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# Some Differential Inequalities in the Complex Plane

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**Abstract.** In the present paper, we obtain some new results by applying well-known Jack's lemma. Moreover, the second-order differential subordinations associated with convex functions are also considered.

## 1. Introduction

Let $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ be the class of analytic functions in $\mathbb{E}$.

For analytic functions $f, g \in \mathcal{H}$, we say that $f$ is subordinate to $g$ in $\mathbb{E}$, written $f \prec g$ or $f(z) \prec g(z)$ if and only if there exists an analytic functions $w \in \mathcal{H}$ such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$ for $z \in \mathbb{E}$. Therefore we note that $f \prec g$ in $\mathbb{E}$ implies that $f(\mathbb{E}) \subset g(\mathbb{E})$. Furthermore, if $g$ is univalent in $\mathbb{E}$, then the subordination principle [1] says that

$$f \prec g \quad \text{if and only if} \quad f(0) = g(0) \quad \text{and} \quad f(|z| < r) \subset g(|z| < r) \quad \text{for all } r \in (0, 1].$$

For a positive integer $p$, we denote by $\mathcal{A}(p)$ the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

which are analytic in $\mathbb{E}$ and $\mathcal{A}(1) \equiv \mathcal{A}$. The subclass of $\mathcal{A}$ consisting of convex functions of order $\alpha$ $(0 \leq \alpha < 1)$ is denoted by $\mathcal{K}(\alpha)$. An analytic characterization of $\mathcal{K}(\alpha)$ is given by

$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{A} : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leq \alpha < 1; z \in \mathbb{E}) \right\}.$$

In this paper, we obtain some interesting properties of certain analytic functions by using Fukui and Sakaguchi's [2] results, which is a generalization of well-known Jack's lemma [3]. Furthermore, we improve a result obtained by Miller and Mocanu [5] in connection with the second-order differential subordination.

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## 2. Main Results

To prove the main results, we need the following lemma due to Fukui and Sakaguchi [2].

**Lemma 2.1.** (Fukui and Sakaguchi [2]) Let $w \in \mathcal{A}(p)$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z = z_0$, then we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geq p,$$

where $k$ is a real number.

Applying Lemma 2.1 (see, also [3] and [2]), we will obtain some results.

**Theorem 2.2.** *Let $\alpha \in \mathbb{C}$ and $p \in \mathbb{N}$ with $\Re\{\alpha\} \geq -p$. And let $w \in \mathcal{A}(p)$. Suppose that*

$$|\alpha w(z) + zw'(z)| < \Re\{\alpha\} + p, \quad z \in \mathbb{E}.$$

*Then we have*

$$|w(z)| < 1, \quad z \in \mathbb{E}.$$

*Proof.* If there exists a point $z_0$ in $\mathbb{E}$ such that

$$|w(z)| < 1 \quad \text{for} \quad |z| < |z_0|$$

and

$$|w(z_0)| = 1,$$

then from Lemma 2.1, we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geq p.$$

Then it follows that

$$\begin{aligned} |\alpha w(z_0) + z_0 w'(z_0)| &= |w(z_0)| \left| \alpha + \frac{z_0 w'(z_0)}{w(z_0)} \right| \\ &= |\alpha + k| \geq \Re\{\alpha\} + p. \end{aligned}$$

It contradicts hypothesis and therefore it completes the proof. □

**Corollary 2.3.** (Miller [4]) Let $w(z)$ be analytic in $\mathbb{E}$ with $w(0) = 0$ and suppose that

$$\left| \frac{1}{2} w(z) + zw'(z) \right| < 1, \quad z \in \mathbb{E}.$$

Then we have

$$|w(z)| < 1, \quad z \in \mathbb{E}.$$

**Theorem 2.4.** *Let $w \in \mathcal{A}(p)$ and suppose that*

$$\left| w(z)^2 + w(z) + zw'(z) \right| < p, \quad z \in \mathbb{E}.$$

*Then we have*

$$|w(z)| < 1, \quad z \in \mathbb{E}.$$

*Proof.* If there exists a point $z_0 \in \mathbb{E}$ such that

$$|w(z)| < 1 \quad \text{for} \quad |z| < |z_0|$$

and

$$|w(z_0)| = 1.$$

If we take $w(z_0) = e^{i\theta}$, where $\theta$ is a real number, then from Lemma 1, we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geq p.$$

Let us put

$$u(z) = w(z)e^{-i\theta}.$$

Then it follows that $|u(z_0)| = |w(z_0)|$ and $|u(z)| = |w(z)|$. Therefore, we have

$$\frac{z_0 u'(z_0)}{u(z_0)} = k \geq p.$$

Then it follows that

$$\begin{aligned} & w(z)^2 + w(z) + zw'(z) = u(z)^2 e^{i2\theta} + u(z)e^{i\theta} + zu'(z)e^{i\theta} \\ =& u(z)e^{i\theta}\left(u(z)e^{i\theta} + 1 + \frac{zu'(z)}{u(z)}\right) \end{aligned}$$

and so, we have

$$\begin{aligned} & \left|w(z_0)^2 + w(z_0) + z_0 w'(z_0)\right| \\ =& \left|u(z_0)e^{i\theta}\right| \left|u(z_0)e^{i\theta} + 1 + \frac{z_0 u'(z_0)}{u(z_0)}\right| \\ =& \left|e^{i\theta} + 1 + k\right| \geq |1 + k| - \left|e^{i\theta}\right| \\ \geq& 1 + p - |w(z_0)| = p. \end{aligned}$$

