Filomat 31:9 (2017), 2837–2849
https://doi.org/10.2298/FIL1709837E

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Novel Subclass of Analytic Functions Specified by a Family of Fractional Derivatives in the Complex Domain

Zainab Esa[a], H. M. Srivastava[b], Adem Kılıçman[c], Rabha W. Ibrahim[d]

[a]*Department of Mathematics, University Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*
[b]*Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*
and
China Medical University, Taichung 40402, Taiwan, Republic of China
[c]*Department of Mathematics, University Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*
[d]*Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia*

Abstract. In this paper, by making use of a certain family of fractional derivative operators in the complex domain, we introduce and investigate a new subclass $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$ of analytic and univalent functions in the open unit disk $\mathbb{U}$. In particular, for functions in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$, we derive sufficient coefficient inequalities and coefficient estimates, distortion theorems involving the above-mentioned fractional derivative operators, and the radii of starlikeness and convexity. In addition, some applications of functions in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$ are also pointed out.

1. Introduction

Let $\mathcal{H}$ be the class of functions which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{H}[a,k]$ denote the subclass of $\mathcal{H}$ consisting of analytic functions of the form:

$$f(z) = a + \sum_{j=k}^{\infty} a_j z^j = a + a_k z^k + a_{k+1} z^{k+1} + \cdots .$$

We denote by $\mathcal{A}(k)$ the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{\nu=k+1}^{\infty} a_\nu z^\nu \qquad (z \in \mathbb{U};\ k \in \mathbb{N} := \{1,2,3,\cdots\}), \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 26A33
Keywords. Analytic functions; Univalent functions; Fractional integral and fractional derivative operators; Srivastava-Owa operator of fractional derivative; Coefficient inequalities and coefficient estimates; Distortion theorems; Radii of convexity and starlikeness; Modified convolution.
Received: 10 September 2015; Accepted: 06 March 2016
Communicated by Dragan S. Djordjević
Email addresses: esazainab@yahoo.com (Zainab Esa), harimsri@math.uvic.ca (H. M. Srivastava), akilic@upm.edu.my (Adem Kılıçman), rabhaibrahim@yahoo.com (Rabha W. Ibrahim)

which are analytic in the open unit disk $\mathbb{U}$. In particular, we write

$$\mathcal{A}(1) =: \mathcal{A}.$$

Let $\mathcal{S}(k)$ denote the subclass of $\mathcal{A}(k)$ consisting of functions which are univalent in $\mathbb{U}$. Then, by definition, a function $f(z)$ belonging to the univalent function class $\mathcal{S}(k)$ is said to be a starlike function of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{2}$$

Furthermore, a function $f(z)$ in the univalent function class $\mathcal{S}(k)$ is said to be a convex function of order α $(0 \leqq \alpha < 1)$ in in $\mathbb{U}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{3}$$

We denote by $\mathcal{S}^*(k,\alpha)$ and $\mathcal{K}(k,\alpha)$ the classes of all functions in $\mathcal{S}(k)$ which are, respectively, starlike of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ and convex of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$.

Let $\mathcal{P}(k)$ denote the subclass of $\mathcal{S}(k)$ consisting of functions $f(z)$ which are analytic *and* univalent in $\mathbb{U}$ with negative coefficients, that is, of the form:

$$f(z) = z - \sum_{\nu=k+1}^{\infty} a_\nu z^\nu \qquad (z \in \mathbb{U};\ a_\nu \geqq 0). \tag{4}$$

For $0 \leqq \alpha < 1$ and $k \in \mathbb{N}$, we write

$$\mathcal{P}^*(k,\alpha) := \mathcal{S}^*(k,\alpha) \cap \mathcal{P}(k) \qquad \text{and} \qquad \mathcal{L}(k,\alpha) := \mathcal{K}(k,\alpha) \cap \mathcal{P}(k). \tag{5}$$

Chatterjea [3] studied the classes $\mathcal{P}^*(k,\alpha)$ and $\mathcal{L}(k,\alpha)$, which are, respectively, starlike and convex of order α in $\mathbb{U}$. Subsequently, Srivastava *et al.* [12] observed and remarked that some of the results of Chatterjea [3] would follow immediately by trivially setting

$$a_k = 0 \qquad (k \in \mathbb{N} \setminus \{1\} = \{2,3,4,\cdots\})$$

in the corresponding earlier results of Silverman [11, p. 110, Theorem 2; p. 111, Corollary 2] (see, for details, [12, p. 117]).

