Filomat 31:8 (2017), 2509–2515
DOI 10.2298/FIL1708509A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Generalization of the m-Topology on $C(X)$ Finer than the m-Topology

F. Azarpanah[a], F. Manshoor[b], R. Mohamadian[a]

[a] *Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran*
[b] *Department of Mathematics, Abadan Branch, Islamic Azad University, Abadan, Iran*

Abstract. It is well known that the component of the zero function in $C(X)$ with the m-topology is the ideal $C_\psi(X)$. Given any ideal $I \subseteq C_\psi(X)$, we are going to define a topology on $C(X)$ namely the m^I-topology, finer than the m-topology in which the component of 0 is exactly the ideal I and $C(X)$ with this topology becomes a topological ring. We show that compact sets in $C(X)$ with the m^I-topology have empty interior if and only if $X \setminus \bigcap Z[I]$ is infinite. We also show that nonzero ideals are never compact, the ideal I may be locally compact in $C(X)$ with the m^I-topology and every Lindelöf ideal in this space is contained in $C_\psi(X)$. Finally, we give some relations between topological properties of the spaces X and $C_m(X)$. For instance, we show that the set of units is dense in $C_m(X)$ if and only if X is strongly zero-dimensional and we characterize the space X for which the set $r(X)$ of regular elements of $C(X)$ is dense in $C_m(X)$.

1. Introduction

Throughout this paper we denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real-valued continuous functions on a completely regular Hausdorff space X. The m-topology on $C(X)$ is defined by taking the set of the form

$$B(f, u) = \{g \in C(X) : |f(x) - g(x)| < u(x), \forall x \in X\}$$

as a base for a neighborhood system at f, for each $f \in C(X)$ and $u \in U^+(X)$, where $U^+(X)$ is the set of all positive elements of $C(X)$. $C(X)$ endowed with the m-topology is denoted by $C_m(X)$ which is a Hausdorff topological ring. The m-topology is first introduced in the late 40s in [8] and later the research in this area became active over the last 20 years, for example, the works in [2], [3], [6] and [10].

Compact sets and connected sets in $C_m(X)$ are investigated in [2] and it is shown that the component of 0 in $C_m(X)$ is the ideal $C_\psi(X)$. Clearly the connected sets (component of 0) in $C(X)$ with a topology finer that the m-topology are also connected in $C_m(X)$ (is contained in $C_\psi(X)$). In this paper, for a given ideal I contained in $C_\psi(X)$, we define a topology on $C(X)$, namely the m^I-topology, in which the component of 0 is exactly the ideal I. This topology is finer than the m-topology and makes $C(X)$ a topological ring. We denote the space $C(X)$ with the m^I-topology by $C_{m^I}(X)$. More generally, if I is an arbitrary ideal in $C(X)$, the m^I-topology is defined similarly and we show that the component of 0 in the space $C_{m^I}(X)$ is $C_\psi(X) \cap I$. We

2010 *Mathematics Subject Classification*. Primary 54C35; Secondary 54C40
Keywords. m-topology, m^I-topology, component, Lindelöf, almost P-space, strongly zero-dimensional, μ-compact, cellularity
Received: 28 October 2015; Revised: 02 February 2016; Accepted: 04 February 2016
Communicated by Ljubiša D.R. Kočinac
Email addresses: azarpanah@ipm.ir (F. Azarpanah), Avazerood@yahoo.com (F. Manshoor), mohamadian_r@scu.ac.ir (R. Mohamadian)

also investigate compact sets in $C(X)$ with the m^I-topology and it turns out that whenever $I \not\subseteq C_\psi(X)$, every compact set in $C_{m^I}(X)$ has an empty interior.

For each $f \in C(X)$, the set zeros of f is called the zero-set of f and is denoted by $Z(f)$, $\mathrm{coz} f = X \setminus Z(f)$ and $\mathrm{cl}_X \mathrm{coz} f$ is called the support of f. We also denote the sets $\{x \in X : f(x) > 0\}$ and $\{x \in X : f(x) < 0\}$ by $\mathrm{pos} f$ and $\mathrm{neg} f$ respectively. An ideal I in $C(X)$ is called a z-ideal if whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. We recall that $C_\psi(X)$ ($C_K(X)$) is a z-ideal in $C(X)$ consisting of all functions with pseudocompact (compact) support and it is well-known that $f \in C_\psi(X)$ if and only if $X \setminus Z(f)$ is relatively pseudocompact, i.e., every function in $C(X)$ is bounded on $X \setminus Z(f)$, see Theorem 2.1 in [11]. It is well-known that $C_K(X) = O^{\beta X \setminus X}$ and $C_\psi(X) = M^{\beta X \setminus \upsilon X}$, where βX is the Stone-Čech compactification and υX is the Hewitt realcompactification of X, see [5]. For terminology and notations, the reader is referred to [4] and [5].

