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Research on Some New Results Arising from Multiple q-Calculus

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Abstract. In this paper, we develop the theory of the multiple q-analogue of the Heine's binomial formula, chain rule and Leibniz's rule. We also derive many useful definitions and results involving multiple q-antiderivative and multiple q-Jackson's integral. Finally, we list here multiple q-analogue of some elementary functions including trigonometric functions and hyperbolic functions. This may be a good consideration in developing the multiple q-calculus in combinatorics, number theory and other fields of mathematics.

1. Introduction

In the year 1910, Jackson [6] first considered the q-difference calculus (or the so-called quantum calculus), which is an old subject. From Jackson's time to the present, this theory was widely-investigated in the theory of special functions, differential equations (also fractional differential equations), and other related theories: that is, quantum calculus (also known as q-calculus) was one of the most active area of research in the physics and mathematics. While one takes care of q-calculus with one base q, Nalci and Pashaev [10] concerned with multiple q-calculus for the functions including independent several variables. Thereby, the necessity of multiple q-calculus has been emerged in several physical and mathematical problems.

We now review briefly some concepts of the multiple q-calculus taken in [10].

Throughout the paper, the indexes i and j will be considered as

$$i = 1, 2, \cdots, N \text{ and } j = 1, 2, \cdots, N.$$

Let $\overrightarrow{q} := (q_1, q_2, \cdots, q_N)$. Then the multiple q-number (a generalization of q-number) is defined by

$$[n]_{q_i,q_j} := \frac{q_i^n - q_j^n}{q_i - q_j}.$$

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It is clear that $[n]_{q_i,q_j} = [n]_{q_j,q_i}$. These numbers are represented as

$$\left([n]_{q_i,q_j}\right) = \begin{pmatrix} [n]_{q_1,q_1} & [n]_{q_1,q_2} & \cdots & [n]_{q_1,q_N} \\ [n]_{q_2,q_1} & [n]_{q_2,q_2} & \cdots & [n]_{q_2,q_n} \\ \cdots & \cdots & \cdots & \cdots \\ [n]_{q_N,q_1} & [n]_{q_N,q_2} & \cdots & [n]_{q_N,q_N} \end{pmatrix} \tag{1}$$

where i denotes the number of rows and j denotes the number of columns. One can see that the diagonal terms of the matrix can be considered as the limit $q_i \to q_j$: that is,

$$\lim_{q_i \to q_j} [n]_{q_i,q_j} = nq_j^{n-1}. \tag{2}$$

In view of multiple q-calculus, multiple q-derivative is defined by the following linear operator:

$$D_{q_i,q_j} f(x) = \frac{f(q_i x) - f\left(q_j x\right)}{\left(q_i - q_j\right) x}, \tag{3}$$

representing $N \times N$ matrix of multiple q-derivative operators $D := \left(D_{q_i,q_j}\right)$ which is symmetric, $D_{q_i,q_j} = D_{q_j,q_i}$. The multiple q-analogue of $(x-a)^n$ is given by

$$\begin{aligned} (x-a)^n_{q_i,q_j} &= \begin{cases} (x - q_i^{n-1}a)(x - q_i^{n-2}q_j a)\cdots(x - q_i q_j^{n-2}a)(x - q_j^{n-1}a), & \text{if } n \geq 1 \\ 1, & \text{if } n = 0 \end{cases} \\ &= \sum_{k=0}^{n} \binom{n}{k}_{q_i,q_j} (-1)^k \left(q_i q_j\right)^{\frac{k(k-1)}{2}} x^{n-k} a^k \qquad (xa = ax) \end{aligned} \tag{4}$$

where the notations $\binom{n}{k}_{q_i,q_j}$ (called multiple q-Gauss Binomial coefficients) and $[n]_{q_i,q_j}!$ (called multiple q-factorial) are defined by

$$\begin{aligned} \binom{n}{k}_{q_i,q_j} &= \frac{[n]_{q_i,q_j}!}{[n-k]_{q_i,q_j}!\,[k]_{q_i,q_j}!} \qquad (n \geq k) \\ [n]_{q_i,q_j}! &= [n]_{q_i,q_j}\,[n-1]_{q_i,q_j}\cdots[2]_{q_i,q_j}\,[1]_{q_i,q_j} \qquad (n \in \mathbb{N}). \end{aligned}$$

