Filomat 31:8 (2017), 2267–2279
DOI 10.2298/FIL1708267P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Extremes of Gaussian Processes with a Smooth Random Trend

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Abstract. Let $\xi(t)$, $t \in \mathbf{R}$, be a Gaussian zero mean stationary process, and $\eta(t)$ another random process, smooth enough, being independent of $\xi(t)$. We will consider the process $\xi(t) + \eta(t)$ such that conditioned on $\eta(t)$ it is a Gaussian process. We want to establish an asymptotic exact result for

$$\mathbb{P}\left(\sup_{t\in[0,T]} (\xi(t) + \eta(t)) > u\right), \text{ as } u \to \infty,$$

where $T > 0$.

1. Introduction

For a random element (X, Y), the component Y is called a *random environment* and the component X a *random element in the random environment*, if the properties of X are the subject of primary interest, whereas the properties of Y can be of substantial influence on X (see [13]). The mathematical modelling of a random process in random environment is tightly connected with fundamental notions of conditional expectation and conditional probability.

Gaussian processes in a random environment i.e. Gaussian processes with random parameters (mean, covariance) are called conditionally Gaussian processes, and they significantly extend the class of processes whose extremes can be studied with those techniques applied to Gaussian processes. In order to calculate the exact asymptotic behavior of the probability of large extremes of conditionally Gaussian process it is necessary to calculate this probability using a fixed, non-random parameter, and then average the behavior over all the states of the parameter.

The asymptotic theory for large extremes of Gaussian processes and fields is well known, see [10]. In the paper [6] the authors show how the asymptotic theory for large extremes of conditionally Gaussian processes with a random variance is related to the corresponding theory for Gaussian processes. In this paper the authors considered the process $X(t) = \xi(t)\eta(t)$, where $\xi(t)$ is a Gaussian zero mean stationary process, and $\eta(t)$ another random process, smooth enough, being independent of $\xi(t)$, and presented asymptotic exact result for $\mathbb{P}\left(\sup_{t\in[0,T]} \xi(t)\eta(t) > u\right)$, as $u \to \infty$. Under the same conditions on $\eta(t)$ we will consider process $X(t) = \xi(t) + \eta(t)$.

2010 *Mathematics Subject Classification*. Primary 60G15, 60G60
Keywords. conditionally Gaussian process, Gaussian process, stationary random process, random expectation
Received: 20 October 2015; Accepted: 15 January 2016
Communicated by Svetlana Janković
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The paper [1] (as far as we are aware) started to investigate the extremes of conditionally Gaussian processes. In this paper it is considered the process $X(t) = A^{1/2} \cdot G(t)$, where $G(t)$ is a stationary Gaussian process and A is a stable random variable independent of $G(\cdot)$ and the paper dealt with the expected number of upcrossings of a large level u. To the best of knowledge the paper [6] started to develop an asymptotic theory for large extremes of conditionally Gaussian processes.

Applications of conditionally Gaussian processes can be seen in financial, optimization and control problems, as in [4, 7, 8].

2. Definitions, Auxiliary Results, Main Result

Let $(X(t), Y)$, $t \in \mathbf{R}$, be a random element, where $X(t)$ is a random process taking values in $\mathbf{R}$, and Y is another arbitrary random element.

Definition 2.1 The random process $X(t)$ is a *conditionally Gaussian process* if the conditional distribution of $X(\cdot)\,|\,Y$ is Gaussian.

Let $\xi(t)$, $t \in [0, T]$, $T > 0$, be a centered Gaussian stationary process, and $\eta(t)$ another random process, being independent of $\xi(t)$. Then, if we set $X(t) := \xi(t) + \eta(t)$ and $Y := \eta$, the process $X(t)$ is a conditionally Gaussian process, and

$$\mathsf{E}(\xi(t) + \eta(t)\,|\,\eta(t)) = \eta(t),$$

so motivated by the properties of $\eta(t)$ and [6] we will call the process $X(t)$ *Gaussian process with a smooth random trend.*

Write

$$\Psi(x) = \frac{1}{\sqrt{2\pi}x} \exp\left\{-\frac{x^2}{2}\right\},$$

and from now on H_α denotes Pickands' constant (see [9, 10]) and $\sigma(G) := \operatorname{ess\,sup}(G)$ for any random element G.

We will assume that the covariance function $r(t)$ of the stationary Gaussian process $\xi(t)$, with the expectation of zero, satisfies

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \text{ as } t \to 0, \tag{1}$$

for some $\alpha \in (0, 2]$, and

$$r(t) < 1, \text{ for all } t > 0. \tag{2}$$

We will assume that the process $\eta(t)$ satisfies conditions

η1. $\eta(t)$ is non-negative and $0 < \sigma < \infty$, where $\sigma := \sigma(\eta(t))$.