2. m^I-Topology on $C(X)$

Let I be an ideal (not necessarily proper) of $C(X)$. For each $f \in C(X)$ and $u \in U^+(X)$, we define the subset $B(f, I, u)$ of $C(X)$ as follows:

$$B(f, I, u) = \{g \in C(X) : |f - g| < u \text{ and } g \equiv f (\mathrm{mod} I)\}.$$

We define the m^I-topology on $C(X)$ by taking the family $\{B(f, I, u) : u \in U^+(X)\}$ as a base for a neighborhood system at f for each $f \in C(X)$. The set $C(X)$ endowed with the m^I-topology is denoted by $C_{m^I}(X)$. To see that $\{B(f, I, u) : u \in U^+(X)\}$ is a base at f, it is evident that $f \in B(f, I, u)$, $B(f, I, u \wedge v) \subseteq B(f, I, u) \cap B(f, I, v)$, for all $u, v \in U^+(X)$ and whenever $g \in B(f, I, u)$ for some $u \in U^+(X)$, then $B(g, I, v) \subseteq B(f, I, u)$, where $v = u - |f - g| \in U^+(X)$. If $I = C(X)$, then the m^I-topology and the m-topology coincide and whenever $I \subseteq J$ are two ideals in $C(X)$, it is clear that the m^I-topology is finer than the the m^J-topology. This implies that for each ideal I in $C(X)$, the m^I-topology is finer than the m-topology.

Proposition 2.1. *The space $C_{m^I}(X)$ is a topological ring.*

Proof. If $u \in U^+(X)$ and $B(f + g, I, u)$ is a neighborhood at $f + g$, then we consider the neighborhoods $B(f, I, \frac{u}{2})$ and $B(g, I, \frac{u}{2})$ at f and g respectively. Now suppose that $h \in B(f, I, \frac{u}{2})$ and $k \in B(g, I, \frac{u}{2})$, then we have $|h - f| < \frac{u}{2}$, $|k - g| < \frac{u}{2}$, $h - f \in I$ and $k - g \in I$. Hence we have $|(h + k) - (f + g)| < u$ and $(h + k) - (f + g) \in I$, i.e., the function $+$ is continuous. For the continuity of $\times$, let $B(fg, I, u)$ be a neighborhood at fg and take $v = \frac{u}{2(1+|g|)}$ and $w = \frac{u}{2(|f|+v+1)}$. Now if $h \in B(f, I, v)$ and $k \in B(g, I, w)$, then $|h - f| < v$ and $|k - g| < w$ imply that $|gh - fg| < \frac{u}{2}$ and $|hk - hg| < \frac{u|h|}{2(|f|+v+1)} < \frac{u(|f|+v)}{2(|f|+v+1)} < \frac{u}{2}$. On the other hand $f(k - g) \in I$ and $k(h - f) \in I$ imply that $hk - fg \in I$ and we are through. □

We need the following results in the sequel.

Proposition 2.2. *The following statements hold.*

(a) *Every ideal containing I is a closed-open set in $C_{m^I}(X)$.*
(b) *If I is a z-ideal and J is a closed ideal in $C_{m^I}(X)$, then $I \cap J$ is also a z-ideal.*
(c) *Every maximal ideal is closed in $C_{m^I}(X)$.*
(d) *$C^*(X) \cap I$ is a closed-open set in $C_{m^I}(X)$.*
(e) *The closure of every proper ideal in $C_{m^I}(X)$ is a proper ideal.*

Proof. (a) Let $I \subseteq J$ and $f \in \mathrm{cl}_{m^I} J$, where $\mathrm{cl}_{m^I} J$ means the closure of J in $C_{m^I}(X)$. Hence there exists $j \in J$ such that $j \in B(f, I, 1)$. Thus $f - j \in I \subseteq J$, so $f \in J$. This implies that J is closed. On the other hand if $g \in J$, then $B(g, I, u) \subseteq J$, for all $u \in U^+(X)$, i.e., J is open.

(b) Let $Z(g) \subseteq Z(f)$ and $g \in I \cap J$. Since I is a z-ideal, it is enough to show that $f \in J$. For each $u \in U^+(X)$, we define

$$h(x) = \begin{cases} \frac{f(x)+u(x)}{g(x)} & f(x) \leq -u(x) \\ 0 & |f(x)| \leq u(x) \\ \frac{f(x)-u(x)}{g(x)} & u(x) \leq f(x). \end{cases}$$

Clearly $h \in C(X)$ and $|f - gh| < u$. Since $f - gh \in I$, $f \in \mathrm{cl}_{m^I} J$. But J is closed, hence $f \in J$.

(c) In fact every closed ideal in $C_m(X)$ is also closed in $C_{m^I}(X)$.

(d) For each $f \in C^*(X) \cap I$, we have $B(f, I, 1) \subseteq C^*(X) \cap I$. Now if $h \in \mathrm{cl}_{m^I}(C^*(X) \cap I)$, then there exists $f \in B(f, I, 1) \cap C^*(X) \cap I$. Hence $|f - h| < 1$ and $f - h \in I$, whence $h \in C^*(X) \cap I$.