The *modified* convolution of two analytic functions $f(z)$ and $\psi(z)$ in the class $\mathcal{P}(k)$ is defined by (see [10])

$$f * \psi(z) := z - \sum_{\nu=k+1}^{\infty} a_\nu \lambda_\nu z^\nu =: \psi * f(z),$$

where $f(z)$ is given by (4) and $\psi(z)$ is defined as follows:

$$\psi(z) = z - \sum_{\nu=k+1}^{\infty} \lambda_\nu z^\nu \qquad (\lambda_\nu \geqq 0;\ k \in \mathbb{N}). \tag{6}$$

Definition 1. The fractional integral of order ς is defined, for a function $f(z)$, by

$$\mathcal{I}_z^\varsigma f(z) := \frac{1}{\Gamma(\varsigma)} \int_0^z f(\zeta)(z-\zeta)^{\varsigma-1}\, \mathrm{d}\zeta, \tag{7}$$

where $0 \leqq \varsigma < 1$, the function $f(z)$ is analytic in a simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{\varsigma-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Here, and in what follows, we refer to $\mathcal{I}_z^{\varsigma} f(z)$ as the Srivastava-Owa operator of fractional integral. Similarly, we have the following definition of the Srivastava-Owa operator of fractional derivative (see also [7]).

Definition 2. The fractional derivative of order ς is defined, for a function $f(z)$, by

$$\mathfrak{D}_z^{\varsigma} f(z) := \frac{1}{\Gamma(1-\varsigma)} \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \int_0^z f(\zeta)(z-\zeta)^{-\varsigma} \mathrm{d}\zeta \right\}, \tag{8}$$

where $0 \leqq \varsigma < 1$, the function $f(z)$ is analytic in a simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\varsigma}$ is removed as in Definition 1.

Now, by using Definition 2, the Srivastava-Owa fractional derivative of order $n+\varsigma$ can easily be defined as follows:

$$\mathfrak{D}_z^{n+\varsigma} f(z) := \frac{\mathrm{d}^n}{\mathrm{d}z^n} \{\mathfrak{D}_z^{\varsigma} f(z)\} \qquad (0 \leqq \varsigma < 1;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \cdots\}), \tag{9}$$

which readily yields

$$\mathfrak{D}_z^{0+\varsigma} f(z) = \mathfrak{D}_z^{\varsigma} f(z) \qquad \text{and} \qquad \mathfrak{D}_z^{1+\varsigma} f(z) = \frac{\mathrm{d}}{\mathrm{d}z} \{\mathfrak{D}_z^{\varsigma} f(z)\} \qquad (0 \leqq \varsigma < 1).$$

Recently, by applying the Srivastava-Owa definition (9), Tremblay [6] introduced and studied an interesting fractional derivative operator $\mathfrak{T}^{\tau,\mu}$, which was defined in the complex domain and whose properties in several spaces were discussed systematically (see, for details, [5] and [6]).

Definition 3. For $0 < \tau \leqq 1$, $0 < \mu \leqq 1$ and $0 \leqq \tau - \mu < 1$, the Tremblay operator $\mathfrak{T}^{\tau,\mu}$ of a function $f \in \mathcal{A}$ is defined for all $z \in \mathbb{U}$ by

$$\mathfrak{T}^{\tau,\mu} f(z) := \frac{\Gamma(\mu)}{\Gamma(\tau)} z^{1-\mu}\, \mathfrak{D}_z^{\tau-\mu}\, z^{\tau-1} f(z) \qquad (z \in \mathbb{U}). \tag{10}$$

In the special case when $\tau = \mu = 1$ in (10), we have

$$\mathfrak{T}^{1,1} f(z) = f(z). \tag{11}$$

We note also that $\mathfrak{D}_z^{\tau-\mu}$ represents a Srivastava-Owa operator of fractional derivative of order $\tau - \mu$ $(0 \leqq \tau - \mu < 1)$, which is given by Definition 2.

The main purpose of this paper is to present coefficient inequalities and coefficient estimates, distortion theorems, and the radii of starlikeness and convexity, for functions belonging to the class $\mathcal{P}_{\tau,\mu}(k, \delta, \gamma)$ which we introduce in Section 2 below. We also consider some other interesting results involving closure and convolution of functions in the class $\mathcal{P}_{\tau,\mu}(k, \delta, \gamma)$.

2. A Set of Main Results

In this section, we define a new analytic class $\mathcal{P}_{\tau,\mu}(k, \delta, \gamma)$ by considering the fractional derivative operator given by Definition 3 and establish a sufficient condition for a function $f(z) \in \mathcal{P}(k)$ to be in the function class $\mathcal{P}_{\tau,\mu}(k, \delta, \gamma)$. The following two lemmas will be needed in our investigation.

Lemma 1. *Let the function $f(z)$ defined by* (4) *belong to the class $\mathcal{P}(k)$ $(k \in \mathbb{N})$. Then*

$$\mathfrak{T}^{\tau,\mu} f(z) = \frac{\tau}{\mu} z - \sum_{\nu=m+1}^{\infty} \frac{\Gamma(\nu+\tau)\Gamma(\mu)}{\Gamma(\nu++\mu)\Gamma(\tau)} a_\nu z^\nu,$$

where $0 < \tau \leqq 1$, $0 < \mu \leqq 1$ and $0 \leqq \tau - \mu < 1$.