The multiple q-exponential functions are introduced by

$$e_{q_iq_j}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{q_i,q_j}!} x^n \text{ and } E_{q_iq_j}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{q_i,q_j}!} \left(q_i q_j\right)^{\frac{n(n-1)}{2}} x^n \tag{5}$$

whose multiple q-derivatives, respectively, are as follows:

$$D_{q_i,q_j} e_{q_iq_j}(x) = e_{q_iq_j}(x) \text{ and } D_{q_i,q_j} E_{q_iq_j}(x) = E_{q_iq_j}(q_i q_j x).$$

Under circumstance commutative x and y $(xy = yx)$, we have addition formula

$$e_{q_iq_j}(x+y)_{q_iq_j} = e_{q_iq_j}(x) E_{q_iq_j}(x). \tag{6}$$

The multiple q-integral (a generalization of Jackson's integral) is given by

$$\int f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x = (q_i - q_j) \sum_{k=0}^{\infty} \frac{q_j^k x}{q_i^{k+1}} f\left(\frac{q_j^k}{q_i^{k+1}} x\right). \tag{7}$$

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a formal power series. Then it has multiple q-integral representation as follows:

$$\int f(x)\, d_{\frac{q_j}{q_i}} x = \sum_{k=0}^{\infty} q_i^{k+1} a_k \frac{x^{k+1}}{[k+1]_{q_i,q_j}} + C$$

where C is a constant.

In the special cases for q_i and q_j, the notations given in this part reduce to the notations of known q-calculus (see, for details, [8], [9], [5], [11], [7], [2], [3], [4], [12], [13], [14], [15]). Recently, Nalci and Pashaev [10] have represented multiple q-calculus and investigated many important notions and results in the course of developing multiple q-calculus along the traditional lines of q-calculus. In [1], Acikgoz *et al*. also considered some new identities involving a new class of some special polynomials in the light of multiple q-calculus. They also derived a further investigation of some new identities related to multiple q-Jackson integral.

In this paper, we develop the theory of the multiple q-analogue of the Heine's binomial formula, chain rule and Leibniz's rule. We also derive many useful definitions and results involving multiple q-antiderivative and multiple q-Jackson's integral. Finally, we list here multiple q-analogue of some elementary functions including trigonometric functions and hyperbolic functions. This may be a good consideration in developing the multiple q-calculus in combinatorics, number theory and other fields of mathematics.

2. Generalizations of some Elementary Functions belonging to q-Calculus

As it has been q-calculus, there doesn't exist a general chain rule for multiple q-derivatives. That is, if we consider the function $f(u(x))$, where $u = u(x) = \lambda x^{\mu}$ with λ, μ being constants, we have a chain rule as special cases:

$$\begin{aligned} D_{q_i,q_j}\left[f(u(x))\right] &= D_{q_i,q_j}\left[f\left(\lambda x^{\mu}\right)\right] \\ &= \left(D_{q_i^{\mu},q_j^{\mu}} f\right)(u(x)) D_{q_i,q_j} u(x). \end{aligned} \tag{8}$$

Conversely, if we consider the function $u(x) = x^3 + x^2$ or $u(x) = \cos x$, the quantity $u(q_i x)$ and $u(q_j x)$ can not be derived in terms of u in a basic way, and thereby it is impossible to write a general chain rule. Now let us investigate the derivative of the function $\frac{1}{(x-a)^n_{q_i,q_j}}$. For any integer n, we have

$$\begin{aligned} D_{q_i,q_j}\left(\frac{1}{(x-a)^n_{q_i,q_j}}\right) &= D_{q_i,q_j}\left(\frac{1}{\left(x - q_i^{-n}\left(q_i^n a\right)\right)^n_{q_i,q_j}}\right) \\ &= -\left(q_j q_i\right)^{-n}[n]_{q_i,q_j}\left(x - \left(q_j q_i\right)^n a\right)^{-n-1}_{q_i,q_j}, \end{aligned}$$

where

$$(x - q_j^n a)^{-n}_{q_i,q_j} = \frac{1}{(x - q_i^{-n} a)^n_{q_i,q_j}}.$$

By the similar way, we have for $n \geq 0$:

$$D_{q_i,q_j}(a-x)^n_{q_i,q_j} = -[n]_{q_i,q_j}(a-q_iq_jx)^{n-1}_{q_i,q_j}$$

and

$$D_{q_i,q_j}\left(\frac{1}{(a-x)^n_{q_i,q_j}}\right) = \frac{[n]_{q_i,q_j}}{(a-q_jx)^{n+1}_{q_i,q_j}}. \tag{9}$$