(e) If J is an ideal, $f, g \in \mathrm{cl}_{m^I} J$ and $u \in U^+(X)$, then there are $h, k \in C(X)$ such that $h \in B(f, I, \frac{u}{2}) \cap J$, $k \in B(g, I, \frac{u}{2}) \cap J$ and $h - f, k - g \in I$. Hence it is clear that $h + k \in B(f + g, I, u) \cap J$ and hence $f + g \in \mathrm{cl}_{m^I} J$. Whenever $f \in \mathrm{cl}_{m^I} J$, $g \in C(X)$ and $u \in U^+(X)$, then there exists $h \in B(f, I, \frac{u}{1+|g|}) \cap J$, i.e., $|h - f| < \frac{u}{1+|g|}$ and $h - f \in I$. Hence $|gh - fg| < \frac{|g|u}{1+|g|} < u$ and $gh - fg \in I$ which means that $B(fg, I, u) \cap J \neq \emptyset$, so $fg \in \mathrm{cl}_{m^I} J$. $\square$

Remark 2.3. Whenever $S \subseteq C(X)$ and $C(X)$ is endowed with the m-topology (m^I-topology), then S may be considered as a subspace of $C_m(X)$ ($C_{m^I}(X)$) with the relative topology. We should emphasize here that m-topology and m^I-topology on I coincide. In fact whenever $u \in U^+(X)$ and $f \in I$, we have $B(f, I, u) \cap I = B(f, I, u) = B(f, u) \cap I$.

3. Connectedness in $C_{m^I}(X)$

In this section we characterize the components of $C_{m^I}(X)$ and investigate the disconnectedness of $C_{m^I}(X)$. To this end, we need the following lemmas.

Lemma 3.1. *$f \in C_\psi(X) \cap I$, if and only if the function $\varphi_f : \mathbb{R} \to C_{m^I}(X)$ defined by $\varphi_f(r) = rf$, for all $r \in \mathbb{R}$, is continuous.*

Proof. Let $u \in U^+(X)$. Since $f \in C_\psi(X)$, u is bounded away from zero on $X \setminus Z(f)$ for, $\frac{1}{u}$ is bounded on $X \setminus Z(f)$. Thus we may assume that $u(x) > \alpha > 0$, for all $x \in X \setminus Z(f)$. If $|f| < M$, then $(r - \frac{\alpha}{M}, r + \frac{\alpha}{M}) \subseteq \varphi_f^{-1}(B(rf, I, u))$. So whenever $|r - s| < \frac{\alpha}{M}$, then we have $|rf - sf| < \frac{\alpha}{M}|f| < \alpha < u$. It is also evident that $rf - sf \in I$ for, $f \in I$, hence φ_f is continuous. Conversely suppose that φ_f is continuous. Hence for every $u \in U^+(X)$, there exists $\delta > 0$ such that $(-\delta, \delta) \subseteq \varphi_f^{-1}(B(0, I, u))$. This means that for each $0 \neq s \in (-\delta, \delta)$, we have $|sf| < u$ and $sf \in I$, so $f \in I$. Now whenever $g \in C^*(X)$, by taking $u = \frac{1}{1+|g|} \in U^+(X)$, we have $|sf| < \frac{1}{1+|g|}$ or $|gf| < (1 + |g|)|f| < \frac{1}{s}$, for each $0 \neq s \in (-\delta, \delta)$. This implies that $fg \in C^*(X)$, for all $g \in C^*(X)$ and hence by Lemma 2.10 in [7], we have $f \in C_\psi(X)$, therefore $f \in C_\psi(X) \cap I$. $\square$

Corollary 3.2. *$f \in C_\psi(X)$, if and only if the function $\varphi_f : \mathbb{R} \to C_m(X)$ defined by $\varphi_f(r) = rf$ is continuous, for all $r \in \mathbb{R}$.*

Whenever every element of an ideal in $C(X)$ is bounded, we call it a bounded ideal. The largest bounded ideal in $C(X)$ exists by the following result, see Corollary 3.10 in [2] for its proof.

Lemma 3.3. *The largest bounded ideal in $C(X)$ is $C_\psi(X)$.*

The following theorem shows that the component of 0 in $C(X)$ with the m-topology is $C_\psi(X)$, see also [2]. This theorem also shows that whenever $I \subseteq C_\psi(X)$, then the component of 0 in $C_{m^I}(X)$ is I.

Theorem 3.4. *The component of 0 in $C_{m^I}(X)$ is $C_\psi(X) \cap I$.*

Proof. For each $f \in C_\psi(X) \cap I$, the function φ_f is continuous by Lemma 3.1. Hence $\varphi_f(\mathbb{R})$ is connected. But $C_\psi(X) \cap I = \bigcup_{f \in C_\psi(X) \cap I} \varphi_f(\mathbb{R})$ means that $C_\psi(X) \cap I$ is also connected. Now suppose that J is a connected ideal in $C_{m^I}(X)$. Since by part (d) of Proposition 2.2, $C^*(X) \cap I$ is a closed-open set in $C_{m^I}(X)$, we have $J \subseteq C^*(X) \cap I$. This implies that J is a bounded ideal and hence $J \subseteq C_\psi(X)$ by Lemma 3.3. Therefore $J \subseteq C_\psi(X) \cap I$, i.e., $C_\psi(X) \cap I$ is the component of 0. $\square$

Since the ideal I is an open-closed set in $C_{m^I}(X)$, by Proposition 2.2, the following corollary is evident.