Proof. By using Definition 3 and Definition 2, we find for $z \in \mathbb{U}$ that

$$\begin{aligned}
\mathfrak{T}^{\tau,\mu} f(z) &= \frac{\Gamma(\mu)}{\Gamma(\tau)} z^{1-\mu} \, \mathfrak{D}_z^{\tau-\mu} z^{\tau-1} f(z) \\
&= \frac{\Gamma(\mu)}{\Gamma(\tau)} z^{1-\mu} \, \mathfrak{D}_z^{\tau-\mu} \, z^{\tau-1} \left(z - \sum_{\nu=m+1}^{\infty} a_\nu z^\nu \right) \\
&= \frac{\Gamma(\mu)}{\Gamma(\tau)} z^{1-\mu} \, \mathfrak{D}_z^{\tau-\mu} \left(z^\tau - \sum_{\nu=k+1}^{\infty} a_\nu z^{\nu+\tau-1} \right) \\
&= \frac{\Gamma(\mu)}{\Gamma(\tau)} z^{1-\mu} \left(\frac{\Gamma(\tau+1)}{\Gamma(\mu+1)} z^\mu - \sum_{\nu=k+1}^{\infty} \frac{\Gamma(\nu+\tau)}{\Gamma(\nu+\mu)} a_\nu z^{\nu+\mu-1} \right) \\
&= \frac{\tau}{\mu} z - \sum_{\nu=k+1}^{\infty} \frac{\Gamma(\nu+\tau)\Gamma(\mu)}{\Gamma(\nu+\mu)\Gamma(\tau)} a_\nu z^\nu,
\end{aligned}$$

which proves Lemma 1. □

Lemma 2. *Let the function* $f(z)$ *defined by* (4) *belong to the class* $\mathcal{P}(k)$ $(k \in \mathbb{N})$. *Then*

$$\left(\mathfrak{T}^{\tau,\mu} f(z)\right)' = \frac{\tau}{\mu} - \sum_{\nu=k+1}^{\infty} \frac{\nu\Gamma(\nu+\tau)\Gamma(\mu)}{\Gamma(\nu+\mu)\Gamma(\tau)} a_\nu z^{\nu-1} \qquad (z \in \mathbb{U}),$$

where $0 < \tau \leqq 1$, $0 < \mu \leqq 1$ *and* $0 \leqq \tau - \mu < 1$.

Proof. By using Lemma 1 and the definition 9, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left\{\mathfrak{T}^{\tau,\mu} f(z)\right\} &= \frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{\Gamma(\mu)}{\Gamma(\tau)} z^{1-\mu} \, \mathfrak{D}_z^{\tau-\mu} \, z^{\tau-1} f(z)\right\} \\
&= \frac{\tau}{\mu} - \sum_{\nu=k+1}^{\infty} \frac{\nu\Gamma(\nu+\tau)\Gamma(\mu)}{\Gamma(\nu+\mu)\Gamma(\tau\alpha)} a_\nu z^{\nu-1} \qquad (z \in \mathbb{U}),
\end{aligned}$$

which evidently completes the proof of Lemma 2. □

By employing Lemma 1 and Lemma 2, we now introduce a new class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$ of analytic functions in $\mathbb{U}$ as follows.

Definition 4. Let $0 < \tau \leqq 1$, $0 < \mu \leqq 1$, $0 \leqq \delta < 1$ and $0 \leqq \gamma < 1$. A function $f(z)$ belonging to the analytic function class $\mathcal{P}(k)$ is said to be in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$ if and only if

$$\Re\left(\frac{\Gamma(\mu+1)\Gamma(\tau)}{\Gamma(\tau+1)\Gamma(\mu)} z^{-1}\left[(1-\delta)\,\mathfrak{T}^{\tau,\mu} f(z) + z\delta\left(\mathfrak{T}^{\tau,\mu} f(z)\right)'\right]\right) > \gamma \qquad (z \in \mathbb{U};\ \tau - \mu + \gamma < 1), \tag{12}$$

where $\mathfrak{T}^{\tau,\mu}$ is the fractional derivative operator in the complex domain in Definition 3.

2.1. A Theorem on Coefficient Bounds

Theorem 1. *Let the function* $f(z)$ *be given by* (4). *Then* $f(z)$ *belongs to the class* $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$ *if and only if*

$$\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)} a_\nu \leqq 1-\gamma \qquad (a_\nu \geqq 0;\ 0 \leqq \gamma < 1). \tag{13}$$

The result (13) *is sharp and the extremal function* $f(z)$ *is given by by*

$$f(z) = z - \frac{(1-\gamma)(\mu+1)_k}{(1+\delta k)(\tau+1)_k}\, z^{k+1} \qquad (k \in \mathbb{N}), \tag{14}$$

where $(\lambda)_k$ $(\lambda \in \mathbb{C})$ *denotes the Pochhammer symbol defined by*

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}). \end{cases}$$

Proof. Supposing first that $f(z) \in \mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$, we find from Definition 3 in conjunction with with Lemmas 1 and 2 that

$$\Re\left(1 - \sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)}\, a_\nu\, z^{\nu-1}\right) > \gamma. \tag{15}$$

If we choose z to be real and let $z \to -1$, we have

$$1 - \sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)}\, a_\nu \geqq \gamma \qquad (0 < \tau \leqq 1;\ 0 < \mu \leqq 1),$$

which readily yields the inequality (13) of Theorem 1.