Taking the value $a = 1$ in the Eq. (9), we derive multiple q-derivative of k-th order as follows:

$$D^k_{q_i,q_j}\left(\frac{1}{(1-x)^n_{q_i,q_j}}\right) = \frac{[n]_{q_i,q_j}[n+1]_{q_i,q_j}\cdots[n+k-1]_{q_i,q_j}}{(1-q_j^kx)^{n+k}_{q_i,q_j}}. \tag{10}$$

In the case when $x = 0$ in the Eq. (10) gives

$$[n]_{q_i,q_j}[n+1]_{q_i,q_j}\cdots[n+k-1]_{q_i,q_j}. \tag{11}$$

By the Eq. (11), we have, i.e., a Taylor expansion for $\frac{1}{(1-x)^n_{q_i,q_j}}$ about $x = 0$:

$$\begin{aligned}\frac{1}{(1-x)^n_{q_i,q_j}} &= \sum_{k=0}^{\infty}\frac{[n]_{q_i,q_j}[n+1]_{q_i,q_j}\cdots[n+k-1]_{q_i,q_j}}{[k]_{q_i,q_j}!}x^k \\ &= \sum_{k=0}^{\infty}\frac{(1-Q^n)^k_Q}{(1-Q)^k_Q}q_i^{(n-k)k}x^k \qquad \left(Q=\frac{q_j}{q_i}\right)\end{aligned}$$

which is called Heine's multiple q-Binomial formula.

We now give the multiple q-analogue of Leibniz rule as follows.

Theorem 2.1. *Let $f(x)$ and $g(x)$ be n-times multiple q-differentiable functions. Then $(fg)(x)$ is also n-times multiple q-differentiable and*

$$D^n_{q_i,q_j}(fg)(x) = \sum_{k=0}^{n}\binom{n}{k}_{q_i,q_j} D^k_{q_i,q_j}(f)\left(xq_i^{n-k}\right)D^{n-k}_{q_i,q_j}(g)\left(xq_j^k\right).$$

Proof. The theorem can be easily proved by mathematical induction method. So we omit the proof of theorem. □

Corollary 2.2. *Each multiple q-binomial coefficient is a polynomial including the parameters q_i and q_j of degree $k(n-k)$ whose leading coefficient is* 1.

Proof. It is proved by making use of the same technique in [7]. So we omit the proof. □

Note that the multiple q-binomial coefficients also have combinatorial interpretations like q-binomial coefficients.

3. Multiple q-Antiderivative

Some information and useful methods in this section will be utilized from the book [7].

Definition 3.1. *The function $F(x)$ is a q-antiderivative of $f(x)$ if $D_{q_i,q_j}F(x) = f(x)$. It is shown by*

$$\int f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x.$$

Proposition 3.2. *Let $0 < \frac{q_j}{q_i} < 1$. Then, any function $f(x)$ has at most one multiple q-antiderivative which is continuous at $x = 0$, up to adding a constant.*

Proof. Let us consider F_1 and F_2 as two multiple q-antiderivatives of f, which are both continuous at 0. Let $\varpi = F_1 - F_2$, which also must be continuous at 0. Moreover

$$D_{q_i,q_j}\varpi(x) = D_{q_i,q_j}(F_1(x) - F_2(x)) = f(x) - f(x) = 0$$

implies that $\varpi(q_i x) = \varpi\left(q_j x\right)$ for any x. For some $U > 0$, let

$$\begin{aligned} s &= \inf\left\{\varpi(x) \mid \frac{q_j}{q_i}U \leq x \leq U\right\}, \\ S &= \sup\left\{\varpi(x) \mid \frac{q_j}{q_i}U \leq x \leq U\right\}, \end{aligned}$$

which may be infinity if ϖ is unbounded above and/or below. It should be clear that because of $s \neq S$, $\varpi(0)$ can not be both s and S. It is not problem that we select s or S, so we can suppose $\varpi(0) \neq s$. By the definition of continuous at $x = 0$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$s + \varepsilon \notin \varpi(0, \delta).$$