η2. For some $\varepsilon, \delta > 0$ there exists $\eta''(t)$ for all t with $(t, \eta(t)) \in K(\delta, \varepsilon) := [-\delta, T + \delta] \times [\sigma - \varepsilon, \sigma]$, and that

$$\sup_{(t,\eta(t))\in K(\delta,\varepsilon)} |\eta''(t)| \leqslant c,$$

for some constant c. Moreover, assume that for all t with $(t, \eta(t)) \in K(\delta, \varepsilon)$ $\eta''(t)$ is equicontinuous in the following sense

$$\omega(h) := \sup_{(t,\eta(t))\in K(\delta,\varepsilon)} \sup_{s\in[0,h]:(t+s,\eta(t+s))\in K(\delta,\varepsilon)} \sigma(|\eta''(t+s) - \eta''(t)|) \to 0, \text{ as } h \to 0.$$

η3. For some $\varepsilon, \delta > 0$ the vector $\mathbf{X}_t = (\eta(t), \eta'(t), \eta''(t))$ has a density $f_{\mathbf{X}_t}(x, y, z)$, $x \in [\sigma - \varepsilon, \sigma]$, which is bounded for any $t \in [-\delta, T + \delta]$.

η4. For some $\varepsilon, \delta, \kappa > 0$ almost surely $\eta''(t) \leqslant -\kappa$ for any $(t, x) \in K(\delta, \varepsilon)$ such that $\eta'(t) = 0$ and $\eta''(t) < 0$. Moreover, assume that the function

$$m(t,x) := \int_{-c}^{-\kappa} |z|^{1/2} f_{\eta'(t),\eta''(t)|\,\eta(t)=x}(0,z)dz$$

is continuous in $x = \sigma$ uniformly on t, with $\int_0^T m(t,\sigma)dt > 0$.

Under different assumptions on $\eta(t)$, the asymptotic behavior of the tail of the process $X(t)$ has been evaluated in [14]. In that paper it is assumed that process $\eta(t)$ is almost sure three times continuously differentiable on $\mathbf{R}$ with the property that $\sup_{t\in B}(|\eta(t)| + |\eta'''(t)|) \leqslant C(B)$ for any bounded $B \subset \mathbf{R}$ and for some non-random $C(B) < \infty$. In this paper we assume that the process $\eta(t)$ is two times differentiable and bounded near σ, that is in $K(\delta, \varepsilon)$ and that $\eta''(t)$ is equicontinuous i.e. we assume condition $\boldsymbol{\eta}\mathbf{2}$. Also in [14] it is assumed that the function $\widetilde{m}(t,x) = \mathsf{E}\left(|\eta''(t)|^{1/2} \,\middle|\, \eta(t) = x, \eta'(t) = 0\right)$ is continuous in $x = \sigma$ uniformly on t, with $\widetilde{m}(t,\sigma) > 0$ and $\lim_{x\to\sigma-} f_{\eta'(t)\,|\,\eta(t)=x}(0) = f_{\eta'(t)\,|\,\eta(t)=\sigma}(0) > 0$ uniformly on t. In this paper, we assume conditions $\boldsymbol{\eta}\mathbf{3}$ and $\boldsymbol{\eta}\mathbf{4}$.

We state here (with proofs) two lemmas that will be used in the proof of Theorem Theorem 2.4.

Lemma 2.2 *Let $g(x)$, $x \in [a, \sigma]$, be a bounded function, which is r times continuously differentiable in a neighborhood of σ, such that $g^{(i)}(\sigma) = 0$ for $i = 0, 1, \ldots, r-1$ and $g^{(r)}(\sigma) \neq 0$. Then*

$$\int_a^\sigma g(x)\Psi(u-x)dx = (-1)^r g^{(r)}(\sigma)u^{-1-r}\Psi(u-\sigma)(1+o(1)), \tag{3}$$

as $u \to \infty$. If $g(x) = g_1(x)g_2(x)$, where $g_1(x)$ is a bounded function, continuous from the left at σ with $g_1(\sigma) > 0$ and $g_2(x)$ satisfies the above conditions on g, one can change $g^{(r)}(\sigma)$ in (3) to $g_1(\sigma)g_2^{(r)}(\sigma)$.

Lemma 2.3 *Let $\xi(t)$, $t \in [-T, T]$, $T > 0$, be a stationary Gaussian process with the expectation of zero and with a covariance function $r(t)$ that satisfies (1) and (2) and let $a \in (0, +\infty)$ be constant. Then*

$$\mathbb{P}\left(\max_{t\in[-T,T]}\left(\xi(t) - \frac{at^2}{2}\right) > u\right) = \sqrt{2\pi}\, H_\alpha\, a^{-\frac{1}{2}}\, u^{\frac{2}{\alpha}-\frac{1}{2}}\, \Psi(u)(1+o(1)),\ \textit{as } u \to \infty.$$

Remark. Theorem 13.2.5 of [12] is more general result than Lemma 2.3 but we think that it is useful to have a self-contained proof of this lemma here. Let us mention that in the same paper the asymptotic behavior of the tail of non-centered locally $(\alpha_\mathbf{t}, D_\mathbf{t})-$stationary Gaussian field indexed on a smooth manifold is evaluated.