Conversely, by assuming that the inequality (13) is true, we let $|z| = 1$. We then obtain

$$\begin{aligned} &\left|\frac{\Gamma(\mu+1)\Gamma(\tau)}{\Gamma(\tau+1)\Gamma(\mu)}\, z^{-1}\left[(1-\delta)\mathfrak{T}^{\tau-\mu} f(z) + \delta z\left(\mathfrak{T}^{\tau-\mu} f(z)\right)'\right] - 1\right| \\ &\quad = \left| - \sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)}\, a_\nu\, z^{\nu-1}\right| \\ &\quad \leqq \sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)}\, a_\nu\, |z|^{\nu-1} \\ &\quad \leqq 1-\gamma, \end{aligned} \tag{16}$$

which shows that the function $f(z)$ is in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$.

Finally, it is easily verified that the result is sharp for the function $f(z)$ given by (14). □

Corollary 1. *Let the function* $f(z)$ *given by* (4) *be in the class* $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. *Then*

$$a_{k+1} \leqq \frac{(1-\gamma)(\mu+1)_k}{(1+\delta k)(\tau+1)_k} \tag{17}$$

$$(k = \mathbb{N} \setminus \{1\} = \{2, 3, 4, \cdots\};\ 0 < \mu \leqq 1;\ 0 < \tau \leqq 1;\ 0 \leqq \gamma < 1;\ 0 \leqq \delta < 1).$$

Corollary 2. *The function* $f(z) \in \mathcal{P}(k)$ *is in the class* $\mathcal{P}_{1,1}(k,\delta,\gamma)$ *if and only if*

$$\sum_{\nu=k+1}^{\infty} (1+\delta\nu-\delta)\, a_\nu \leqq 1-\gamma \qquad (0 \leqq \gamma < 1;\ 0 \leqq \delta \leqq 1). \tag{18}$$

Corollary 2 was given by Altintaş *et al.* [1]. In particular, it was given earlier for $k = 1$ by Bhoosnurmath and Swamy [2] for $k = 1$ and by Silverman [11] for $k = \delta = 1$.

2.2. Distortion Theorems

Theorem 2. *Let the function $f(z)$ belong to the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. Then*

$$\begin{aligned}\frac{\beta}{\alpha}\,|z|\left(1-|z|^k\frac{(1-\gamma)(\beta+1)_k(\mu+1)_k}{(1+\delta k)\,(\alpha+1)_k(\tau+1)_k}\right)\\ \leqq \left|\mathfrak{T}^{\beta,\alpha} f(z)\right|\\ \leqq \frac{\beta}{\alpha}\,|z|\left(1+|z|^k\frac{(1-\gamma)(\beta+1)_k(\mu+1)_k}{(1+\delta k)\,(\alpha+1)_k(\tau+1)_k}\right)\end{aligned} \tag{19}$$

$$(z\in\mathbb{U};\ 0<\mu\leqq 1;\ 0<\beta\leqq 1;\ k\in\mathbb{N}_0.$$

Proof. By hypothesis, the function $f(z)$ belongs to the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. Thus, clearly, we find from the inequality (13) in Theorem 1 that

$$\frac{(1+\delta k)\,(\tau+1)_k}{(\mu+1)_k}\sum_{\nu=k+1}^{\infty}a_\nu \leqq \sum_{\nu=k+1}^{\infty}\frac{(1+\delta\nu-\delta))\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)}\,a_\nu, \tag{20}$$

which leads us to

$$\sum_{\nu=k+1}^{\infty}a_\nu \leqq \frac{(1-\gamma)(\mu+1)_k}{(1+\delta k)\,(\tau+1)_k} \tag{21}$$

$$(0<\mu\leqq 1;\ 0<\tau\leqq 1;\ 0\leqq\delta<1;\ 0\leqq\gamma<1;\ k\in\mathbb{N}).$$