Corollary 3.5. *If I is an ideal in $C(X)$ and $I \subseteq C_\psi(X)$, then the quasicomponent of 0 in $C_{m^I}(X)$ is I.*

If $C_\psi(X) \neq (0)$, then $C_\psi(X) = M^{\beta X \setminus \upsilon X}$ is free and hence it is an essential ideal (i.e., intersects every nonzero ideal nontrivially) by Proposition 2.1 in [1]. Now if I is a nonzero ideal in $C(X)$, then $C_\psi(X) \cap I \neq (0)$ and the following corollary is evident.

Corollary 3.6. *The following statements hold.*

(a) *$C_{m^I}(X)$ is a totally disconnected if and only if either $I = (0)$ or $C_\psi(X) = (0)$.*
(b) *If X is pseudocompact, then $C_{m^I}(X)$ is a totally disconnected space if and only if $I = (0)$.*
(c) *Whenever I is a proper ideal in $C(X)$, then $C_{m^I}(X)$ is never connected.*

4. Compactness in $C_{m^I}(X)$

In this section we investigate compact subsets of $C_{m^I}(X)$. Using the next theorem, for an infinite space X, every compact subset of $C(X)$ with the m-topology has an empty interior. To prove the theorem, we first need the following lemma.

Lemma 4.1. *Suppose that u is unit and I is an ideal in $C(X)$.*

(a) *If $\{a_1, a_2, \dots, a_k\} \subseteq X \setminus \bigcap Z[I]$, then for each $1 \leq i \leq k$, there exists $t_i \in I$ such that $|t_i| < u$, $t_i(a_i) = \frac{1}{2}u(a_i)$ and $t_i(a_j) = 0$, for all $j \neq i$.*
(b) *If $X \setminus \bigcap Z[I]$ is finite, then the subspace I of $C_{m^I}(X)$ is homeomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}$.*

Proof. (a) Since $\{a_1, a_2, \dots, a_k\} \subseteq X \setminus \bigcap Z[I]$, for each $1 \leq i \leq k$, there exists $s_i \in I$ such that $s_i(a_i) \neq 0$ and $s_i(a_j) = 0$. Without loss of generality, let $s_i(a_i) = 1$ and $s_i \geq 0$. Now consider the function $t_i = \frac{3}{2}\frac{s_i}{1+2s_i}u$. Clearly we have $t_i \in I$, $t_i(a_i) = \frac{1}{2}u(a_i)$, $t_i(a_j) = 0$ and $|t_i| < \frac{3}{2}\frac{1}{2}u < u$.

(b) Let $X \setminus \bigcap Z[I] = \{a_1, a_2, \dots, a_k\}$. Clearly, each a_i is an isolated point. First we show that $I = (e)$, where $e(a_1) = \cdots = e(a_k) = 1$ and $e(x) = 0$, otherwise. For each $1 \leq i \leq k$, there exists $f_i \in I$ such that $f_i(x_i) \neq 0$, for $x_i \notin \bigcap Z[I]$. Now $h = f_1^2 + \cdots + f_k^2 \in I$ and $Z(h) = \bigcap Z[I]$. But $Z(h) = Z(e)$ is open, hence e is a multiple of h, by 1D in [5], i.e., $e \in I$. On the other hand, for each $f \in I$, we have $Z(e) \subseteq Z(f)$ which means that $f \in (e)$ by 1D in [5] again, i.e., $I = (e)$.

Now corresponding to each $b = (b_1, b_2, \dots, b_k) \in \mathbb{R}^k$, the function f_b defined by $f_b(a_i) = b_i$, for all $i = 1, \dots, k$ and $f_b(\bigcap Z[I]) = \{0\}$ belongs to $I = (e)$. We define $\varphi : \mathbb{R}^k \to I \subseteq C_{m^I}(X)$ by $\varphi(b) = f_b$, for all $b \in \mathbb{R}^k$. Clearly, the function φ is one to one and onto. The function φ is also continuous. In fact for every $f_b \in I$, where $b = (b_1, \dots, b_k) \in \mathbb{R}^k$ and for each positive unit u in $C(X)$, we have $\varphi^{-1}(B(f_b, I, u) \cap I) = \prod_{i=1}^k (b_i - u(a_i), b_i + u(a_i))$. Finally, φ is open for, $\varphi(\pi_i^{-1}(b_i - \varepsilon_i, b_i + \varepsilon_i)) = \{f \in I : |f(a_i) - b_i| < \varepsilon_i\}$ is open in I for each $i = 1, \dots, k$ and $\varepsilon_i > 0$. Therefore, I is homeomorphic to $\mathbb{R}^k$. □

Theorem 4.2. *If I is an ideal in $C(X)$, then every compact subset of $C_{m^I}(X)$ has an empty interior if and only if $X \setminus \bigcap Z[I]$ is infinite.*