Our main result is the next theorem.

Theorem 2.4 *Let $\xi(t)$, $t \in [0, T]$, $T > 0$, be a stationary Gaussian process with the expectation of zero and with a covariance function $r(t)$ that satisfies (1) and (2) and let $\eta(t)$ be a process being independent of the process $\xi(t)$ that satisfies conditions $\boldsymbol{\eta}\mathbf{1} - \boldsymbol{\eta}\mathbf{4}$.*

Let for any fixed $t \in [0, T]$ the density function $f_{\eta(t)}(x)$ of the random variable $\eta(t)$ be r times continuously differentiable in a neighborhood of σ, with $f^{(i)}_{\eta(t)}(\sigma) = 0$, $i = 0, \ldots, r-1$, and $f^{(r)}_{\eta(t)}(\sigma) \neq 0$ for some $r \in \mathbf{Z}^+$.

Then

$$\mathbb{P}\left(\sup_{t\in[0,T]}(\xi(t)+\eta(t))>u\right)=(1+o(1))\sqrt{2\pi}\,H_\alpha\,u^{\frac{2}{\alpha}-\frac{3}{2}-r}\,\Psi(u-\sigma)\int_0^T(-1)^r f^{(r)}_{\eta(t)}(\sigma)\,m(t,\sigma)dt,$$

as $u\to\infty$.

3. Proofs

3.1. *Proof of Lemma 2.2*

If we make the substitution $x=\sigma-y$ we will get

$$\int_a^\sigma g(x)\Psi(u-x)dx=\int_0^{\sigma-a}g(\sigma-y)\Psi(u-\sigma+y)dy$$
$$=\Psi(u-\sigma)\int_0^{\sigma-a}\frac{u-\sigma}{u-\sigma+y}e^{\sigma y-\frac{y^2}{2}}\,g(\sigma-y)\,e^{-uy}dy.$$

We will estimate the right side integral from below and from above, and for both bounds we will derive asymptotic expansions using the idea from the proof of Watson's lemma (see [5]) .

By using the inequality $\frac{u-\sigma}{u-\sigma+y}\leqslant 1$ (here $y\in[0,\sigma-a]$), we find that the integral from the right side is not greater than

$$I:=\int_0^{\sigma-a}g(\sigma-y)\,e^{-(u-\sigma)y}dy.$$

By Taylor's formula

$$g(\sigma-y)=\frac{(-1)^r}{r!}g^{(r)}(\sigma)y^r+R_r(y).$$

There exists a positive constant C_r such that $|R_r(y)|\leqslant C_r y^{r+1}$ in a neighbourhood of 0.

Let

$$I_1:=\frac{(-1)^r}{r!}g^{(r)}(\sigma)\int_0^{\sigma-a}y^r\,e^{-(u-\sigma)y}dy$$

Now we have

$$\int_0^{\sigma-a}y^r\,e^{-(u-\sigma)y}dy=\int_0^{+\infty}y^r\,e^{-(u-\sigma)y}dy-\int_{\sigma-a}^{+\infty}y^r\,e^{-(u-\sigma)y}dy.$$

The first integral is equal

$$\int_0^{+\infty}y^r\,e^{-(u-\sigma)y}dy=(u-\sigma)^{-r-1}\Gamma(r+1)=(u-\sigma)^{-r-1}\,r!,$$

and for the second one we have

$$\int_{\sigma-a}^{+\infty}y^r\,e^{-(u-\sigma)y}dy\leqslant e^{-\frac{(u-\sigma)(\sigma-a)}{2}}\int_{\sigma-a}^{+\infty}y^r\,e^{-\frac{(u-\sigma)y}{2}}dy$$
$$<e^{-\frac{(u-\sigma)(\sigma-a)}{2}}\int_0^{+\infty}y^r\,e^{-y}dy$$
$$=e^{-\frac{(u-\sigma)(\sigma-a)}{2}}\Gamma(r+1),$$

so $\int_{\sigma-a}^{+\infty}y^r\,e^{-(u-\sigma)y}dy=O(e^{-cu})$, for some $c>0$, as $u\to\infty$.

It follows

$$I_1 = (-1)^r g^{(r)}(\sigma)(u-\sigma)^{-r-1} + O(e^{-cu}), \text{ as } u \to \infty.$$

Let

$$I_2 := \int_0^{\sigma-a} R_r(y)\, e^{-(u-\sigma)y} dy.$$

We have

$$|I_2| \leqslant C_r \int_0^{+\infty} y^{r+1}\, e^{-(u-\sigma)y} dy = C_r(u-\sigma)^{-r-2}(r+1)!.$$

Since $O(e^{-cu}) = O((u-\sigma)^{-r-2})$, as $u \to \infty$, for any $r \geqslant 0$, we get

$$I = (-1)^r g^{(r)}(\sigma)(u-\sigma)^{-1-r} + O(e^{-cu}), \text{ as } u \to \infty.$$

It follows

$$\int_a^{\sigma} g(x)\Psi(u-x)dx \leqslant (1+\nu(u))\Psi(u-\sigma)(-1)^r g^{(r)}(\sigma)u^{-1-r}\left(1-\frac{\sigma}{u}\right)^{-1-r},$$

where $\nu(u) \to 0$ as $u \to \infty$.

