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Representations of the (b, c)-Inverses in Rings with Involution

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Abstract. Let R be a ring and $b, c \in R$. The concept of (b, c)-inverses was introduced by Drazin in 2012. In this paper, the existence and the expression of the (b, c)-inverse in a ring with an involution are investigated. A new representation of the (b, c)-inverse based on the group inverse is also presented.

1. Introduction

Throughout this paper, R denotes a ring with identity. The set of all idempotents in R is denoted by $R^{\bullet}$. An involution of R is any map $* : R \to R$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$, $(a + b)^* = a^* + b^*$ for any $a, b \in R$. An element $a \in R$ is self-adjoint if $a^* = a$. An element $q \in R$ is a projection if it is self-adjoint idempotent. Let $p \in R^{\bullet}$, the range projection of p is a projection $p^{\perp} \in R$ such that $p^{\perp}p = p$ and $pp^{\perp} = p^{\perp}$ [14].

An element a of R is said to be (von Neumann) regular if there exists an element a^- of R such that $aa^-a = a$. In this case, a^- is called a $\{1\}$-inverse of a. A solution to $xax = x$ is called an outer inverse of a. An element a^+ of R is a $\{1, 2\}$-inverse of a if $aa^+a = a$ and $a^+aa^+ = a^+$ hold.

In [11] a special outer inverse, called (b, c)-inverse (see Definition 2.1), was introduced in the context of semigroups. It is shown that the Moore-Penrose inverse ([20]), the Drazin inverse ([10]), the Chipman's weighted inverse ([3, pp. 114-176], or see [1, pp. 119-120]), the Bott-Duffin inverse ([2]), the inverse along an element ([15]), the core inverse and dual core inverse ([21]) are all special cases of (b, c)-inverses.

The purpose of this article is to give necessary and sufficient conditions for the existence of the (b, c)-inverses in a ring with an involution, and derive new expressions for them, and then state some new properties for these inverses.

Let $a \in R$ and $p, q \in R^{\bullet}$. An element $b \in R$ is the (p, q)-outer generalized inverse of a if

$$bab = b, \quad ba = p, \quad 1 - ab = q. \tag{1}$$

If the (p, q)-outer generalized inverse b exists, it is unique [8] and denoted $a^{(2)}_{p,q}$. For more details about the (p, q)-outer generalized inverse see [4, 5, 7, 9, 12, 13, 17, 19].

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If $p, q \in R^\bullet$, then arbitrary $x \in R$ can be written as

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q),$$

or in the matrix form

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{p \times q},$$

where $x_{11} = pxq$, $x_{12} = px(1-q)$, $x_{21} = (1-p)xq$, $x_{22} = (1-p)x(1-q)$. If $x = (x_{ij})_{p\times q}$ and $y = (y_{ij})_{p\times q}$, then $x + y = (x_{ij} + y_{ij})_{p\times q}$. Moreover, if $r \in R^\bullet$ and $z = (z_{ij})_{q\times r}$, then one can use usual matrix rules in order to multiply x and z. In a ring R with an involution, notice that

$$x^* = \begin{bmatrix} x_{11}^* & x_{21}^* \\ x_{12}^* & x_{22}^* \end{bmatrix}_{q^* \times p^*}.$$

Recall that an element $a \in R$ is group invertible if there exists an unique element $a^\# \in R$ such that

$$aa^\# a = a, \quad a^\# aa^\# = a^\#, \quad aa^\# = a^\# a.$$

We use $R^\#$ to denote the set of all the group invertible element of R. If $a \in R^\#$, then $a^\pi = 1 - aa^\#$ is the spectral idempotent of a.

The following lemma was proved for Banach algebra elements in [16], it is correct in the ring case by elementary computations.

Lemma 1.1. [18, Lemma 1.4]

(i) Let $x = \begin{bmatrix} a & b \\ 0 & s \end{bmatrix}_{p\times p} \in R$. *If* $a \in (pRp)^\#$, $s \in ((1-p)R(1-p))^\#$ *and* $a^\pi b s^\pi = 0$, *then*

$$x^\# = \begin{bmatrix} a^\# & (a^\#)^2 b s^\pi - a^\# b s^\# + a^\pi b (s^\#)^2 \\ 0 & s^\# \end{bmatrix}_{p\times p}.$$

(ii) Let $x = \begin{bmatrix} a & 0 \\ c & s \end{bmatrix}_{p\times p} \in R$. *If* $a \in (pRp)^\#$, $s \in ((1-p)R(1-p))^\#$ *and* $s^\pi c a^\pi = 0$, *then*

$$x^\# = \begin{bmatrix} a^\# & 0 \\ s^\pi c (a^\#)^2 - s^\# c a^\# + (s^\#)^2 c a^\pi & s^\# \end{bmatrix}_{p\times p}.$$

2. The Representations of (b, c)-inverses in Rings with Involution

In this section, we first recall the definition of the (b, c)-inverse and give some lemmas, and then investigate the representations of this inverse.