Proof. Let $X \setminus \bigcap Z[I]$ be infinite and F be a compact subset of $C_{m^I}(X)$. Suppose that $f \in \operatorname{int}_{m^I} F$, then there exists $u \in U^+(X)$ such that $B(f, I, u) \subseteq F$. Since F is compact, there are $g_1, g_2, \dots, g_n \in F$ such that $F \subseteq \bigcup_{i=1}^n B(g_i, I, \frac{u}{4})$. Since $X \setminus \bigcap Z[I]$ is an infinite set, we may produce a set $\{x_1, x_2, \dots, x_n, x_{n+1}\} \subseteq X \setminus \bigcap Z[I]$ with distinct elements. Now by invoking Lemma 4.1, for $i \in \{1, 2, \dots, n+1\}$, we define the function $t_i \in I$ with $|t_i| < u$, where $t_i(x_i) = \frac{1}{2}u(x_i)$ and $t_i(x_j) = 0$, for all $j \neq i$. If we take $h_i = f + t_i$, then we have $h_i - f = t_i \in I$ and $|h_i - f| = |t_i| < u$, for all $i = 1, 2, \dots, n+1$. Therefore $h_k \in B(f, I, u) \subseteq F \subseteq \bigcup_{i=1}^n B(g_i, I, \frac{u}{4})$, for all $k = 1, \dots, n+1$. This means that for some $1 \leq s \leq n+1$, $B(g_s, I, \frac{u}{4})$ contains at least two of h_i's. Let $h_i, h_j \in B(g_s, I, \frac{u}{4})$, for $i \neq j$. Thus we have $|h_i - h_j| < \frac{u}{2}$ which implies that $|t_i - t_j| < \frac{u}{2}$. But $t_j(x_i) = 0$ implies that $\frac{1}{2}u(x_i) < \frac{1}{2}u(x_i)$, a contradiction. Conversely, suppose that $X \setminus \bigcap Z[I]$ is a finite set, say $\{a_1, a_2, \dots, a_k\}$. By what we have already shown in the proof of Lemma 4.1, the function $\varphi : \mathbb{R}^k \to I \subseteq C_{m^I}(X)$, defined by $\varphi(b) = f_b$, for all $b \in \mathbb{R}^k$ is continuous. Now consider $S = \{f \in I : |f| \leq 1\}$. Clearly $B(0, I, 1) \subseteq S$, implies that $\operatorname{int}_{m^I} S \neq \emptyset$ and $\varphi(\prod_{i=1}^k [-1, 1]) = S$ implies that S is compact and the proof is complete. □

Proposition 4.3. *If I is an ideal in $C(X)$, then I is a locally compact subspace of $C_{m^I}(X)$ if and only if $X \setminus \bigcap Z[I]$ is finite.*

Proof. If I is locally compact, then by Proposition 2.2 and Theorem 4.2, $X \setminus \bigcap Z[I]$ is finite. On the other hand, whenever $X \setminus \bigcap Z[I]$ is finite, then by Lemma 4.1, I as a subspace of $C_{m^I}(X)$ is homeomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}$, so I is locally compact. □

By Lemma 3.3 and Theorem 4.2, the following result is evident. We note that whenever $f \in I \setminus C_\psi(X)$, then f is unbounded, by Lemma 3.3 and hence $X \setminus \bigcap Z[I]$ must be infinite.

Corollary 4.4. *If $I \not\subseteq C_\psi(X)$, then every compact subset of $C_{m^I}(X)$ has an empty interior.*

The following result is also an immediate consequence of our Theorem 4.2 and Proposition 2.1 in [1], see also Proposition 3.2 in [12] for more general case.

Corollary 4.5. *If I is an essential ideal in $C(X)$, then every compact subset of $C_{m^I}(X)$ has an empty interior.*

We conclude this section by the following proposition which investigates the compactness and Lindelöfness of ideals in $C_{m^I}(X)$. For an example of a Lindelöf ideal in $C_m(X)$ (which coincides with $C_{m^I}(X)$, where $I = C(X)$), see Example 4.7 in [2].

Proposition 4.6. *Let J be an ideal in $C(X)$.*

(a) *J is never compact in $C_{m^I}(X)$.*
(b) *If J is Lindelöf in $C_{m^I}(X)$, then $J \subseteq C_\psi(X)$.*

Proof. (a) Let J be compact and $u \in U^+(X)$. Since $J \subseteq \bigcup_{f\in J} B(f, I, u)$, there are $f_1, f_2, \ldots, f_n \in J$ such that $J \subseteq \bigcup_{i=1}^n B(f_i, I, u)$. Suppose that $x_0 \notin \bigcap_{f\in J} Z(f)$ and let $\alpha = \sup\{|f_1(x_0)| + u(x_0), \ldots, |f_n(x_0)| + u(x_0)\}$. Take $f \in J$ such that $f(x_0) = \alpha$ (if $g \in J$ with $g(x_0) \neq 0$, consider $f = \alpha \frac{g}{g(x_0)} \in J$). Thus $f \in B(f_k, I, u)$ for some $1 \leq k \leq n$. Hence $|f| < |f_k| + u$ implies that $\alpha = |f(x_0)| < f_k(x_0)| + u(x_0)$, a contradiction.