Next, by the definition (10) and from (21), we have

$$\begin{aligned}\mathfrak{T}^{\beta,\alpha} f(z) &= \frac{\Gamma(\alpha)}{\Gamma(\beta)}\,z^{1-\alpha}D_z^{\beta-\alpha}\,z^{\beta-1}\,f(z)\\ &= \frac{\beta}{\alpha}\left(z-\sum_{\nu=k+1}^{\infty}\frac{\Gamma(\nu+\beta)\Gamma(\alpha+1)}{\Gamma(\nu+\alpha)\Gamma(\beta+1)}\,a_\nu\,z^\nu\right)\\ &= \frac{\beta}{\alpha}\left(z-\sum_{\nu=k+1}^{\infty}\omega(\nu)a_\nu\,z^\nu\right),\end{aligned}$$

where

$$\omega(\nu)=\frac{(\beta+1)_{\nu-1}}{(\alpha+1)_{\nu-1}}\qquad(\nu=k+1,k+2,k+3,\cdots).$$

Since the function $\omega(\nu)$ can be seen to be non-increasing, we get

$$0<\omega(\nu)\leqq\omega(k+1)=\frac{(\beta+1)_k}{(\alpha+1)_k}. \tag{22}$$

Thus, from the inequalities (22) and (21), we find that

$$\begin{aligned}\left|\mathfrak{T}^{\beta,\alpha} f(z)\right| &\geqq \frac{\beta}{\alpha}\left(|z|-\left|\omega(k+1)\sum_{\nu=k+1}^{\infty}a_\nu z^\nu\right|\right)\\ &\geqq \frac{\beta}{\alpha}\left(|z|-|z|^\nu\omega(k+1)\sum_{\nu=k+1}^{\infty}a_\nu\right)\\ &\geqq \frac{\beta}{\alpha}|z|\left(1-|z|^k\frac{(\beta+1)_k\,(1-\gamma)(\mu+1)_k}{(\alpha+1)_k(1+\delta k)\,(\tau+1)_k}\right),\end{aligned}$$

which proves the first part of the inequality (19). In a similar manner, we can prove the second part of the inequality (19). □

Theorem 3. *Let the function* $f(z)$ *be in the class* $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. *Then*

$$|z| - |z|^{k+1}\ \frac{(1-\gamma)(\mu+1)_k}{(1+\delta k)(\tau+1)_k} \leqq\ |f(z)| \leqq |z| + |z|^{k+1}\ \frac{(1-\gamma)(\mu+1)_k}{(1+\delta k)(\tau+1)_k}. \tag{23}$$

Proof. By using the same method as in Theorem 2, we have

$$\begin{aligned} |f(z)| &\leqq |z| + \sum_{\nu=k+1} a_\nu |z|^\nu \\ &\leqq |z| + |z|^{k+1} \sum_{\nu=k+1} a_\nu \\ &\leqq |z| + |z|^{k+1}\ \frac{(1-\gamma)\Gamma(k+1+\mu)\Gamma(\tau+1)}{(1+\delta k)\Gamma(k+1+\tau)\Gamma(\mu+1)} \\ &= |z| + |z|^{k+1}\ \frac{(1-\gamma)(\mu+1)_k}{(1+\delta k)\,(\tau+1)_k} \end{aligned} \tag{24}$$

and

$$\begin{aligned} |f(z)| &\geqq |z| - |z|^{k+1} \sum_{\nu=k=1} a_\nu \\ &\geqq |z| - |z|^{k+1}\ \frac{(1-\gamma)(\mu+1)_k}{(1+\delta k)\,(\tau+1)_k}. \end{aligned} \tag{25}$$

Consequently, from (24) and (25), we immediately get the inequality (23) of Theorem 3. □

Upon setting $\tau = \mu = 1$ in Theorem 3, we obtain the following corollary.

Corollary 3. *If* $f(z) \in \mathcal{P}_0(k,\delta,\gamma) =: \mathcal{P}(k,\delta,\gamma)$, *then*

$$|z| - |z|^{k+1}\ \left(\frac{1-\gamma}{1+\delta k}\right) \leqq \left|f(z)\right| \leqq |z| + |z|^{k+1}\ \left(\frac{1-\gamma}{1+\delta k}\right)$$

$$(z \in \mathbb{U};\ 0 \leqq \gamma < 1;\ 0 \leqq \delta < 1;\ k \in \mathbb{N}).$$

Moreover, if $\tau = \mu = 1$ and $k = 1$ in Theorem 3, then we have the following known result (see [2]).

Corollary 4. *If* $f(z) \in \mathcal{P}_0(1,\delta,\gamma) =: \mathcal{P}(\delta,\gamma)$, *then*

$$|z| - |z|^2\ \left(\frac{1-\gamma}{1+\delta}\right) \leqq \left|f(z)\right| \leqq |z| + |z|^2\ \left(\frac{1-\gamma}{1+\delta}\right) \qquad (z \in \mathbb{U}).$$

3. Radii of Starlikeness and Convexity

Theorem 4. *If the function* $f(z) \in \mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$, *then* $f(z) \in \mathcal{P}^*_{\tau,\mu}(k,\delta,\gamma)$ *in the disk* $|z| < r_1$, *where*

$$r_1 := \inf_{\nu \geqq k+1} \left\{ \frac{(1-\alpha)(1+\delta\nu-\delta)\Gamma(\nu+\tau)\Gamma(\mu+1)}{(\nu-\alpha)(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)} \right\}^{1/(\nu-1)}.$$