To discuss these matters more formally, we recall the definition of (b, c)-inverse in [11].

Definition 2.1. [11, Definition 1.3] *Let R be any ring and let a, b, $c \in R$. An element $y \in R$ satisfying*

$$y \in (bRy) \cap (yRc), \qquad yab = b \quad \text{and} \quad cay = c \tag{2}$$

is called a (b, c)-inverse of a.

The element a of R has at most one (b, c)-inverse in R, and if the (b, c)-inverse y of a exists, it always satisfies $yay = y$. We denote by $a^{(b,c)}$ the (b, c)-inverse of a.

The (b, c)-inverse can reduces to classical inverse, Drazin inverse, Moore-Penrose inverse, the Bott-Duffin inverse, the inverse along an element, core inverse and dual core inverse denoted by $a^{(1,1)}$, $a^{(a^j,a^j)}$, $a^{(a^*,a^*)}$, $a^{(e,e)}$, $a^{(d,d)}$, $a^{(a,a^*)}$, $a^{(a^*,a)}$ respectively.

The following result is easy to check by the definition of the (b, c)-inverse.

Lemma 2.2. *Let R be any ring and let $a,\ b,\ c \in R$. If a has a (b, c)-inverse, then b and c are both regular.*

Proof. If a has a (b, c)-inverse, using Definition 2.1, there is $y \in R$ such that (2) holds. This means there exist $s, t \in R$ such that $bsy = y = ytc$, $yab = b$, $cay = c$. Therefore, $b = yab = bsyab = bsb$, $c = cay = caytc = ctc$, that is, b and c are both regular. $\square$

Lemma 2.3. [14, Theorem 2.1] *Let R be a ring with an involution and $p \in R^\bullet$. Then the following are equivalent:*

(i) $p + p^* - 1$ *is invertible in R;*
(ii) $p^\perp$ *and* $(p^*)^\perp$ *exist.*

The range projections are unique, given by the formula

$$p^\perp = p(p + p^* - 1)^{-1}, \qquad (p^*)^\perp = (p + p^* - 1)^{-1}p.$$

If a ring R with an involution has the GN-property ($1 + xx^* \in R^{-1}$ for all $x \in R$), then every idempotent has a unique range projection.

Lemma 2.4. [6, Lemma 2.2] *Let $p \in R^\bullet$ such that $p^\perp$ exists. If $f_p = 1 + p - p^\perp$, then $f_p \in R^{-1}$ and $f_p^{-1} = f^*_{1-p^*}$.*

Base on the above facts, we have the following theorem.

Theorem 2.5. *If $b,\ c \in R$ are regular such that $(bb^-)^\perp$ and $(c^-c)^\perp$ exist. Let $p = bb^-$, $q = c^-c$, $f_p = 1 + p - p^\perp$ and $f_q = 1 + q - q^\perp$. For $a \in R$, then*

(i) $a^{(b,\ c)}$ *exists if and only if*

$$a = f_q^{-1} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p \tag{3}$$

and $(a_3)^{(2)}_{p^\perp, 1-q^\perp}$ *exists.*
In this case,

$$a^{(b,\ c)} = f_p^{-1} \begin{bmatrix} 0 & (a_3)^{(2)}_{p^\perp, 1-q^\perp} \\ 0 & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} f_q. \tag{4}$$

(ii) $a^{(b,\ 1-c^-c)}$ *exists if and only if a is represented as in (3) and* $(a_1)^{(2)}_{p^\perp, q^\perp}$ *exists.*
In this case,

$$a^{(b,\ 1-c^-c)} = f_p^{-1} \begin{bmatrix} (a_1)^{(2)}_{p^\perp, q^\perp} & 0 \\ 0 & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} f_q.$$