(b) Let $J \not\subseteq C_\psi(X)$. To prove that J is not Lindelöf, it is enough to show that every open cover of J is uncountable. Suppose that $J \subseteq \bigcup_{n=1}^\infty B(f_n, I, u_n)$, where $f_n \in C(X)$ and $u_n \in U^+(X)$, for all $n \in \mathbb{N}$. Since $J \not\subseteq C_\psi(X)$, there is an unbounded $f \in J$. Now using 1.20 in [5], there exists a copy of $\mathbb{N}$, say a sequence $\{x_n\}$ in X, C-embedded in X on which f is unbounded. Without loss of generality, we suppose that $|f(x_n)| > 1$, for all $n \in \mathbb{N}$. But $\{x_n\}$ is C-embedded, so a function $g \in C(X)$ exists such that $g(x_n) = |f(x_n)| + u(x_n)$. Now $fg \in J \subseteq \bigcup_{n=1}^\infty B(f_n, I, u_n)$ implies that $fg \in B(f_m, I, u_m)$ for some $m \in \mathbb{N}$. Therefore $|g(x_m)| < |f(x_m)g(x_m)| < |f_m(x_m)| + u_m(x_m)$, a contradiction. □

5. Characterizations of the Space X via Properties of Some Subspaces of $C_m(X)$

We devote this section to the special case $I = C(X)$ of m^I-topology on $C(X)$, i.e., to the m-topology on $C(X)$. In this section we investigate some relations between topological spaces X and $C_m(X)$. The set $U(X)$ of units, the set $D(X)$ of zerodivisors, the set $r(X)$ of regulars (nonzerodivisors) and ideals of $C(X)$ are important subspaces of $C_m(X)$. We show that some properties of these subspaces completely determine the space X. For example, we show that $U(X)$ is dense in $C_m(X)$ if and only if X is strongly zero-dimensional and $D(X)$ is closed in $C_m(X)$ if and only if X is an almost P-space. First we recall that a space X is strongly zero-dimensional if for every pair A, B of completely separated subsets of the space X, there exists an open-closed set G such that $A \subseteq G \subseteq X \setminus B$, see Theorem 6.2.5 in [4]. We also recall that a space X is called an almost P-space if every nonempty G_δ-set (zero-set) in X has a nonempty interior. Characterization of the space X for which $r(X)$ ($C_K(X)$) is dense (closed) in $C_m(X)$ is also given in this section.

Proposition 5.1. *$U(X)$ is dense in $C_m(X)$ if and only if X is strongly zero-dimensional.*

Proof. Let X be strongly zero-dimensional, $f \in C(X)$ and u be a positive unit in $C(X)$. Suppose that

$$G = \{x \in X : f(x) \geq \tfrac{1}{2}u(x)\}, \qquad H = \{x \in X : f(x) \leq -\tfrac{1}{2}u(x)\}.$$

Since G and H are two disjoint zero-sets and X is strongly zero-dimensional, there exists an open-closed set K in X such that $G \subseteq K \subseteq X \setminus H$. Now define

$$v(x) = \begin{cases} f(x) + \frac{1}{2}u(x) & x \in K \\ f(x) - \frac{1}{2}u(x) & x \notin K. \end{cases}$$

Clearly v is unit, in fact if $x \in K$, then $x \notin H$ and hence $f(x) > -\frac{1}{2}u(x)$, i.e., $v(x) = f(x) + \frac{1}{2}u(x) > 0$ and if $x \notin K$, then $x \notin G$, so $f(x) - \frac{1}{2}u(x) = v(x) < 0$. Moreover, $|f - v| = \frac{1}{2}u < u$, i.e., $U(X)$ is dense in $C_m(X)$. Conversely, let $U(X)$ be dense in $C_m(X)$ and Z_1 and Z_2 be two disjoint zero-sets. Suppose that $f \in C(X)$ such that $f(Z_1) = \{-1\}$ and $f(Z_2) = \{1\}$. Consider $u = \frac{1}{2}$, then there exists a unit $v \in B(f, \frac{1}{2})$, i.e., $|f - v| < \frac{1}{2}$. Let $K = \{x \in X : v(x) < 0\}$. Since v is unit, K is open-closed. Clearly $Z_1 \subseteq K \subseteq X \setminus Z_2$ which means that X is strongly zero-dimensional. □

Proposition 5.2. *The set $D(X)$ of zerodivisors of $C(X)$ is closed in $C_m(X)$ if and only if X is an almost P-space.*

Proof. It is enough to show that $\mathrm{cl}_m D(X) = C_m(X) \setminus U(X)$. Clearly $U(X)$ is open in $C_m(X)$ for, if $u \in U(X)$, then $B(u, \pi) \subseteq U(X)$, where $\pi = \frac{|u|}{2}$. In fact if $f \in B(u, \pi)$, then $|f - u| < \frac{|u|}{2}$ implies that $Z(f) = \emptyset$, i.e., $f \in U(X)$. Thus $C_m(X) \setminus U(X)$ is closed and hence $\mathrm{cl}_m D(X) \subseteq C_m(X) \setminus U(X)$. Now suppose that $f \in C_m(X) \setminus U(X)$ and π is positive unit. We show that $B(f, \pi) \cap D(X) \neq \emptyset$. Define

$$h(x) = \begin{cases} f(x) + \frac{1}{2}\pi(x) & f(x) \leq -\frac{1}{2}\pi(x) \\ 0 & |f(x)| < \frac{1}{2}\pi(x) \\ f(x) - \frac{1}{2}\pi(x) & |f(x)| \geq \frac{1}{2}\pi(x). \end{cases}$$

Clearly $h \in C(X)$ and $|f - h| < \pi$, i.e., $h \in B(f, \pi)$. On the other hand $G = \{x \in X : |f(x)| < \frac{1}{2}\pi(x)\}$ is a nonempty open set in X, for $\emptyset \neq Z(f) \subseteq G$. Since $G \subseteq Z(h)$, the interior of $Z(h)$ is nonempty and hence $h \in D(X)$, i.e., $B(f, \pi) \cap D(X) \neq \emptyset$. □

In the following proposition we characterize spaces X for which the subset $r(X)$ of $C(X)$ is dense in $C_m(X)$. This proposition shows that for space $X = \mathbb{R}$ and more generally for a perfectly normal space X, the set $r(X)$ is dense in $C_m(X)$. First we prove the following lemma.