Now we will find the lower bound.

By using the inequalities $\frac{u-\sigma}{u-\sigma+y} \geqslant \frac{u-\sigma}{u-a}$ and $e^{\sigma y - \frac{y^2}{2}} \geqslant 1$ (here $y \in [0, \sigma - a]$) we have

$$\int_0^{\sigma-a} \frac{u-\sigma}{u-\sigma+y} e^{\sigma y - \frac{y^2}{2}} g(\sigma-y)\, e^{-uy} dy \geqslant \frac{u-\sigma}{u-a} \int_0^{\sigma-a} g(\sigma-y)\, e^{-uy} dy.$$

By applying the same idea from the above part of the proof to the integral $\int_0^{\sigma-a} g(\sigma-y)\, e^{-uy} dy$ we get

$$\int_a^{\sigma} g(x)\Psi(u-x)dx \geqslant (1-\nu_1(u))\Psi(u-\sigma)\frac{u-\sigma}{u-a}(-1)^r g^{(r)}(\sigma)u^{-1-r},$$

where $\nu_1(u) \to 0$ as $u \to \infty$. Finally, the result follows by using the fact that $\frac{u-\sigma}{u-a} = 1 + o(1)$ and $\left(1-\frac{\sigma}{u}\right)^{-1-r} = 1 + o(1)$ as $u \to \infty$, and the given bounds. □

3.2. *Proof of Lemma 2.3*

Let us denote

$$\Delta := u^{-\frac{2}{\alpha}} S,$$
$$\Delta_k := [k\Delta, (k+1)\Delta]\,, \quad S > 0,$$
$$A_k := \left\{ \sup_{t\in\Delta_k} \left(\xi(t) - \frac{at^2}{2}\right) > u \right\},$$
$$N_t := \left\lfloor \frac{t}{\Delta} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes an integer part of the real number x.

`Upper bound.`

We have

$$\mathbb{P}\left(\max_{t\in[-T,T]} \left(\xi(t) - \frac{at^2}{2}\right) > u\right) \leqslant \sum_{k=-N_T}^{N_T} \mathbb{P}(A_k) \leqslant \sum_{k=-N_T}^{N_T} \mathbb{P}\left(\max_{t\in\Delta_k} \xi(t) > u + \frac{a(k\Delta)^2}{2}\right)$$

$$= \sum_{k=-N_T}^{N_T} \mathbb{P}\left(\max_{t\in\Delta_0} \xi(t) > u + \frac{a(k\Delta)^2}{2}\right)$$

where the last equality follows by using the stationarity of the process $\xi(t)$.

By using Lemma D.1 in [10] or more precisely, a generalization of that lemma, Lemma 13.2.1 in [12], we find that for some function $\gamma(u) \to 0$ as $u \to \infty$

$$\mathbb{P}\left(\max_{t\in\Delta_0} \xi(t) > u + \frac{a(k\Delta)^2}{2}\right) \leqslant H_\alpha(S)\,\Psi\left(u + \frac{a(k\Delta)^2}{2}\right)(1+\gamma(u))$$
$$\leqslant H_\alpha(S)\,\Psi(u)\,\exp\left(-\frac{ua(k\Delta)^2}{2}\right)(1+\gamma(u)).$$

It follows that

$$\mathbb{P}\left(\max_{t\in[-T,T]}\left(\xi(t) - \frac{at^2}{2}\right) > u\right) \leqslant (1+\gamma(u))H_\alpha(S)\,\Psi(u) \sum_{k=-N_T}^{N_T} \exp\left(-\frac{ua(k\Delta)^2}{2}\right)$$
$$= (1+\gamma(u))\,\frac{H_\alpha(S)}{S}\,\Psi(u)\,u^{\frac{2}{\alpha}-\frac{1}{2}}\,a^{-\frac{1}{2}} \sum_{k=-N_T}^{N_T} \left(\Delta\sqrt{ua}\right)\exp\left(-\frac{ua(k\Delta)^2}{2}\right)$$
$$\leqslant (1+\gamma(u))\,\frac{H_\alpha(S)}{S}\,\Psi(u)\,u^{\frac{2}{\alpha}-\frac{1}{2}}\,a^{-\frac{1}{2}} \int_{-T\sqrt{ua}}^{T\sqrt{ua}} \exp\left(-\frac{t^2}{2}\right)dt.$$

Since

$$\int_{-T\sqrt{ua}}^{T\sqrt{ua}} \exp\left(-\frac{t^2}{2}\right)dt = (1+o(1))\sqrt{2\pi}, \text{ as } u \to \infty,$$

then by letting $S \to \infty$ we get

$$\mathbb{P}\left(\max_{t\in[-T,T]}\left(\xi(t) - \frac{at^2}{2}\right) > u\right) \leqslant (1+\gamma_1(u))\,\sqrt{2\pi}\,H_\alpha\,a^{-\frac{1}{2}}\,u^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(u), \tag{4}$$

for some function $\gamma_1(u)$ ($\gamma_1(u) \to 0$, as $u \to \infty$).