Proof. We must show that the condition in (3) holds true. Indeed, since

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqq \frac{z-\sum\limits_{\nu=k+1}^{\infty} \nu a_\nu |z|^\nu}{z-\sum\limits_{\nu=2}^{\infty} a_\nu |z|^\nu} \leqq 1-\alpha \qquad (z \in \mathbb{U}) \tag{26}$$

and

$$\sum_{\nu=k+1}^{\infty} (\nu-\alpha) a_\nu |z|^{\nu-1} \leqq 1-\alpha \qquad (z \in \mathbb{U}), \tag{27}$$

we find that

$$\frac{(\nu-\alpha)|z|^{\nu-1}}{1-\alpha} \leqq \frac{(1+\delta\nu-\delta)\Gamma(\nu+\tau)\Gamma(\mu+1)}{(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)} \qquad (\nu \geqq k+1),$$

that is, that

$$|z| \leqq \left\{\frac{(1-\alpha)(1+\delta\nu-\delta)\Gamma(\nu+\tau)\Gamma(\mu+1)}{(\nu-\alpha)(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)}\right\}^{1/(\nu-1)},$$

which proves Theorem 4. □

Corollary 5. *If the function* $f(z) \in \mathcal{P}_{1,1}(k,\delta,\gamma)$, *then* $f(z)$ *is starlike of order* α *in the disk* $|z| < r_2$, *where*

$$r_2 := \inf_{\nu \geqq k}\left\{\frac{(1-\alpha)(1+\delta\nu-\delta)}{(\nu-\alpha)(1-\gamma)}\right\}^{1/(\nu-1)}.$$

In its special case when $k = 1$, Corollary 5 was proven by Altintaş *et al.* [1]. Moreover, Corollary 5 was given earlier by Bhoosnurmath and Swamy [2] for $k = 1$ and $\alpha = 0$, and by Silverman [11] when $k = 1$ and $\delta = \gamma = 0$.

Theorem 5. *If the function* $f(z) \in \mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$, *then* $f(z) \in \mathcal{K}_{\tau,\mu}(k,\delta,\gamma)$ *in the disk* $|z| < r_3$, *where*

$$r_3 := \inf_{\nu \geqq k+1}\left\{\frac{(1-\alpha)(1+\delta\nu-\delta)\Gamma(\nu+\tau)\Gamma(\mu+1)}{\nu(\nu-\alpha)(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)}\right\}^{1/(\nu-1)}.$$

Proof. For the function $f(z)$ given by (4), we must show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1-\alpha \qquad (z \in \mathbb{U}).$$

First of all, we find from (4) that

$$\begin{aligned}\left|\frac{zf''(z)}{f'(z)}\right| &= \left|\frac{-\sum\limits_{\nu=k+1}^{\infty} \nu(\nu-1) a_\nu z^{\nu-1}}{1-\sum\limits_{\nu=k+1}^{\infty} \nu a_\nu z^{\nu-1}}\right| \\ &\leqq \frac{\sum\limits_{\nu=k+1}^{\infty} \nu(\nu-1) a_\nu |z|^{\nu-1}}{1-\sum\limits_{\nu=k+1}^{\infty} \nu a_\nu |z|^{\nu-1}} \\ &\leqq 1-\alpha \qquad (z \in \mathbb{U}),\end{aligned}$$

if

$$\sum_{\nu=k+1}^{\infty} \nu(\nu-1)a_\nu|z|^{\nu-1} \leqq (1-\alpha)\left(1-\sum_{\nu=k+1}^{\infty} \nu a_\nu|z|^{\nu-1}\right) \qquad (z \in \mathbb{U}), \tag{28}$$

that is, if

$$\sum_{\nu=k+1}^{\infty} \nu(\nu-\alpha)a_\nu|z|^{\nu-1} \leqq 1-\alpha \qquad (z \in \mathbb{U}). \tag{29}$$

From the last inequality (29), together with Theorem 1, we thus find that

$$\frac{\nu(\nu-\alpha)|z|^{\nu-1}}{(1-\alpha)} \leqq \frac{(1+\delta\nu-\delta)\Gamma(\nu+\tau)\Gamma(\mu+1)}{(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)} \qquad (\nu \geqq k+1),$$

that is, that

$$|z| \leqq \left\{\frac{(1-\alpha)(1+\delta\nu-\delta)\Gamma(\nu+\tau)\Gamma(\mu+1)}{\nu(\nu-\alpha)(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)}\right\}^{1/(\nu-1)},$$

which evidently proves Theorem 5. □

Corollary 6. *If the function $f(z) \in \mathcal{P}_0(k,\delta,\gamma)$, then $f(z)$ is convex of order α in the disk $|z| < r_4$, where*

$$r_4 := \inf_{\nu \geqq k+1} \left\{\frac{(1-\alpha)(1+\delta\nu-\delta)}{\nu(\nu-\alpha)(1-\gamma)}\right\}^{1/(\nu-1)}.$$

For $k = 1$, Corollary 6 was proved by Altintaş *et al.* [1]. Further special cases of Corollary 6 were given earlier by Bhoosnurmath and Swamy [2] when $k = 1$ and $\alpha = 0$, and by Silverman [11] for $k = 1$ and $\delta = \gamma = 0$.