Lemma 5.3. *Let A and B be two disjoint sets. A and B can be separated by disjoint cozero-sets whose union is dense if and only if there exists $g \in r(X)$ such that $A \subseteq \mathrm{pos}g$ and $B \subseteq \mathrm{neg}g$.*

Proof. If there is such $g \in r(X)$, then pos$g$ and neg$g$ are cozero-sets whose union is dense for, $\mathrm{int}_X Z(g) = \emptyset$. Conversely, suppose that A and B are separated by disjoint cozero-sets coz$h$ and coz$k$ respectively whose union is dense. Define

$$g(x) = \begin{cases} |h(x)| & x \in \mathrm{coz}h \\ 0 & x \in Z(h) \cap Z(k) \\ -|k(x)| & x \in \mathrm{coz}k. \end{cases}$$

Clearly $g \in C(X)$, $\mathrm{int}_X Z(g) = \emptyset$ (i.e. $g \in r(X)$), $A \subseteq \mathrm{pos}g$ and $B \subseteq \mathrm{neg}g$. □

Proposition 5.4. *$r(X)$ is dense in $C_m(X)$ if and only if disjoint zero-sets in X can be separated by disjoint cozero-sets whose union is dense in X.*

Proof. Suppose that $r(X)$ is dense in $C_m(X)$ and $Z(f) \cap Z(g) = \emptyset$. Consider $h \in C(X)$ such that $|h| \leq \alpha$, $\alpha > 0$ and $h(Z(f)) = \{\alpha\}$, $h(Z(g)) = \{-\alpha\}$. Since $r(X)$ is dense, there exists $k \in r(X) \cap B(h, \alpha)$. Hence $h - \alpha < k < h + \alpha$ and $\mathrm{int}_X Z(k) = \emptyset$. If $x \in Z(f)$, then $k(x) > h(x) - \alpha = \alpha - \alpha = 0$ and if $x \in Z(g)$, then $k(x) < h(x) + \alpha = -\alpha + \alpha = 0$, i.e., $Z(f) \subseteq \mathrm{pos}k$, $Z(g) \subseteq \mathrm{neg}k$. Now by our lemma, we are through.

Conversely, suppose that disjoint zero-sets can be separated by disjoint cozero-sets whose union is dense in X. Let $f \in C(X)$ and π be a positive unit in $C(X)$. By our lemma, there exists $g \in r(X)$ such that $\{x \in X : f(x) \geq \frac{\pi}{2}(x)\} \subseteq \mathrm{pos}g$ and $\{x \in X : f(x) \leq -\frac{\pi}{2}(x)\} \subseteq \mathrm{neg}g$ and we consider $|g| \leq \frac{\pi}{2}$. Now define $h = [(f + \frac{\pi}{2}) \wedge g] \vee (f - \frac{\pi}{2})$. Clearly $h \geq f - \frac{\pi}{2}$. Hence for each $x \in X$, either $h(x) = f(x) - \frac{\pi(x)}{2} \leq f(x) + \frac{\pi(x)}{2}$ or $h(x) = [f(x) + \frac{\pi(x)}{2}] \wedge g \leq f(x) + \frac{\pi(x)}{2}$. Therefore $f - \frac{\pi}{2} \leq h \leq f + \frac{\pi}{2}$ and hence $h \in B(f, \pi)$. On the other hand, if $h(x) = 0$, then $f(x) \neq \pm\frac{\pi(x)}{2}$. Whenever $f(x) = -\frac{\pi}{2}(x)$, then $g(x) < 0$, so $h(x) = g(x) \vee (f(x) - \frac{\pi}{2}(x)) = g(x) \vee -\pi(x) = g(x) < 0$ (note that $g(x) \geq -\frac{\pi}{2}(x)$). If $f(x) = \frac{\pi}{2}(x)$, then $g(x) > 0$, $f(x) + \frac{\pi}{2}(x) = \pi(x) > \frac{\pi}{2}(x) \geq g(x)$ and hence $h(x) = g(x) \vee (f(x) - \frac{\pi}{2}(x)) = g(x) \vee 0 = g(x) > 0$. Also, $f(x) < -\frac{\pi}{2}(x)$ and $f(x) > \frac{\pi}{2}(x)$ do not happen. In fact $f(x) < -\frac{\pi}{2}(x)$ implies $g(x) < 0$, hence $h(x) < 0$ and $f(x) > \frac{\pi}{2}(x)$ implies $g(x) > 0$, so $h(x) > 0$. Therefore $f(x) - \frac{\pi(x)}{2} < h(x) < f(x) + \frac{\pi(x)}{2}$ and this means that $g(x) = 0$. Consequently, $Z(h) \subseteq Z(g)$ and hence $\mathrm{int}_X Z(h) = \emptyset$, since $g \in r(X)$. This implies that $B(f, \pi) \cap r(X) \neq \emptyset$, i.e., $r(X)$ is dense in $C_m(X)$. □

In the following result, we observe that for any space X satisfying countable chain condition, i.e., for any space X with countable cellularity $c(X)$, the set $r(X)$ is also dense in $C_m(X)$. The smallest cardinal number $\mathfrak{a} \geq \aleph_0$ such that every family of pairwise disjoint nonempty open subsets of X has cardinality less than or equal to $\mathfrak{a}$, is called the cellularity of the space X and is denoted by $c(X)$. If $c(X) = \aleph_0$, we say X satisfies the countable chain condition.