It follows

$$\limsup_{u\to\infty} \frac{\mathbb{P}\left(\max_{t\in[-T,T]}\left(\xi(t) - \frac{at^2}{2}\right) > u\right)}{u^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(u)} \leqslant \sqrt{2\pi}\,H_\alpha\,a^{-\frac{1}{2}}.$$

`Lower bound.`

By Bonferroni's inequality we have

$$\mathbb{P}\left(\max_{t\in[-T,T]}\left(\xi(t) - \frac{at^2}{2}\right) > u\right) \geqslant \sum_{k=-N_T}^{N_T-1} \mathbb{P}\left(\max_{t\in\Delta_k} \xi(t) > u + \frac{a((k+1)\Delta)^2}{2}\right) - \sum_{-N_T\leqslant i<j\leqslant N_T-1} \mathbb{P}\left(A_iA_j\right) \tag{5}$$

The first sum from the right side is bounded below by the same term as in (4) but with $1-\gamma_1(u)$ instead of $1+\gamma_1(u)$

$$\sum_{k=-N_T}^{N_T-1} \mathbb{P}\left(\max_{t\in\Delta_k} \xi(t) > u + \frac{a((k+1)\Delta)^2}{2}\right) \geqslant (1-\gamma_1(u))\,\sqrt{2\pi}\,H_\alpha\,a^{-\frac{1}{2}}\,u^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(u). \tag{6}$$

Now let us consider the double sum. We have

$$\mathbb{P}\left(A_iA_j\right) \leqslant \mathbb{P}\left(\max_{t\in\Delta_i}\xi(t) > u + \frac{a(i\Delta)^2}{2},\ \max_{t\in\Delta_j}\xi(t) > u + \frac{a(j\Delta)^2}{2}\right) \tag{7}$$

Let $i < j$ be such that the segments Δ_i and Δ_j are non-adjacent. Let us denote $k := j - i$. Note that in this case $k > 1$. Then, by stationarity and Lemma 6.3 in [10] (or the proof of Lemma 9.14 in [11]), $C_1 > 0$ exists such that for all u large enough

$$\begin{aligned}
&\mathbb{P}\left(\max_{t\in\Delta_i}\xi(t) > u + \frac{a(i\Delta)^2}{2},\ \max_{t\in\Delta_j}\xi(t) > u + \frac{a(j\Delta)^2}{2}\right)\\
&\leqslant \mathbb{P}\left(\max_{t\in\Delta_0}\xi(t) > u + \frac{a(i\Delta)^2}{2},\ \max_{t\in\Delta_k}\xi(t) > u + \frac{a(i\Delta)^2}{2}\right)\\
&\leqslant C_1\, S^2\, \Psi(u)\, \exp\left\{-\frac{ua(i\Delta)^2}{2}\right\}\, \exp\left\{-\frac{(k-1)^\alpha S^\alpha}{8}\right\}.
\end{aligned}$$

It follows

$$\begin{aligned}
\sum_{\substack{-N_T\leqslant i<j\leqslant N_T-1\\ j-i>1}} \mathbb{P}\left(A_iA_j\right) &\leqslant C_1\, S^2\, \Psi(u) \sum_{i=-N_T}^{N_T-1} \exp\left\{-\frac{ua(i\Delta)^2}{2}\right\} \sum_{k:=j-i>1} \exp\left\{-\frac{(k-1)^\alpha S^\alpha}{8}\right\}\\
&\leqslant C_2\, a^{-\frac{1}{2}}\, S\, \exp\left\{-\frac{S^\alpha}{8}\right\}\, u^{\frac{2}{\alpha}-\frac{1}{2}}\, \Psi(u),
\end{aligned} \tag{8}$$

for some constant $C_2 > 0$.