However there exists for some sufficiently N such that $\left(\frac{q_j}{q_i}\right)^N U < \delta$, which implies that

$$s + \varepsilon \in (s, S) \subset \varpi\left[\frac{q_j}{q_i}U, U\right] = \varpi\left[\left(\frac{q_j}{q_i}\right)^{N+1}U, \left(\frac{q_j}{q_i}\right)^{N}U\right] \subset \varpi(0, \delta),$$

bringing about a contradiction. So, we have $s = S$, ϖ is a constant in that $\varpi\left[\frac{q_j}{q_i}U, U\right]$, which shows that $F_1 - F_2$ is also constant everywhere. □

4. Multiple q-Jackson Integral

By the expression of the Eq. (7), we develop a more general formula:

$$\int f\left(\frac{x}{q_i}\right) D_{q_i,q_j} g\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x = \sum_{k=0}^{\infty} f\left(\frac{q_j^k}{q_i^{k+1}}x\right)\left(g\left(\frac{q_j^k}{q_i^k}x\right) - g\left(\frac{q_j^{k+1}}{q_i^{k+1}}x\right)\right).$$

Theorem 4.1. *Let $q_i, q_j \in (0,1)$ with $0 < \frac{q_j}{q_i} < 1$ and let $\left|f(x)x^{\tau}\right|$ be bounded on the interval $(0, A]$ for some $0 \leq \tau < 1$. Then the Jackson integral defined by (7) converges to a function $F(x)$ on $(0, A]$, which is a multiple q-antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x = 0$ with $F(0) = 0$.*

Proof. *Suppose* $\left|f(x)x^{\tau}\right| < M$ *on* $(0,A]$ *and fix* $0 < x \leq A$. *Then for* $k \geq 0$,

$$\begin{aligned}\left|f\left(\frac{q_j^k}{q_i^{k+1}}x\right)\left(\frac{q_j^k}{q_i^{k+1}}x\right)^{\tau}\right| &< M\\ \left|f\left(\frac{q_j^k}{q_i^{k+1}}x\right)\right| &< M\left(\frac{q_j^k}{q_i^{k+1}}x\right)^{-\tau}.\end{aligned}$$

Hence, for any $0 < x \leq A$, *we get*

$$\left|\left(\frac{q_j^k}{q_i^{k+1}}\right)f\left(\frac{q_j^k}{q_i^{k+1}}x\right)\right| < Mx^{-\tau}\frac{1}{\left(q_i^{1-\tau}\right)}\left(\frac{q_j^{1-\tau}}{q_i^{1-\tau}}\right)^k. \tag{12}$$

If we write in the following sum including Jackson integral that is majorized by a convergent geometric series. Then, (7) converges pointwise to some functions. Namely, one can see without difficulty that $F(0) = 0$. *It is the fact that* $F(x)$ *is continuous at* $x = 0$, *i.e.,* $F(x)$ *approaches zero as* $x \longrightarrow 0$ *using (12), for* $0 < x \leq A$ *as*

$$\begin{aligned}\left|(q_i - q_j)\sum_{k=0}^{\infty}\frac{q_j^k x}{q_i^{k+1}}f\left(\frac{q_j^k}{q_i^{k+1}}x\right)\right| &< \left|q_i - q_j\right||x|\sum_{k=0}^{\infty}\frac{q_j^k}{q_i^{k+1}}f\left(\frac{q_j^k}{q_i^{k+1}}x\right)\\ &< \left|q_i - q_j\right|\frac{1}{\left(q_i^{1-\tau}\right)}\frac{Mx^{1-\tau}}{1-\left(\frac{q_j}{q_i}\right)^{1-\tau}}.\end{aligned}$$

□

We now give the following theorem in order to verify $F(x)$ being a multiple q-antiderivative of $f(x)$.

Theorem 4.2. *The definition of q-multiple Jackson integral given in* (7) *presents a q-antiderivatives of* $f(x)$.

Proof. It is sufficient to check that

$$\begin{aligned}D_{q_i,q_j}F(x) &= \frac{1}{\left(q_i - q_j\right)x}\left((q_i - q_j)\sum_{\tau=0}^{\infty}\frac{q_j^{\tau}x}{q_i^{\tau}}f\left(\frac{q_j^{\tau}}{q_i^{\tau}}x\right) - (q_i - q_j)\sum_{\tau=0}^{\infty}\frac{q_j^{\tau+1}x}{q_i^{\tau+1}}f\left(\frac{q_j^{\tau+1}}{q_i^{\tau+1}}x\right)\right)\\ &= f(x).\end{aligned}$$