This contradicts hypothesis of Theorem 2.4 and therefore, it completes the proof. □

**Corollary 2.5.** (Miller [4]) Let $w(z)$ be analytic in $\mathbb{E}$ with $w(0) = 0$ and suppose that

$$\left|w(z)^2 + w(z) + zw'(z)\right| < 1, \quad z \in \mathbb{E}.$$

Then we have

$$|w(z)| < 1, \quad z \in \mathbb{E}.$$

Applying the same method as in the proof of Theorem 2.4, we have the following theorems.

**Theorem 2.6.** *Let $w \in \mathcal{A}(p)$ and suppose that*

$$\left|w(z)^2 + w(z) + zw'(z)\right| < R\left|R - 1 - p\right|, \quad z \in \mathbb{E},$$

*where $0 < R$ and $R \neq 1 + p$. Then we have*

$$|w(z)| < R, \quad z \in \mathbb{E}.$$

**Theorem 2.7.** *Let $w \in \mathcal{A}(p)$ and suppose that*

$$|w(z)|\, e^{|zw'(z)|} < e^{p}, \quad z \in \mathbb{E}.$$

*Then we have*

$$|w(z)| < 1, \quad z \in \mathbb{E}.$$

**Corollary 2.8.** (Miller [4]) Let $w(z)$ be analytic in $\mathbb{E}$ with $w(0) = 0$. Then

$$|w(z)|\, e^{|zw'(z)|} < 1$$

implies that

$$|w(z)| < 1, \quad z \in \mathbb{E}.$$

**Theorem 2.9.** *Let $h \in \mathcal{K}(\alpha)$. Suppose that $B(z)$ is analytic in $\mathbb{E}$ with $\Re\{B(z)\} \geq A(1-\alpha)$, where $A \geq 0$. If $q \in \mathcal{A}$, then*

$$Az^2q''(z) + B(z)zq'(z) + q(z) \prec h(z), \quad z \in \mathbb{E}$$

*implies that*

$$q(z) \prec h(z), \quad z \in \mathbb{E}.$$

*Proof.* Assume that $q \not\prec h$. Then there exist points $z_0 \in \mathbb{E}$ and $\zeta_0 \in \partial\mathbb{E}$, and $m \geq 1$ such that

$$q(z_0) = h(\zeta_0),$$
$$z_0q'(z_0) = m\zeta_0h'(\zeta_0) \tag{1}$$

and

$$\Re\left\{1 + \frac{z_0q''(z_0)}{q'(z_0)}\right\} \geq m\Re\left\{1 + \frac{\zeta_0h''(\zeta_0)}{h'(\zeta_0)}\right\}. \tag{2}$$

Since $h \in \mathcal{K}(\alpha)$, we have

$$\Re\left\{1 + \frac{\zeta_0h''(\zeta_0)}{h'(\zeta_0)}\right\} \geq \alpha, \quad \text{for} \quad |\zeta_0| = 1, \tag{3}$$

and therefore, from (1), (2) and (3), we obtain

$$\Re\left\{\frac{z_0^2q''(z_0)}{\zeta_0h'(\zeta_0)}\right\} \geq m(m\alpha - 1). \tag{4}$$

From (4), we have

$$\begin{aligned}
&\Re\left\{\frac{Az_0^2q''(z_0) + B(z_0)z_0q'(z_0) + q(z_0) - h(\zeta_0)}{\zeta_0h'(\zeta_0)}\right\} \\
&\geq m(A(m\alpha - 1) + \Re\{B(z_0)\}) \\
&\geq m(m-1)A\alpha \geq 0.
\end{aligned} \tag{5}$$

Using (5), we have

$$Az_0^2q''(z_0) + B(z_0)z_0q'(z_0) + q(z_0) = h(\zeta_0) + \lambda\zeta_0h'(\zeta_0),$$

where $\Re(\lambda) \geq 0$. Since $\zeta_0h'(\zeta_0)$ is the outward normal to the boundary of the convex domain $h(\overline{\mathbb{E}})$ at $h(\zeta_0)$, we obtain

$$Az^2q''(z) + B(z)zq'(z) + q(z) \not\prec h(z) \quad (z \in \mathbb{E}),$$

which contradicts to our hypothesis. This completes the proof of theorem. □

**Corollary 2.10.** (Miller and Mocanu [7]) Let $h(z)$ be convex in $\mathbb{E}$ and let $A \geq 0$. Suppose that $B(z)$ is analytic in $\mathbb{E}$ with $\Re\{B(z)\} \geq A$. If $q(z)$ is analytic in $\mathbb{E}$ and $q(0) = h(0) = 0$, then the condition