Let $i < j$ be such that the segments Δ_i and Δ_j are adjacent. In this case, $j = i + 1$. Then, by stationarity and the monotonicity of probabilities with respect to expanding sets

$$\begin{aligned}
&\mathbb{P}\left(\max_{t\in\Delta_i}\xi(t) > u + \frac{a(i\Delta)^2}{2},\ \max_{t\in\Delta_{i+1}}\xi(t) > u + \frac{a(j\Delta)^2}{2}\right)\\
&\leqslant \mathbb{P}\left(\max_{t\in\Delta_0}\xi(t) > u + \frac{a(i\Delta)^2}{2},\ \max_{t\in\Delta_1}\xi(t) > u + \frac{a(i\Delta)^2}{2}\right)\\
&\leqslant \mathbb{P}\left(\max_{t\in u^{-\frac{2}{\alpha}}[0,S]}\xi(t) > u + \frac{a(i\Delta)^2}{2},\ \max_{t\in u^{-\frac{2}{\alpha}}[S+\sqrt{S},2S]}\xi(t) > u + \frac{a(i\Delta)^2}{2}\right) + \mathbb{P}\left(\max_{t\in u^{-\frac{2}{\alpha}}[0,\sqrt{S}]}\xi(t) > u + \frac{a(i\Delta)^2}{2}\right)
\end{aligned} \tag{9}$$

By using Lemma 6.3 in [10] (or the proof of Lemma 9.14 in [11]), there exists $C_3 > 0$ such that for all u large enough

$$\begin{aligned}
&\sum_{i=-N_T}^{N_T-1} \mathbb{P}\left(\max_{t\in u^{-\frac{2}{\alpha}}[0,S]}\xi(t) > u + \frac{a(i\Delta)^2}{2},\ \max_{t\in u^{-\frac{2}{\alpha}}[S+\sqrt{S},2S]}\xi(t) > u + \frac{a(i\Delta)^2}{2}\right)\\
&\qquad\leqslant C_3\, a^{-\frac{1}{2}}\, S\, \exp\left\{-\frac{(\sqrt{S})^\alpha}{8}\right\}\, u^{\frac{2}{\alpha}-\frac{1}{2}}\, \Psi(u).
\end{aligned} \tag{10}$$

By using Lemma D.1 and by Lemma 6.8 in [10] there exists a constant G such that $H_\alpha(\sqrt{S}) \leqslant G\sqrt{S}$ we have

$$\sum_{i=-N_T}^{N_T-1} \mathbb{P}\left(\max_{t\in u^{-\frac{2}{\alpha}}[0,\sqrt{S}]}\xi(t) > u + \frac{a(i\Delta)^2}{2}\right) \leqslant \frac{C_4}{\sqrt{S}}\, a^{-\frac{1}{2}}\, u^{\frac{2}{\alpha}-\frac{1}{2}}\, \Psi(u). \tag{11}$$

for some constant $C_4 > 0$.

Finally, by using the relations (5)–(11) we get

$$\mathbb{P}\left(\max_{t\in[-T,T]}\left(\xi(t)-\frac{at^2}{2}\right)>u\right)\geqslant(1-\gamma_1(u))\ \sqrt{2\pi}\,H_\alpha\,a^{-\frac{1}{2}}\,u^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(u)-$$
$$-\left(C_2\,S\,\exp\left\{-\frac{S^\alpha}{8}\right\}+C_3\,S\,\exp\left\{-\frac{(\sqrt{S})^\alpha}{8}\right\}+\frac{C_4}{\sqrt{S}}\right)a^{-\frac{1}{2}}\,u^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(u).$$

It follows (by letting $S\to\infty$)

$$\liminf_{u\to\infty}\frac{\mathbb{P}\left(\max_{t\in[-T,T]}\left(\xi(t)-\frac{at^2}{2}\right)>u\right)}{u^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(u)}\geqslant\sqrt{2\pi}\,H_\alpha\,a^{-\frac{1}{2}}.\qquad\square$$

3.3. *Proof of Theorem 2.4*

Let t be a point of local maximum of $\eta(t)$ with $\eta(t)\geqslant\sigma-\varepsilon(u)$, where $0<\varepsilon(u)<\frac{\varepsilon}{2}$ and $\varepsilon(u)\to0$ as $u\to\infty$. Using Taylor's formula and the fact that $\eta'(t)=0$ we have

$$\eta(s)=\eta(t)+\frac{(s-t)^2}{2}\eta''(t+\theta(s-t)),\tag{12}$$

for some $\theta\in(0,1)$.

We will say that point s is *connected* with point t if

$$(y,\eta(y))\in K(\delta,\varepsilon),\ \text{for every } y\in I:=\{x\in\mathbf{R}:\ x=\lambda s+(1-\lambda)t,\ \text{for some } \lambda\in[0,1]\}.$$

Using the equation (12) and condition $\boldsymbol{\eta}\mathbf{2}$ we get for s connected with t

$$\eta(s)\geqslant\eta(t)-\frac{c}{2}(t-s)^2\geqslant\sigma-\varepsilon(u)-\frac{c}{2}(t-s)^2.$$

If $\sigma-\varepsilon(u)-\frac{c}{2}(t-s)^2\geqslant\sigma-\varepsilon$, then any s with $|s-t|\leqslant\sqrt{\frac{2}{c}(\varepsilon-\varepsilon(u))}$ is connected with t.