This completes the proof of the Theorem. □

Notice that the multiple q-differentiation is valid provided that $x \in (0,A]$ and $0 < \frac{q_j}{q_i} < 1$, then $x\frac{q_j}{q_i} \in (0,A]$. By Proposition 3.2, if the hypothesis of Theorem 4.1 is satisfied, the q-multiple Jackson integral gives the unique multiple q-antiderivative being continuous at $x = 0$, up to adding a constant. On the other hand, if we know that $F(x)$ is a multiple q-antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, $F(x)$ must be given, up to adding a constant. By q-multiple Jackson's formula (7), since a partial sum of the q-multiple Jackson integral is

$$\begin{aligned}(q_i - q_j)\sum_{\tau=0}^{N}\frac{q_j^{\tau}x}{q_i^{\tau+1}}f\left(\frac{q_j^{\tau}}{q_i^{\tau+1}}x\right) &= (q_i - q_j)\sum_{\tau=0}^{N}\frac{q_j^{\tau}x}{q_i^{\tau+1}}D_{q_i,q_j}F(x)\mid_{\frac{q_j^{\tau}}{q_i^{\tau+1}}x}\\ &= F(x) - F\left(\frac{q_j^{N+1}}{q_i^{N+1}}x\right),\end{aligned}$$

approaching to $F(x) - F(0)$ as $N \longrightarrow \infty$, by the continuity of $F(x)$ at the case $x = 0$. We now give an example to see in which the q-multiple Jackson formula fails. Let $f(x) = 1/x$. We have

$$\int \frac{1}{x} d_{\frac{q_j}{q_i}} x = \frac{(q_i - q_j)}{\log(\frac{q_j}{q_i})} \log(x)$$

since

$$D_{q_i,q_j} \log x = \frac{\log(q_i x) - \log(q_j x)}{(q_i - q_j)x} = \frac{\log(\frac{q_j}{q_i})}{(q_i - q_j)} \frac{1}{x}.$$

However, the q-multiple Jackson formula gives

$$\int \frac{1}{x} d_{\frac{q_j}{q_i}} x = \frac{(q_i - q_j)}{q_i} \sum_{k=0}^{\infty} 1 = \infty.$$

Finally, the formula fails because $f(x)x^{\tau}$ is not bounded for any $0 \leq \tau < 1$. Note that $\log x$ is not continuous at the case $x = 0$.

5. Multiple q-Trigonometric Functions

The multiple q-analogues of the sine, cosine, tangent and cotangent functions can be defined in the same manner with their well known Euler expressions of the exponential functions.

Definition 5.1. *Let* $\mathbf{i} = \sqrt{-1}$. *Then two pairs of multiple q-trigonometric functions are defined by*

$\sin_{q_i,q_j} x := \frac{e_{q_iq_j}(\mathbf{i}x) - e_{q_iq_j}(-\mathbf{i}x)}{2\mathbf{i}}$	$SIN_{q_i,q_j} x := \frac{E_{q_iq_j}(\mathbf{i}x) - E_{q_iq_j}(-\mathbf{i}x)}{2\mathbf{i}}$	(13)
$\cos_{q_i,q_j} x := \frac{e_{q_iq_j}(\mathbf{i}x) + e_{q_iq_j}(-\mathbf{i}x)}{2}$	$COS_{q_i,q_j} x := \frac{E_{q_iq_j}(\mathbf{i}x) + E_{q_iq_j}(-\mathbf{i}x)}{2}$	
$\tan_{q_i,q_j} x := \frac{\sin_{q_i,q_j} x}{\cos_{q_i,q_j} x}$	$TAN_{q_i,q_j} x := \frac{SIN_{q_i,q_j} x}{COS_{q_i,q_j} x}$	
$\cot_{q_i,q_j} x := \frac{\cos_{q_i,q_j} x}{\sin_{q_i,q_j} x}$	$COT_{q_i,q_j} x := \frac{COS_{q_i,q_j} x}{SIN_{q_i,q_j} x}.$	

Note that one can represent $N \times N$ matrix of the multiple q-trigonometric functions in view of Eq. (1).