(iii) $a^{(1-bb^-,\ c)}$ *exists if and only if a is represented as in (3) and* $(a_4)^{(2)}_{1-p^\perp, 1-q^\perp}$ *exists.*
In this case,

$$a^{(1-bb^-,\ c)} = f_p^{-1} \begin{bmatrix} 0 & 0 \\ 0 & (a_4)^{(2)}_{1-p^\perp, 1-q^\perp} \end{bmatrix}_{p^\perp\times(1-q^\perp)} f_q.$$

(iv) $a^{(1-bb^-,\ 1-c^-c)}$ *exists if and only if a is represented as in (3) and* $(a_2)^{(2)}_{1-p^\perp, q^\perp}$ *exists.*
In this case,

$$a^{(1-bb^-,\ 1-c^-c)} = f_p^{-1} \begin{bmatrix} 0 & 0 \\ (a_2)^{(2)}_{1-p^\perp, q^\perp} & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} f_q.$$

Proof. We only give the proof of item (i), the rest are left to the reader by using similar techniques.

(i). Necessity. Suppose that $a^{(b,c)}$ exists, by Definition 2.1, there exists $y \in R$ such that (2) holds. This means $y = bsy = ytc$, $yab = b$, $cay = c$ for some $s, t \in R$. As b and c are regular and $p = bb^-$, $q = c^-c$, then $py = bb^-bsy = bsy = y$, $yabb^- = bb^-$, $c^-cay = c^-c$, $yq = yc^-c = ytcc^-c = ytc = y$, that is,

$$y = py, \quad yap = p, \quad qay = q, \quad y(1-q) = 0. \tag{5}$$

As $p = bb^-$, $q = c^-c$ are idempotents, we have the following representations of p and q:

$$p = \begin{bmatrix} p^\perp & p_1 \\ 0 & 0 \end{bmatrix}_{p^\perp \times p^\perp}, \quad q = \begin{bmatrix} 0 & 0 \\ q_1 & q^\perp \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)}. \tag{6}$$

Assume that

$$a = \begin{bmatrix} a_1 & a_2 + a_1p_1 \\ a_3 - q_1a_1 & a_4 + a_3p_1 - q_1a_2 - q_1a_1p_1 \end{bmatrix}_{(1-q^\perp)\times p^\perp},$$

where $a_1 \in (1-q^\perp)Rp^\perp$, $a_2 \in (1-q^\perp)R(1-p^\perp)$, $a_3 \in q^\perp Rp^\perp$, $a_4 \in q^\perp R(1-p^\perp)$. And suppose

$$a^{(b,c)} = y = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p^\perp \times (1-q^\perp)}.$$

From (5), $y = py$ gives $b_3 = b_4 = 0$. Since $y(1-q) = 0$, we obtain $b_1 = b_2q_1$. The equality $yap = p$ implies $b_2a_3p^\perp = p^\perp$, $b_2a_3p_1 = p_1$, that is $b_2a_3 = p^\perp$. Similarly, $qay = q$ can reduce to $a_3b_2 = q^\perp$. Note that $yay = y$ implies $b_2a_3b_2 = b_2$. By (1), we get at once $b_2 = (a_3)^{(2)}_{p^\perp,(1-q^\perp)}$, and

$$a^{(b,c)} = \begin{bmatrix} (a_3)^{(2)}_{p^\perp,(1-q^\perp)}q_1 & (a_3)^{(2)}_{p^\perp,(1-q^\perp)} \\ 0 & 0 \end{bmatrix}_{p^\perp \times (1-q^\perp)}.$$

Furthermore,

$$\begin{aligned} f_q^{-1} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p &= \begin{bmatrix} 1-q^\perp & 0 \\ -q_1 & q^\perp \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{(1-q^\perp)\times p^\perp} \begin{bmatrix} p^\perp & p_1 \\ 0 & 1-p^\perp \end{bmatrix}_{p^\perp \times p^\perp} \\ &= \begin{bmatrix} a_1 & a_2 + a_1p_1 \\ a_3 - q_1a_1 & a_4 + a_3p_1 - q_1a_2 - q_1a_1p_1 \end{bmatrix}_{(1-q^\perp)\times p^\perp} \\ &= a, \end{aligned}$$

and

$$\begin{aligned} f_p^{-1} \begin{bmatrix} 0 & (a_3)^{(2)}_{p^\perp,1-q^\perp} \\ 0 & 0 \end{bmatrix}_{p^\perp \times (1-q^\perp)} f_q &= \begin{bmatrix} p^\perp & -p_1 \\ 0 & 1-p^\perp \end{bmatrix}_{p^\perp \times p^\perp} \begin{bmatrix} 0 & (a_3)^{(2)}_{p^\perp,1-q^\perp} \\ 0 & 0 \end{bmatrix}_{p^\perp \times (1-q^\perp)} \begin{bmatrix} 1-q^\perp & 0 \\ q_1 & q^\perp \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} \\ &= \begin{bmatrix} (a_3)^{(2)}_{p^\perp,(1-q^\perp)}q_1 & (a_3)^{(2)}_{p^\perp,(1-q^\perp)} \\ 0 & 0 \end{bmatrix}_{p^\perp \times (1-q^\perp)} \\ &= a^{(b,c)}. \end{aligned}$$