Let $h>0$ be such that s is connected with t with $|s-t|<h$ and $\omega(h)<\frac{\kappa}{2}$. Using mean value theorem for η' on I (if we assume that $s>t$) we have

$$\eta'(s)-\eta'(t)=\eta''(t+\theta(s-t))(s-t),\tag{13}$$

for some $\theta\in(0,1)$. (If $s<t$ we have $\eta'(t)-\eta'(s)=\eta''(t+\theta(t-s))(s-t)$, for some $\theta\in(0,1)$.) Then

$$\left|\eta''(t+\theta(s-t))-\eta''(t)\right|\leqslant\omega(h).\tag{14}$$

It follows that the right hand part of (13) is $\leqslant(\eta''(t)+\omega(h))(s-t)$ which is (by using the second part of condition $\boldsymbol{\eta}\mathbf{2}$) $\leqslant-\frac{\kappa}{2}(s-t)$. It follows that

$$|\eta'(s)|\geqslant\frac{\kappa}{2}|s-t|>0,$$

which means that there are no local maxima of η in $[t-h,t+h]$ other than t, with trajectories in $K(\delta,\varepsilon)$. So, all points of local maxima in $K(\delta,\varepsilon(u))$ are in distance at least $2h$.

Let $0<h^*<\min\left\{h,\delta,\sqrt{\frac{\varepsilon}{c}}\right\}$ and s a is point such that $|s-t|<h^*$. Then, by using (12) and (14) we have

$$\eta(t)+\frac{(s-t)^2}{2}(\eta''(t)-\omega(h^*))\leqslant\eta(s)\leqslant\eta(t)+\frac{(s-t)^2}{2}(\eta''(t)+\omega(h^*)).\tag{15}$$

Set (see [6])

$$D_u(\delta) := \{(x, y) \in K(\delta, \varepsilon(u)) : \ y = \eta(x)\}$$

consists of small "hats" with only one point of maximum for each "hat", and the points of maximum are separated by at least $2h^*$. If $(s, \eta(s)), (t, \eta(t)) \in D_u(\delta)$ belong to the same "hat" (where t is the point of local maximum of η), then

$$\sigma - \varepsilon(u) \leqslant \eta(s) \leqslant \eta(t) + \frac{(s-t)^2}{2}\left(-\kappa + \frac{\kappa}{2}\right) \leqslant \sigma - \frac{\kappa}{4}(s-t)^2,$$

hence $|s - t| \leqslant 2\sqrt{\frac{\varepsilon(u)}{\kappa}}$, so it follows that the width of the base of each "hat" is at most $2\delta(u) := 4\sqrt{\frac{\varepsilon(u)}{\kappa}}$.

Let s_1 be the first local maximum of η in $[0, T]$ (with $\eta(s_1) \geqslant \sigma - \varepsilon(u)$) and s_M the last one. We will introduce the random sets

$$L := [0, T] \cap \bigcup_{s \in \mathcal{M}(\varepsilon(u))} [s - \delta(u), s + \delta(u)],$$

$$L_+ := L \cup [0, s_1 \mathbf{1}_{A_1}] \cup [s_M \mathbf{1}_{A_M}, T\mathbf{1}_{A_M}],$$

where $\mathcal{M}(\varepsilon(u))$ is a set of local maximum points of the process $\eta(t)$ which are above $\sigma - \varepsilon(u)$, $A_1 = \{\eta(0) \geqslant \sigma - \varepsilon(u), \eta'(0) < 0\}$ and $A_M = \{\eta(T) \geqslant \sigma - \varepsilon(u), \eta'(T) > 0\}$.

If $t \in [0, T]\backslash L_+$ then $\eta(t) < \sigma - \varepsilon(u)$, so we have

$$\begin{aligned}
\mathbb{P}\left(\max_{t\in[0,T]\backslash L_+} (\xi(t) + \eta(t)) > u \,|\, \eta\right) &\leqslant \mathbb{P}\left(\max_{t\in[0,T]\backslash L_+} \xi(t) > u - (\sigma - \varepsilon(u))\right) \\
&\leqslant \mathbb{P}\left(\max_{t\in[0,T]} \xi(t) > u - (\sigma - \varepsilon(u))\right) \\
&= O\left(u^{\frac{2}{\alpha}} \Psi(u - (\sigma - \varepsilon(u)))\right), \text{ as } u \to \infty.
\end{aligned}$$

where the last equality follows from Theorem D.2 in [10].

By using the total probability rule and the previous inequality we have

$$\begin{aligned}
\mathbb{P}\left(\max_{t\in[0,T]} (\xi(t) + \eta(t)) > u\right) &= \mathsf{E}\left(\mathbb{P}\left(\max_{t\in[0,T]} (\xi(t) + \eta(t)) > u \,|\, \eta\right)\right) \\
&= \mathsf{E}\left(\mathbb{P}\left(\max_{t\in L_+} (\xi(t) + \eta(t)) > u \,|\, \eta\right)\right) + O\left(u^{\frac{2}{\alpha}} \Psi(u - (\sigma - \varepsilon(u)))\right).
\end{aligned}$$