Definition 5.2. *Two pairs of multiple q-hyperbolic functions are defined by*

$\sinh_{q_i,q_j} x = \frac{e_{q_iq_j}(x) - e_{q_iq_j}(-x)}{2}$	$SINH_{q_i,q_j} x = \frac{E_{q_iq_j}(x) - E_{q_iq_j}(-x)}{2}$	(14)
$\cosh_{q_i,q_j} x = \frac{e_{q_iq_j}(x) + e_{q_iq_j}(-x)}{2}$	$COSH_{q_i,q_j} x = \frac{E_{q_iq_j}(x) + E_{q_iq_j}(-x)}{2}$	
$\tanh_{q_i,q_j} x = \frac{\sinh_{q_i,q_j} x}{\cosh_{q_i,q_j} x}$	$TANH_{q_i,q_j} x = \frac{SINH_{q_i,q_j} x}{COSH_{q_i,q_j} x}$	
$\coth_{q_i,q_j} x = \frac{\cosh_{q_i,q_j} x}{\sinh_{q_i,q_j} x}$	$COTH_{q_i,q_j} x = \frac{COSH_{q_i,q_j} x}{SINH_{q_i,q_j} x}.$	

By Definition 5.2, we readily see that

$e_{q_iq_j}(x) = \cosh_{q_i,q_j} x + \sinh_{q_i,q_j} x$	$E_{q_iq_j}(x) = COSH_{q_i,q_j} x + SINH_{q_i,q_j} x$

Note that one can represent $N \times N$ matrix of the multiple q-hyperbolic functions in view of Eq. (1).

We now list intriguing identities for trigonometric and hyperbolic functions under the theory of multiple q-theory as follows.

$\sin_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{q_i,q_j}!} x^{2n+1}$	$\sinh_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{[2n+1]_{q_i,q_j}!}$
$SIN_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{q_i,q_j}!} \left(q_i q_j\right)^{\frac{(2n+1)2n}{2}} x^{2n+1}$	$SINH_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{\left(q_i q_j\right)^{\frac{(2n+1)2n}{2}} x^{2n+1}}{[2n+1]_{q_i,q_j}!}$
$\cos_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{q_i,q_j}!} x^{2n}$	$\cosh_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{x^{2n}}{[2n]_{q_i,q_j}!}$
$COS_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{q_i,q_j}!} \left(q_i q_j\right)^{\frac{2n(2n-1)}{2}} x^{2n}$	$COSH_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{\left(q_i q_j\right)^{\frac{2n(2n-1)}{2}} x^{2n}}{[2n]_{q_i,q_j}!}$
$\sec_{q_i,q_j} x := \frac{1}{\cos_{q_i,q_j} x}$	$\csc_{q_i,q_j} x := \frac{1}{\sin_{q_i,q_j} x}$
$SEC_{q_i,q_j} x := \frac{1}{COS_{q_i,q_j} x}$	$CSC_{q_i,q_j} x := \frac{1}{SIN_{q_i,q_j} x}$
$sech_{q_i,q_j} x := \frac{1}{\cosh_{q_i,q_j} x}$	$csch_{q_i,q_j} x := \frac{1}{\sinh_{q_i,q_j} x}$
$SECH_{q_i,q_j} x := \frac{1}{COSH_{q_i,q_j} x}$	$CSCH_{q_i,q_j} x := \frac{1}{SINH_{q_i,q_j} x}$

$e_{q_i q_j}(x+y)_{q_i,q_j} = \cosh_{q_i,q_j}(x+y)_{q_i,q_j} + \sinh_{q_i,q_j}(x+y)_{q_i,q_j}$
$E_{q_i q_j}(x+y)_{q_i,q_j} = COSH_{q_i,q_j}(x+y)_{q_i,q_j} + SINH_{q_i,q_j}(x+y)_{q_i,q_j}$
$\sinh_{q_i,q_j}(x+y)_{q_i,q_j} = \sinh_{q_i,q_j} x COSH_{q_i,q_j} y + \cosh_{q_i,q_j} x SINH_{q_i,q_j} y$
$\cosh_{q_i,q_j}(x+y)_{q_i,q_j} = \cosh_{q_i,q_j} x COSH_{q_i,q_j} y + \sinh_{q_i,q_j} x SINH_{q_i,q_j} y$
$SINH_{q_i,q_j}(x+y)_{q_i,q_j} = \sinh_{q_i,q_j} x COSH_{q_i,q_j} y + \cosh_{q_i,q_j} x SINH_{q_i,q_j} y$
$COSH_{q_i,q_j}(x+y)_{q_i,q_j} = \cosh_{q_i,q_j} x COSH_{q_i,q_j} y + \sinh_{q_i,q_j} x SINH_{q_i,q_j} y$
$\sin_{q_i,q_j}(x+\mathbf{i}y)_{q_i,q_j} = \sin_{q_i,q_j} x COSH_{q_i,q_j} y + \mathbf{i} \cos_{q_i,q_j} x SINH_{q_i,q_j} y$
$\cos_{q_i,q_j}(x+\mathbf{i}y)_{q_i,q_j} = \cos_{q_i,q_j} x COSH_{q_i,q_j} y + \mathbf{i} \sin_{q_i,q_j} x SINH_{q_i,q_j} y$