Proposition 5.5. *If $c(X) = \aleph_0$, then $r(X)$ is dense in $C_m(X)$.*

Proof. Let $f \in C(X)$ and π be a positive unit in $C(X)$. For every $a \in (0, 1)$, we define $Z_a = \{x \in X : \frac{f}{\pi}(x) = a\}$. Clearly $Z_a \cap Z_b = \emptyset$, for all $a, b \in (0, 1)$ and $a \neq b$. Since $c(X) = \aleph_0$, then $\mathrm{int}_X Z_a = \emptyset$ for some $a \in (0, 1)$. Now we consider $h = f - a\pi$. Since $Z(h) = Z_a$, then $h \in r(X)$ and $|h - f| = a\pi < \pi$, i.e., $h \in B(f, \pi) \cap r(X)$. □

We conclude the paper with the following result which characterizes the space X for which the ideal $C_K(X)$ is closed in $C_m(X)$. We recall that a space X is called μ-compact if $C_K(X) = I(X) := \bigcap_{p \in \beta X \setminus X} M^p$, see [9] for more details of such spaces.

Proposition 5.6. *The ideal $C_K(X)$ is closed in $C_m(X)$ if and only if X is μ-compact.*

Proof. It is enough to show that $\mathrm{cl}_m C_K(X) = I(X)$. Since $C_K(X) = \bigcup_{p \in \beta X \setminus X} O^p$, we have $C_K(X) \subseteq \bigcup_{p \in \beta X \setminus X} M^p = I(X)$. But $I(X)$ is closed, so $\mathrm{cl}_m C_K(X) \subseteq I(X)$. Now suppose that $f \in I(X)$, then $\beta X \setminus X \subseteq \mathrm{cl}_{\beta X} Z(f)$. For every positive unit π in $C(X)$, we must show that $B(f, \pi) \cap C_K(X) \neq \emptyset$. Consider the function h defined in the proof of Proposition 5.2 and the zero-set $H = \{x \in X : |f(x)| \geq \frac{\pi(x)}{2}\}$, so $H = Z(g)$, for some $g \in C(X)$. Clearly $Z(f) \subseteq X \setminus Z(g) \subseteq Z(h)$, for if $f(x) = 0$, then $x \notin H$, hence $x \in X \setminus Z(g)$ and this implies that $|f(x)| < \frac{\pi(x)}{2}$, so $x \in Z(h)$. Now $\mathrm{cl}_{\beta X} Z(h)$ is a neighborhood of $\mathrm{cl}_{\beta X} Z(f)$ and we have $\beta X \setminus X \subseteq \mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z(h)$, therefore $h \in \bigcup_{p \in \beta X \setminus X} O^p = C_K(X)$. On the other hand $|f - h| < \pi$, i.e., $h \in B(f, \pi)$ which means that $h \in B(f, \pi) \cap C_K(X)$. □

References

[1] F. Azarpanah, Intersection of essential ideals in $C(X)$, Proc. Amer. Math. Soc. 125 (1997) 2149–2154.
[2] F. Azarpanah, F. Manshoor, R. Mohamadian, Connectedness and compactness in $C(X)$ with the m-topology and generalized m-topology, Topology Appl. 159 (2012) 3486–3493.
[3] E. van Douwen, Nonnormality or hereditary paracompactness of some spaces of real functions, Topology Appl. 39 (1991) 3–32.
[4] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[5] L. Gillman, M. Jerison, Rings of Continuous Functions, Springer, 1976.
[6] J. Gomez-Pérez, W.W. McGovern, The m-topology on $C_m(X)$ revisited, Topology Appl. 153 (2006) 1838–1848.
[7] M. Henriksen, B. Mitra, $C(X)$ can sometimes determine X without X being realcompact, Comment. Math. Unive. Carolina, 46(4) (2005) 711–720.
[8] E. Hewitt, Rings of real-valued continuous functions I, Trans. Amer. Math. Soc. 48(64) (1948) 54–99.
[9] D.G. Johnson, M. Mandelker, Functions with pseudocompact support, Gen. Topology Appl. 3 (1973) 331–338.
[10] G. Di Maio, L'. Hola, D. Holý, D. McCoy, Topology on the space of continuous functions, Topology Appl. 86 (1998) 105–122.
[11] M. Mandelker, Support of continuous functions, Trans. Amer. Math. Soc. 156 (1971) 73–83.
[12] A. Taherifar, Essential ideals in subrings of $C(X)$ that contain $C^*(X)$, Filomat 29 (2015) 1631–1637.