4. Further Results for the Function Class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$

In this section, we prove some results for the closure of functions and the convolution of functions in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$.

Theorem 6. *Let each of the functions $f_1(z)$ and $f_2(z)$ given by*

$$f_1(z) = z - \sum_{\nu=k+1}^{\infty} a_{\nu,1}z^\nu \qquad (a_{\nu,1} \geqq 0;\ k \in \mathbb{N})$$

and

$$f_2(z) = z - \sum_{\nu=k+1}^{\infty} a_{\nu,2}z^\nu \qquad (a_{\nu,2} \geqq 0;\ k \in \mathbb{N})$$

be in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. Then the function $\Phi(z)$ given by

$$\Phi(z) = z - \frac{1}{2}\sum_{\nu=k+1}^{\infty} (a_{\nu,1}+a_{\nu,2})z^\nu$$

is also in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$.

Proof. By the hypothesis that each of the functions $f_1(z)$ and $f_2(z)$ is in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$, we get

$$\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)}\, a_{\nu,1} \leqq 1-\gamma \tag{30}$$

and

$$\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)}\, a_{\nu,2} \leqq 1-\gamma, \tag{31}$$

so that, obviously,

$$\frac{1}{2}\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)}\, (a_{\nu,1}+a_{\nu,2}) \leqq 1-\gamma, \tag{32}$$

which proves the assertion that $\Phi(z) \in \mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. □

Theorem 7. *Let the functions* $f_j(z)$ $(j = 1,\cdots,p)$ *defined by*

$$f_j(z) = z - \sum_{\nu=k+1}^{\infty} a_{\nu,j} z^{\nu} \qquad (a_{\nu,j} \geqq 0;\ k \in \mathbb{N})$$

be in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$*. Then the function* $\Theta(z)$ *defined by*

$$\Theta(z) := \sum_{j=1}^{p} q_j f_j(z) \qquad (q_j \geqq 0) \tag{33}$$

is also in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$*, where*

$$\sum_{j=1}^{\infty} q_j = 1 \qquad (q_j \geqq 0).$$

Proof. By the definition (33) of the function $\Theta(z)$, we have

$$\begin{aligned}\Theta(z) &= \sum_{j=1}^{p} q_j \left(z - \sum_{\nu=k+1}^{\infty} a_{\nu,j} z^{\nu} \right)\\ &= \sum_{j=1}^{p} q_j z - \sum_{\nu=k+1}^{\infty} \left(\sum_{j=1}^{p} q_j a_{\nu,j} z^{\nu} \right)\\ &= z - \sum_{\nu=k+1}^{\infty} \left(\sum_{j=1}^{p} q_j a_{\nu,j} \right) z^{\nu}.\end{aligned}$$

Since $f_j(z) \in \mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$ $(j = 1,\cdots,p)$, we also have

$$\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta))\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{\Gamma(\nu+\mu)\Gamma(\tau+1)} a_{\nu,j} \leqq 1-\gamma \qquad (j = 1,\cdots,p).$$

The remainder of the proof of Theorem 7 (which is based essentially upon Theorem 1) is fairly straightforward and is, therefore, omitted here. □

Theorem 8. *Let the function $f(z)$ given by* (4) *and the function $\hbar(z)$ defined by*

$$\hbar(z) = z - \sum_{\nu=k+1}^{\infty} \lambda_\nu z^\nu \qquad (\lambda_\nu \geqq 0;\ k \in \mathbb{N})$$

be in the same class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. Then the function $\Delta(z)$ defined by

$$\begin{aligned}\Delta(z) &= (1-\eta)f(z) + \eta\hbar(z)\\ &= z - \sum_{\nu=k+1}^{\infty} \rho_\nu z^\nu.\end{aligned}$$

is also in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$.