Sufficiency. If a has the form (3), let $y = f_p^{-1} \begin{bmatrix} 0 & (a_3)^{(2)}_{p^\perp,1-q^\perp} \\ 0 & 0 \end{bmatrix}_{p^\perp \times (1-q^\perp)} f_q$, then

$$\begin{aligned} y &= \begin{bmatrix} p^\perp & -p_1 \\ 0 & 1-p^\perp \end{bmatrix}_{p^\perp \times p^\perp} \begin{bmatrix} 0 & (a_3)^{(2)}_{p^\perp,1-q^\perp} \\ 0 & 0 \end{bmatrix}_{p^\perp \times (1-q^\perp)} \begin{bmatrix} 1-q^\perp & 0 \\ q_1 & q^\perp \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} \\ &= \begin{bmatrix} (a_3)^{(2)}_{p^\perp,(1-q^\perp)}q_1 & (a_3)^{(2)}_{p^\perp,(1-q^\perp)} \\ 0 & 0 \end{bmatrix}_{p^\perp \times (1-q^\perp)}. \end{aligned}$$

So we have

$$\begin{aligned} bb^-y = py &= \begin{bmatrix} p^\perp & p_1 \\ 0 & 0 \end{bmatrix}_{p^\perp\times p^\perp} \begin{bmatrix} (a_3)^{(2)}_{p^\perp,(1-q^\perp)}q_1 & (a_3)^{(2)}_{p^\perp,(1-q^\perp)} \\ 0 & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} \\ &= \begin{bmatrix} (a_3)^{(2)}_{p^\perp,(1-q^\perp)}q_1 & (a_3)^{(2)}_{p^\perp,(1-q^\perp)} \\ 0 & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} = y, \end{aligned}$$

$$\begin{aligned} yc^-c = yq &= \begin{bmatrix} (a_3)^{(2)}_{p^\perp,(1-q^\perp)}q_1 & (a_3)^{(2)}_{p^\perp,(1-q^\perp)} \\ 0 & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} \begin{bmatrix} 0 & 0 \\ q_1 & q^\perp \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} \\ &= \begin{bmatrix} (a_3)^{(2)}_{p^\perp,(1-q^\perp)}q_1 & (a_3)^{(2)}_{p^\perp,(1-q^\perp)} \\ 0 & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} = y, \end{aligned}$$

$$\begin{aligned} yabb^- = yap &= f_p^{-1}\begin{bmatrix} 0 & (a_3)^{(2)}_{p^\perp,1-q^\perp} \\ 0 & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p \begin{bmatrix} p^\perp & p_1 \\ 0 & 0 \end{bmatrix}_{p^\perp\times p^\perp} \\ &= \begin{bmatrix} p^\perp & p_1 \\ 0 & 0 \end{bmatrix}_{p^\perp\times p^\perp} = p = bb^-, \end{aligned}$$

$$\begin{aligned} c^-cay &= qay \\ &= \begin{bmatrix} 0 & 0 \\ q_1 & q^\perp \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} f_q^{-1} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{(1-q^\perp)\times p^\perp} \begin{bmatrix} 0 & (a_3)^{(2)}_{p^\perp,1-q^\perp} \\ 0 & 0 \end{bmatrix}_{p^\perp\times(1-q^\perp)} f_q \\ &= \begin{bmatrix} 0 & 0 \\ q_1 & q^\perp \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} = q = c^-c. \end{aligned}$$

Therefore, we have $y \in bRy \cap yRc$, and $yab = b, cay = c$, that is, $a^{(b,c)}$ exists. □

Theorem 2.6. *If b, $c \in R$ are regular such that $(bb^-)^\perp$ and $(c^-c)^\perp$ exist. Let $p = bb^-$, $q = c^-c$, $f_p = 1 + p - p^\perp$ and $f_q = 1 + q - q^\perp$. For $d \in R$, then*

