binary codes from the ψ_2 -images of cyclic and QC-codes over R_2 . $R_2 = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, having 16 elements, we use the hexadecimal notation to denote the elements of R_2 in numerical form. For this we fix the basis $\{uv, v, u, 1\}$ for R_2 and represent the element $auv + bv + cu + d$ as the 4-bit $abcd$ which, then is represented by its hexadecimal value. Thus for example $uv + u + 1$ is represented by b , which stands for the numerical value of 11, while $v + u + 1$ is represented by 7.

Table 2: Optimal binary codes from ψ_2 -images of cyclic codes over R_2

n	Generator of the code	Binary Image under ψ_2	As 8-QC in Database in [7]
2	(11)	[16, 4, 8]	New
3	(088)	[24, 2, 16]	New
3	(246)	[24, 4, 12]	New
3	(135)	[24, 8, 8]	New
3	(019)	[24, 9, 8]	New
4	(0282)	[32, 4, 16]	—
4	(1199)	[32, 5, 16]	New
4	(1119)	[32, 6, 16]	New
5	(11111)	[40, 4, 20]	New
5	(02442)	[40, 8, 16]	New
6	(aec26c)	[48, 4, 24]	New
6	(088088)	[48, 2, 32]	New
7	(0888008)	[56, 3, 32]	New
8	(ceec4e6c)	[64, 6, 32]	—

Table 3: Optimal binary codes from ψ_2 -images of ℓ -QC codes of length $\ell \cdot m$ over R_2

ℓ	m	Generator of the code	Binary Image under ψ_2	As ℓ -QC in Database in [7]
2	2	(2c 2c)	[32, 4, 16]	New 16-QC
2	2	(5F 57)	[32, 5, 16]	New 16-QC
2	3	(088 088)	[48, 2, 32]	New 16-QC
2	3	(066 6e8)	[48, 4, 24]	New 16-QC
2	3	(246 2c6)	[48, 5, 24]	New 16-QC
2	4	(aaa2 4e4e)	[64, 5, 32]	New 16-QC
2	4	(1573 bf51)	[64, 6, 32]	New 16-QC
2	4	(f539 b579)	[64, 7, 32]	New 16-QC
3	2	(08 08 88)	[48, 2, 32]	—
3	3	(231 f87 bc7)	[72, 8, 32]	New 24-QC

Remark 5.6. The codes in Tables 1-3 have different properties even if some of them have the same parameters. The binary codes in Table 1 are equivalent to 6-QC codes while the binary codes in Table 2 all are 8-QC codes and the ones in Table 3 are 16-QC or 24-QC according as $\ell = 2$ or 3.

6. Conclusion

It is known in the literature of coding theory that quasi-cyclic codes (and more generally quasi-twisted codes) tend to have good parameters as they satisfy a modified version of the Gilbert-Varshamov bound. Many optimal codes have constructions that are related to quasi-cyclic codes.

In this work, we demonstrated that, considering QT codes over R_k with respect to the homogeneous weight lead to many optimal (self-orthogonal) codes. Our constructions have resulted much more than what has been put into Tables 1-3 (with many different weight enumerators and automorphism groups), however, we have given a sample of these to illustrate the effectiveness of our constructions. We have found many new quasi-cyclic codes that can be added to the database of known binary quasi-cyclic codes in [7]. We have also come up with rather simple constructions for many optimal binary codes in the database [11], that otherwise have rather complicated constructions. We believe such an approach can be applied to other Frobenius rings as well.

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