Proof. In view of the hypotheses of Theorem Theorem8, we find by using Theorem 1 that

$$\begin{aligned}\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu)}{\Gamma(\nu+\mu)\Gamma(\tau)}\,\rho_\nu &= (1-\eta)\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu)}{\Gamma(\nu+\mu)\Gamma(\tau)}\,a_\nu z^{\nu-1}\\ &\quad + \eta\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu)}{\Gamma(\nu+\mu)\Gamma(\tau)}\,\lambda_\nu z^{\nu-1}\\ &\leqq (1-\eta)(1-\gamma) + \eta(1-\gamma)\\ &\leqq 1-\gamma.\end{aligned}$$

Hence $\Delta(z) \in \mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. □

Theorem 9. *Let the function $f(z)$ given by* (4) *and the function $\psi(z)$ defined by* (6) *be in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$. Then the function $\Omega(z)$ given by the following modified Hadamard product*:

$$\Omega(z) := f * \psi(z) = z - \sum_{\nu=k+1}^{\infty} a_\nu \lambda_\nu z^\nu$$

is in the class $\mathcal{P}_{\tau,\mu}(k,\delta,\xi)$, where

$$\xi \leqq 1 - \frac{(1-\gamma)^2\,(\mu+1)_k}{(1+\delta k)\,(\tau+1)_k}.$$

Proof. With a view to finding the *largest* ξ, by supposing that $\Omega(z) \in \mathcal{P}_{\tau,\mu}(k,\delta,\xi)$, we have

$$\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{(1-\xi)\Gamma(\nu+\mu)\Gamma(\tau+1)} a_\nu\lambda_\nu \leqq 1. \tag{34}$$

Since $f(z), \psi(z) \in \mathcal{P}_{\tau,\mu}(k,\delta,\gamma)$, we know that

$$\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)}\,a_\nu \leqq 1$$

and

$$\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)}\,\lambda_\nu \leqq 1.$$

Thus, by using the Cauchy-Schwarz inequality, we obtain

$$\sum_{\nu=k+1}^{\infty} \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)} \sqrt{\lambda_\nu a_\nu} \leqq 1,$$

which implies that

$$\begin{aligned}\frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)} \sqrt{\lambda_\nu a_\nu}& \\ \leqq \frac{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}{(1-\xi)\Gamma(\nu+\mu)\Gamma(\tau+1)} a_\nu\lambda_\nu \leqq 1 &\qquad (\nu \geqq k+1).\end{aligned}$$

that is, that

$$\sqrt{\lambda_\nu a_\nu} \leqq \frac{1-\xi}{1-\gamma} \qquad (\nu \geqq k+1).$$

We note also that

$$\sqrt{\lambda_\nu a_\nu} \leqq \frac{(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)}{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}.$$

We now need to show that

$$\frac{(1-\gamma)\Gamma(\nu+\mu)\Gamma(\tau+1)}{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)} \leqq \frac{1-\xi}{1-\gamma} \tag{35}$$

or, equivalently, that

$$\xi \leqq 1 - \frac{(1-\gamma)^2\,\Gamma(\nu+\mu)\Gamma(\tau+1)}{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)}.$$

Upon letting

$$\Xi(\nu) := 1 - \frac{(1-\gamma)^2\,\Gamma(\nu+\mu)\Gamma(\tau+1)}{(1+\delta\nu-\delta)\,\Gamma(\nu+\tau)\Gamma(\mu+1)},$$

we can easily see that the function $\Xi(\nu)$ is non-decreasing in ν. We thus obtain

$$\xi \leqq \Xi(k+1) \leqq 1 - \frac{(1-\gamma)^2\,(\mu+1)_k}{(1+\delta k)\,(\tau+1)_k} \qquad (k \in \mathbb{N}).$$

Finally, the result asserted by Theorem 9 is sharp with the extremal function given by

$$f(z) = \psi(z) = z - \frac{(1-\gamma)(\mu+1)_k}{(1+\delta k)\,(\tau+1)_k}\, z^{k+1} \qquad (k \in \mathbb{N}), \tag{36}$$

which evidently completes the proof of Theorem 9. □

References

[1] O. Altintaş, H. Irmak and H. M. Srivastava, A subclass of analytic functions defined by using certain operators of fractional calculus, *Comput. Math. Appl.* **30** (1) (1995), 1–9.
[2] S. S. Bhoosnurmath and S. R. Swamy, Certain classes of analytic functions with negative coefficients, *Indian J. Math.* **27** (1985), 89–98.
[3] S. K. Chatterjea, On starlike functions, *J. Pure Math.* **1** (1) (1981), 23–26.

[4] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band **259**, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
[5] R. W. Ibrahim and J. M. Jahangiri, Boundary fractional differential equation in a complex domain, *Boundary Value Prob.* **2014** (2014), Article ID 66, 1–11.
[6] R. W. Ibrahim, Z. Esa and A. Kılıçman, On geometrical and topological properties of Tremblay operator among various spaces, Preprint 2015.
[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.
[8] S. Owa, On the distortion theorems. I, *Kyungpook Math. J.* **18** (1978), 53–59.
[9] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39** (1987), 1057–1077.
[10] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* **49** (1975), 109–115.
[11] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109–116.
[12] H. M. Srivastava, S. Owa and S. K. Chatterjea, A note on certain classes of starlike functions, *Rend. Sem. Mat. Univ. Padova* **77** (1987), 115–124.