(i) $d^{(1-c^-c,\ 1-bb^-)}$ *exists if and only if*

$$d = f_p^{-1}\begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}_{p^\perp\times(1-q^\perp)} f_q \tag{7}$$

and $(d_3)^{(2)}_{1-q^\perp,p^\perp}$ exists.
In this case,

$$d^{(1-c^-c,\ 1-bb^-)} = f_q^{-1}\begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,p^\perp} \\ 0 & 0 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p. \tag{8}$$

(ii) $d^{(1-c^-c,\ bb^-)}$ *exists if and only if d is represented as in (7) and $(d_1)^{(2)}_{1-q^\perp,1-p^\perp}$ exists.*
In this case,

$$d^{(1-c^-c,\ bb^-)} = f_q^{-1}\begin{bmatrix} (d_1)^{(2)}_{1-q^\perp,1-p^\perp} & 0 \\ 0 & 0 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p.$$

(iii) $d^{(c^-c,\ 1-bb^-)}$ *exists if and only if d is represented as in (7) and $(d_4)^{(2)}_{q^\perp,p^\perp}$ exists.*
In this case,

$$d^{(c^-c,\ 1-bb^-)} = f_q^{-1}\begin{bmatrix} 0 & 0 \\ 0 & (d_4)^{(2)}_{q^\perp,p^\perp} \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p.$$

(iv) $d^{(c^-c,\, bb^-)}$ *exists if and only if d is represented as in (7) and* $(d_2)^{(2)}_{q^\perp,1-p^\perp}$ *exists.*
In this case,

$$d^{(c^-c,\, bb^-)} = f_q^{-1}\begin{bmatrix} 0 & 0 \\ (d_2)^{(2)}_{q^\perp,1-p^\perp} & 0 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p.$$

Proof. We only give the proof of item (i), the rest are left to the reader by using similar techniques.

(i). Necessity. Assume that $d^{(1-c^-c,\, 1-bb^-)}$ exists, according to Definition 2.1, there is $y \in R$ such that $y \in (1-c^-c)Ry \cap yR(1-bb^-)$, $yd(1-c^-c) = 1-c^-c$, $(1-bb^-)dy = 1-bb^-$. So there exist $s, t \in R$ such that

$$y = (1-c^-c)sy = yt(1-bb^-),\ yd(1-c^-c) = 1-c^-c,\ (1-bb^-)dy = 1-bb^-.$$

Since $p = bb^-$, $q = c^-c$, we know $(1-q)y = (1-c^-c)(1-c^-c)sy = (1-c^-c)sy = y$, $yp = ybb^- = yt(1-bb^-)bb^- = 0$. So we have

$$y = (1-q)y,\ yp = 0,\ yd(1-q) = 1-q,\ (1-p)dy = 1-p.$$

Let p and q be represented as in (6). Denote by

$$d = \begin{bmatrix} d_1 + d_2q_1 - p_1d_3 - p_1d_4q_1 & d_2 - p_1d_4 \\ d_3 + d_4q_1 & d_4 \end{bmatrix}_{p^\perp\times(1-q^\perp)},$$

where $d_1 \in p^\perp R(1-q^\perp)$, $d_2 \in p^\perp Rq^\perp$, $d_3 \in (1-p^\perp)R(1-q^\perp)$, $d_4 \in (1-p^\perp)Rq^\perp$. And suppose

$$d^{(1-c^-c,1-bb^-)} = y = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{(1-q^\perp)\times p^\perp}.$$

From $yp = 0$, we have $b_1 = 0$. As $y = (1-q)y$, we get $b_3 = -q_1b_1 = 0$, $b_4 = -q_1b_2$. The condition $yd(1-q) = 1-q$ implies $b_2d_3 = 1-q^\perp$, $q_1b_2d_3 = q_1$. The equality $(1-p)dy = 1-p$ gives $p_1d_3b_2 = p_1$, $d_3b_2 = 1-p^\perp$. Notice that $ydy = y$, which implies $b_2d_3b_2 = b_2$. By (1), we can deduce that $b_2 = (d_3)^{(2)}_{1-q^\perp,\, p^\perp}$. So we have

$$d^{(1-c^-c,1-bb^-)} = \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,\, p^\perp} \\ 0 & -q_1(d_3)^{(2)}_{1-q^\perp,\, p^\perp} \end{bmatrix}_{(1-q^\perp)\times p^\perp}.$$