$\sin_{q_i,q_j}(-x) = -\sin_{q_i,q_j} x$	$SIN_{q_i,q_j}(-x) = -SIN_{q_i,q_j} x$
$\cos_{q_i,q_j}(-x) = \cos_{q_i,q_j} x$	$COS_{q_i,q_j}(-x) = COS_{q_i,q_j} x$
$\tan_{q_i,q_j}(-x) = -\tan_{q_i,q_j} x$	$TAN_{q_i,q_j}(-x) = -TAN_{q_i,q_j} x$
$\cot_{q_i,q_j}(-x) = -\cot_{q_i,q_j} x$	$COT_{q_i,q_j}(-x) = -COT_{q_i,q_j} x$
$\sec_{q_i,q_j}(-x) = \sec_{q_i,q_j} x$	$SEC_{q_i,q_j}(-x) = SEC_{q_i,q_j} x$
$\csc_{q_i,q_j}(-x) = -\csc_{q_i,q_j} x$	$CSC_{q_i,q_j}(-x) = -CSC_{q_i,q_j} x$
$\sinh_{q_i,q_j}(-x) = -\sinh_{q_i,q_j} x$	$SINH_{q_i,q_j}(-x) = -SINH_{q_i,q_j} x$
$\cosh_{q_i,q_j}(-x) = \cosh_{q_i,q_j} x$	$COSH_{q_i,q_j}(-x) = COSH_{q_i,q_j} x$
$\tanh_{q_i,q_j}(-x) = -\tanh_{q_i,q_j} x$	$TANH_{q_i,q_j}(-x) = -TANH_{q_i,q_j} x$
$\coth_{q_i,q_j}(-x) = -\coth_{q_i,q_j} x$	$COTH_{q_i,q_j}(-x) = -COTH_{q_i,q_j} x$
$sech_{q_i,q_j}(-x) = sech_{q_i,q_j} x$	$SECH_{q_i,q_j}(-x) = SECH_{q_i,q_j} x$
$csch_{q_i,q_j}(-x) = -csch_{q_i,q_j} x$	$CSCH_{q_i,q_j}(-x) = -CSCH_{q_i,q_j} x$

$D_{q_i,q_j}\sin_{q_i,q_j}x=\cos_{q_i,q_j}x$	$\int \sin_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=-\cos_{q_i,q_j}x+C$
$D_{q_i,q_j}SIN_{q_i,q_j}x=COS_{q_i,q_j}\left(q_iq_jx\right)$	$\int SIN_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=-q_iq_jCOS_{q_i,q_j}\left(\frac{x}{q_iq_j}\right)+C$
$D_{q_i,q_j}\cos_{q_i,q_j}x=-\sin_{q_i,q_j}x$	$\int \cos_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\sin_{q_i,q_j}x+C$
$D_{q_i,q_j}COS_{q_i,q_j}x=-SIN_{q_i,q_j}\left(q_iq_jx\right)$	$\int COS_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=q_iq_jSIN_{q_i,q_j}\left(\frac{x}{q_iq_j}\right)+C$
$D_{q_i,q_j}\sinh_{q_i,q_j}x=\cosh_{q_i,q_j}$	$\int \sinh_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\cosh_{q_i,q_j}+C$
$D_{q_i,q_j}SINH_{q_i,q_j}x=COSH_{q_i,q_j}\left(q_iq_jx\right)$	$\int SINH_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=q_iq_jCOSH_{q_i,q_j}\left(\frac{x}{q_iq_j}\right)+C$
$D_{q_i,q_j}\cosh_{q_i,q_j}=\sinh_{q_i,q_j}x$	$\int \cosh_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\sinh_{q_i,q_j}x+C$
$D_{q_i,q_j}COSH_{q_i,q_j}x=SINH_{q_i,q_j}\left(q_iq_jx\right)$	$\int COSH_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=q_iq_jSINH_{q_i,q_j}\left(\frac{x}{q_iq_j}\right)+C$

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