$$Az^2q''(z) + B(z)zq'(z) + q(z) \prec h(z), \quad z \in \mathbb{E}$$

implies that

$$q(z) \prec h(z), \quad z \in \mathbb{E}.$$

**Theorem 2.11.** *Let $A \geq 0$ and let $B(z)$ be analytic in $\mathbb{E}$ with $\Re\{B(z)\} \geq -A$ in $\mathbb{E}$. If $q \in \mathcal{A}(p)$ and*

$$\left|Az^2q''(z) + B(z)zq'(z) + (1 - B(z))q(z)\right| < 1 + A(p-1)^2, \quad z \in \mathbb{E}, \tag{6}$$

*then we have*

$$q(z) \prec z^p, \quad z \in \mathbb{E}.$$

*Proof.* The left hand side of (6) has a zero of order $p$ at $z = 0$ and so, applying the Schwarz's lemma, we have

$$\left|Az^2q''(z) + B(z)zq'(z) + (1 - B(z))q(z)\right| \leq (1 + A(p-1)^2)|z|^p.$$

If there exists a point $z_0 \in \mathbb{E}$ such that

$$\left|q(z)\right| < |z_0|^p \quad \text{for} \quad |z| < |z_0|$$

and

$$|q(z_0)| = |z_0|^p$$

as the Fig. 1, then from Lemma 2.1, we have

$$\frac{z_0q'(z_0)}{q(z_0)} = k \geq p.$$

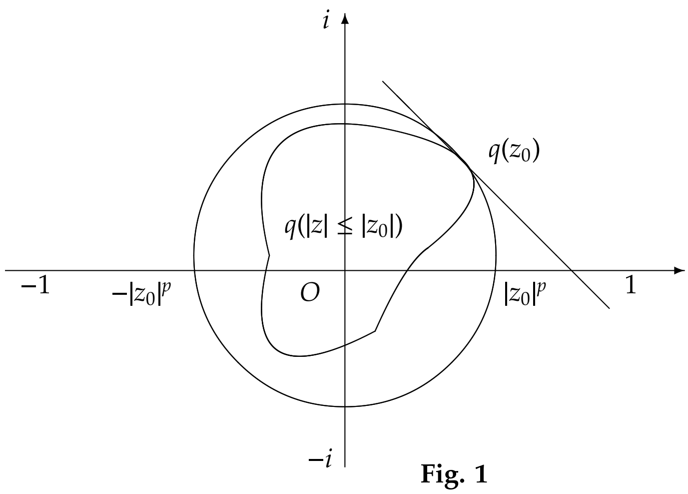


**Fig. 1**

Then the function $w(z) = z^p$ takes the following equalities

$$\frac{z_0w'(z_0)}{w(z_0)} = p$$

and

$$\Re\left\{1+\frac{z_0w''(z_0)}{w'(z_0)}\right\}=p.$$

On the other hand, we suppose that the image curve of the circle $|z|=|z_0|^p$ under the mapping $W(z)=q(z)$ comes in touch at the point $W=q(z_0)$ on the circle $|z|=|z_0|^p$. Therefore, from Lemma 2.1, we have

$$\frac{z_0q'(z_0)}{q(z_0)}=k\geq p$$

and from the geometric property of analytic function and Miller and Mocanu [5, p. 158] (see, also [6, p. 201]),

$$\Re\left\{1+\frac{z_0p''(z_0)}{p'(z_0)}\right\}\geq k\geq p.$$

Then it follows that

$$\begin{aligned}
&\left|Az_0^2q''(z_0)+B(z_0)z_0q'(z_0)+(1-B(z_0))q(z_0)\right|\\
&=\left|q(z_0)\right|\left|A\frac{z_0q''(z_0)}{q'(z_0)}\frac{z_0q'(z_0)}{q(z_0)}+B(z_0)\frac{z_0q'(z_0)}{q(z_0)}+(1-B(z_0))\right|\\
&=\left|q(z_0)\right|\left|A\frac{z_0q''(z_0)}{q'(z_0)}k+B(z_0)k+(1-B(z_0))\right|\\
&\geq\left|q(z_0)\right|\left(\Re\left\{A\left(1+\frac{z_0q''(z_0)}{q'(z_0)}\right)k-Ak+B(z_0)k+1-B(z_0)\right\}\right)\\
&\geq\left|q(z_0)\right|\left(Ak^2-Ak+1+(k-1)\Re B(z_0)\right)\\
&\geq\left|q(z_0)\right|\left|1+Ak^2-Ak-(k-1)A\right|\\
&=|z_0|^p\left\{1+A(k-1)^2\right\}\\
&\geq|z_0|^p\left\{1+A(p-1)^2\right\}.
\end{aligned}$$

This contradicts hypothesis and therefore, it completes the proof. □

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