Therefore,

$$f_p^{-1}\begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}_{p^\perp\times(1-q^\perp)} f_q = \begin{bmatrix} d_1 + d_2q_1 - p_1d_3 - p_1d_4q_1 & d_2 - p_1d_4 \\ d_3 + d_4q_1 & d_4 \end{bmatrix}_{p^\perp\times(1-q^\perp)} = d.$$

And

$$f_q^{-1}\begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,p^\perp} \\ 0 & 0 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p = \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,\, p^\perp} \\ 0 & -q_1(d_3)^{(2)}_{1-q^\perp,\, p^\perp} \end{bmatrix}_{(1-q^\perp)\times p^\perp} = d^{(1-c^-c,1-bb^-)}.$$

Sufficiency. If d has the form (7), let $y = f_q^{-1}\begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,p^\perp} \\ 0 & 0 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p$, we have

$$\begin{aligned} y &= \begin{bmatrix} 1-q^\perp & 0 \\ -q_1 & q^\perp \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,p^\perp} \\ 0 & 0 \end{bmatrix}_{(1-q^\perp)\times p^\perp} \begin{bmatrix} p^\perp & p_1 \\ 0 & 1-p^\perp \end{bmatrix}_{p^\perp\times p^\perp} \\ &= \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,\, p^\perp} \\ 0 & -q_1(d_3)^{(2)}_{1-q^\perp,\, p^\perp} \end{bmatrix}_{(1-q^\perp)\times p^\perp}. \end{aligned}$$

So we have

$$
\begin{aligned}
(1-c^-c)y &= \begin{bmatrix} 1-q^\perp & 0 \\ -q_1 & 0 \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,\, p^\perp} \\ 0 & -q_1(d_3)^{(2)}_{1-q^\perp,\, p^\perp} \end{bmatrix}_{(1-q^\perp)\times p^\perp} \\
&= \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,\, p^\perp} \\ 0 & -q_1(d_3)^{(2)}_{1-q^\perp,\, p^\perp} \end{bmatrix}_{(1-q^\perp)\times p^\perp} = y,
\end{aligned}
$$

$$
\begin{aligned}
y(1-bb^-) &= \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,\, p^\perp} \\ 0 & -q_1(d_3)^{(2)}_{1-q^\perp,\, p^\perp} \end{bmatrix}_{(1-q^\perp)\times p^\perp} \begin{bmatrix} 0 & -p_1 \\ 0 & 1-p^\perp \end{bmatrix}_{p^\perp\times p^\perp} \\
&= \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,\, p^\perp} \\ 0 & -q_1(d_3)^{(2)}_{1-q^\perp,\, p^\perp} \end{bmatrix}_{(1-q^\perp)\times p^\perp} = y,
\end{aligned}
$$

$$
\begin{aligned}
yd(1-c^-c) &= f_q^{-1} \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,p^\perp} \\ 0 & 0 \end{bmatrix}_{(1-q^\perp)\times p^\perp} \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}_{p^\perp\times(1-q^\perp)} f_q \begin{bmatrix} 1-q^\perp & 0 \\ -q_1 & 0 \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} \\
&= \begin{bmatrix} 1-q^\perp & 0 \\ -q_1 & 0 \end{bmatrix}_{(1-q^\perp)\times(1-q^\perp)} = 1-c^-c,
\end{aligned}
$$

$$
\begin{aligned}
(1-bb^-)dy &= \begin{bmatrix} 0 & -p_1 \\ 0 & 1-p^\perp \end{bmatrix}_{p^\perp\times p^\perp} f_p^{-1} \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}_{p^\perp\times(1-q^\perp)} \begin{bmatrix} 0 & (d_3)^{(2)}_{1-q^\perp,p^\perp} \\ 0 & 0 \end{bmatrix}_{(1-q^\perp)\times p^\perp} f_p \\
&= \begin{bmatrix} 0 & -p_1 \\ 0 & 1-p^\perp \end{bmatrix}_{p^\perp\times p^\perp} = 1-bb^-.
\end{aligned}
$$

Therefore, we have $y \in (1-c^-c)Ry \cap yR(1-bb^-)$, $yd(1-c^-c) = 1-c^-c$, $(1-bb^-)dy = 1-bb^-$, that is, $d^{(1-c^-c,1-bb^-)}$ exists. □

The next theorem gives some properties of the (b,c)-inverse.

Theorem 2.7. *Let $a,\ b,\ c \in R$. Then*

(i) $a^{(b,c)}$ exists if and only if $(a^*)^{(c^*,b^*)}$ exists. In addition, $(a^{(b,c)})^* = (a^*)^{(c^*,b^*)}$.
(ii) If $a^{(b,c)}$ exists, then $(a^{(b,c)})^2 = a^{(b,c)}$ if and only if $a^{(b,c)}b = b$.

Proof. (i). By Definition 2.1, $a^{(b,c)}$ exists if and only if there exists $y \in R$ such that

$$y \in bRy \cap yRc,\ yab = b,\ cay = c,$$

which is equivalent to there is $y \in R$ such that

$$y^* \in c^*Ry^* \cap y^*Rb^*,\ y^*a^*c^* = c^*,\ b^*a^*y^* = b^*.$$

That is, $(a^*)^{(c^*,b^*)}$ exists, and $(a^{(b,c)})^* = (a^*)^{(c^*,b^*)}$.

(ii). If $a^{(b,c)}$ exists and $(a^{(b,c)})^2 = a^{(b,c)}$, it follows that

$$b = a^{(b,c)}ab = (a^{(b,c)})^2ab = a^{(b,c)}(a^{(b,c)}ab) = a^{(b,c)}b.$$

Conversely, if $a^{(b,c)}$ exists and $a^{(b,c)}b = b$, from Definition 2.1, there is $s \in R$ such that $a^{(b,c)} = bsa^{(b,c)} = a^{(b,c)}bsa^{(b,c)} = (a^{(b,c)})^2$. □

In the following result, we consider $b = c = e \in R^\bullet$.

Theorem 2.8. *Let $a \in R$ and $e \in R^\bullet$ such that $e^\perp$ exists. Let $f_e = 1 + e - e^\perp$. If $a^{(e,e)}$ exists, then*

(i) $a^{(e,e)}a = aa^{(e,e)}$ *if and only if*

$$a = f_e^{-1} \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \end{bmatrix}_{e^\perp \times e^\perp} f_e. \tag{9}$$

(ii) $a^{(e,e)}$ *is self-adjoint if and only if* e *and* $e^\perp a e^\perp$ *are self-adjoint.*

Proof. The proof is left to the reader since it is same as the proof of [18, Theorem 2.5]. □

Now we will present a representation of the (b, c)-inverse based on group inverse.

Theorem 2.9. *If* $b,\ c \in R$ *are regular such that* $(bb^-)^\perp$ *and* $(c^-c)^\perp$ *exist. Let* $p = bb^-$, $q = c^-c$, $f_p = 1 + p - p^\perp$ *and* $f_q = 1 + q - q^\perp$. *Suppose* $a, d \in R$ *such that* $a^{(bb^-,\ 1-c^-c)}$ *and* $d^{(1-c^-c,\ bb^-)}$ *exist. Then*

(i) $pd(1-q)a,\ pd(1-q)ap \in R^\#$,

$$a^{(bb^-,\ 1-c^-c)} = [pd(1-q)a]^\# pd(1-q) = [pd(1-q)ap]^\# pd(1-q),$$

$$d^{(1-c^-c,\ bb^-)} = (1-q)a[pd(1-q)ap]^\#.$$

(ii) $apd(1-q),\ (1-q)apd(1-q) \in R^\#$,

$$a^{(bb^-,\ 1-c^-c)} = pd(1-q)[apd(1-q)]^\# = pd[(1-q)apd(1-q)]^\#,$$

$$d^{(1-c^-c,\ bb^-)} = [(1-q)apd(1-q)]^\#(1-q)ap.$$

(iii) $pd(1-q)ap,\ (1-q)apd(1-q) \in R^\#$,

$$a^{(bb^-,\ 1-c^-c)} = pd(1-q)[(1-q)apd(1-q)]^\#,$$

$$d^{(1-c^-c,\ bb^-)} = (1-q)ap[pd(1-q)ap]^\#.$$

Proof. The proof is left to the reader since it is same as the proof of [18, Theorem 2.6]. □

Let R be a ring with an involution and has the GN-property (that is, for all $x \in R$, $1 + x^*x$ is invertible), then we can omit the assumption $p^\perp$ and $q^\perp$ exist in the previous results.

Let $\mathcal{A}$ be a C^*-algebra with unit 1. An element x of $\mathcal{A}$ is positive (denoted by $x \geq 0$) if $x^* = x$ and $\sigma(x) \subseteq [0, +\infty)$, where $\sigma(x)$ denotes the spectrum of x.

Theorem 2.10. *Let* $a, d \in \mathcal{A}$. *If* $b,\ c \in \mathcal{A}$ *are regular such that* $(bb^-)^\perp$ *and* $(c^-c)^\perp$ *exist. Take* $p = bb^-$ *and* $q = c^-c$. *Suppose* $a^{(bb^-,\ 1-c^-c)}$ *exists and* $a^{(bb^-,\ 1-c^-c)} \geq 0$, *then* $(a + dd^*)^{(bb^-,\ 1-c^-c)}$ *exists and*

$$(a + dd^*)^{(bb^-,\ 1-c^-c)} = a^{(bb^-,\ 1-c^-c)} - a^{(bb^-,\ 1-c^-c)}d(1 + d^*a^{(bb^-,\ 1-c^-c)}d)^{-1}d^*a^{(bb^-,\ 1-c^-c)}.$$

Proof. The condition $a^{(bb^-,\ 1-c^-c)} \geq 0$ gives $a^{(bb^-,\ 1-c^-c)} = s^*s$, for some $s \in \mathcal{A}$. So we have $d^*a^{(bb^-,\ 1-c^-c)}d = d^*s^*sd = (sd)^*sd$, which implies that $1 + d^*a^{(bb^-,\ 1-c^-c)}d$ is invertible. Denote by $x = a + dd^*$ and $y = a^{(bb^-,\ 1-c^-c)} - a^{(bb^-,\ 1-c^-c)}d(1+d^*a^{(bb^-,\ 1-c^-c)}d)^{-1}d^*a^{(bb^-,\ 1-c^-c)}$. Then, it is easy to get $bb^-y = y$, $y(1-c^-c) = y$ since $bb^-a^{(bb^-,\ 1-c^-c)} = a^{(bb^-,\ 1-c^-c)}$ and $a^{(bb^-,\ 1-c^-c)}(1-c^-c) = a^{(bb^-,\ 1-c^-c)}$. Notice that $a^{(bb^-,\ 1-c^-c)}abb^- = bb^-$ and $(1-c^-c)aa^{(bb^-,\ 1-c^-c)} = 1 - c^-c$, so we have

$$\begin{aligned} yxbb^- &= a^{(bb^-,\ 1-c^-c)}abb^- + a^{(bb^-,\ 1-c^-c)}dd^*bb^- - a^{(bb^-,\ 1-c^-c)}d(1 + d^*a^{(bb^-,\ 1-c^-c)}d)^{-1} \\ &\quad \times(d^*a^{(bb^-,\ 1-c^-c)}abb^- + d^*a^{(bb^-,\ 1-c^-c)}dd^*bb^-) \\ &= bb^- + a^{(bb^-,\ 1-c^-c)}dd^*bb^- - a^{(bb^-,\ 1-c^-c)}d(1 + d^*a^{(bb^-,\ 1-c^-c)}d)^{-1} \\ &\quad \times(1 + d^*a^{(bb^-,\ 1-c^-c)}d)d^*bb^- \\ &= bb^-, \end{aligned}$$

and

$$\begin{aligned}(1-c^-c)xy &= (1-c^-c)aa^{(bb^-,\,1-c^-c)} + (1-c^-c)dd^*a^{(bb^-,\,1-c^-c)} - (1-c^-c)\\ &\quad \times(aa^{(bb^-,\,1-c^-c)}d + dd^*a^{(bb^-,\,1-c^-c)}d)(1+d^*a^{(bb^-,\,1-c^-c)}d)^{-1}d^*a^{(bb^-,\,1-c^-c)}\\ &= 1-c^-c + (1-c^-c)dd^*a^{(bb^-,\,1-c^-c)} - (1-c^-c)d(1+d^*a^{(bb^-,\,1-c^-c)}d)\\ &\quad \times(1+d^*a^{(bb^-,\,1-c^-c)}d)^{-1}d^*a^{(bb^-,\,1-c^-c)}\\ &= 1-c^-c.\end{aligned}$$

Thus, we obtain $y \in bb^-\mathcal{A}y \cap y\mathcal{A}(1-c^-c)$ and $yxbb^- = bb^-, (1-c^-c)xy = 1-c^-c$, so we can conclude that $x^{(bb^-,\,1-c^-c)}$ exists and $x^{(bb^-,\,1-c^-c)} = y$. □

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