Now we choose $\varepsilon(u) = \dfrac{\ell + \frac{2}{\alpha}}{u - \sigma} \cdot \ln u$, with a large positive ℓ ($> \frac{3}{2} + r - \frac{2}{\alpha}$), such that

$$u^{\frac{2}{\alpha}} \Psi(u - (\sigma - \varepsilon(u))) \sim u^{-\ell} \Psi(u - \sigma), \text{ as } u \to \infty.$$

It follows

$$\begin{aligned}
\mathbb{P}\left(\max_{t\in[0,T]} (\xi(t) + \eta(t)) > u\right) \leqslant\ & \mathsf{E}\left(\sum_{t\in\mathcal{M}(\varepsilon(u))\cap[0,T]} \mathbb{P}\left(\max_{s\in[t-h^*, t+h^*]} (\xi(s) + \eta(s)) > u \,|\, \eta\right)\right) \\
&+ \mathsf{E}\left(\mathbb{P}\left(\left(\max_{s\in[0, s_1 - h^*]} (\xi(s) + \eta(s)) > u\right) \cap A_1 \,|\, \eta\right)\right) \\
&+ \mathsf{E}\left(\mathbb{P}\left(\left(\max_{s\in[s_M + h^*, T]} (\xi(s) + \eta(s)) > u\right) \cap A_M \,|\, \eta\right)\right) \\
&+ O\left(u^{-\ell} \Psi(u - \sigma)\right).
\end{aligned}$$

If A_1 occurs then the largest negative point of local maximum of $\eta(t)$, is larger than $-\delta(u)$ by the structure of the set $D_u(\delta)$. If A_M occurs then the smallest point of local maximum of $\eta(t)$ after T, is smaller than $T+\delta(u)$ (the same observation as in [6]). It follows that

$$\mathbb{P}\left(\left(\max_{s\in[0,s_1-h^*]}(\xi(s)+\eta(s))>u\right)\cap A_1\,|\,\eta\right)\leqslant\sum_{t\in\mathcal{M}(\varepsilon(u))\cap[-\delta(u),0]}\mathbb{P}\left(\max_{s\in[t-h^*,t+h^*]}(\xi(s)+\eta(s))>u\,|\,\eta\right)$$

and

$$\mathbb{P}\left(\left(\max_{s\in[s_M+h^*,T]}(\xi(s)+\eta(s))>u\right)\cap A_1\,|\,\eta\right)\leqslant\sum_{t\in\mathcal{M}(\varepsilon(u))\cap[T,T+\delta(u)]}\mathbb{P}\left(\max_{s\in[t-h^*,t+h^*]}(\xi(s)+\eta(s))>u\,|\,\eta\right).$$

Now, if we set $\mathcal{M}:=\mathcal{M}(\varepsilon(u))\cap[-\delta(u),T+\delta(u)]$, we get

$$\mathbb{P}\left(\max_{t\in[0,T]}(\xi(t)+\eta(t))>u\right)\leqslant\mathsf{E}\left(\sum_{t\in\mathcal{M}}\mathbb{P}\left(\max_{s\in[t-h^*,t+h^*]}(\xi(s)+\eta(s))>u\,|\,\eta\right)\right)+O\left(u^{-\ell}\Psi(u-\sigma)\right).$$

From the right inequality (15) and Lemma 2.3 we get

$$\begin{aligned}
&\mathbb{P}\left(\max_{s\in[t-h^*,t+h^*]}(\xi(s)+\eta(s))>u\,|\,\eta\right)\\
&\leqslant\mathbb{P}\left(\max_{s\in[t-h^*,t+h^*]}\left(\xi(s)+\eta(t)+\frac{(s-t)^2}{2}(\eta''(t)+\omega(h^*))\right)>u\,|\,\eta\right)\\
&\leqslant\mathbb{P}\left(\max_{s\in[t-h^*,t+h^*]}\left(\xi(s)-\frac{(s-t)^2}{2}(-\eta''(t))\left(1-\frac{\omega(h^*)}{\kappa}\right)\right)>u-\eta(t)\,|\,\eta\right)\\
&\leqslant\sqrt{2\pi}\,H_\alpha\left(-\eta''(t)\left(1-\frac{\omega(h^*)}{\kappa}\right)\right)^{-\frac{1}{2}}u^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(u-\eta(t))\,(1+\gamma(u)),
\end{aligned}$$

where $\gamma(u)$ ($\downarrow 0$ as $u\to\infty$) can be chosen non-randomly. Indeed, if we set

$$\gamma^*(u,\eta(t),\eta''(t)):=\sup_{x\geqslant u}\left|\frac{\mathbb{P}\left(\max_{s\in[t-h^*,t+h^*]}\left(\xi(s)-\frac{(s-t)^2}{2}(-\eta''(t))\left(1-\frac{\omega(h^*)}{\kappa}\right)\right)>x-\eta(t)\,|\,\eta\right)}{\sqrt{2\pi}\,H_\alpha\left(-\eta''(t)\left(1-\frac{\omega(h^*)}{\kappa}\right)\right)^{-\frac{1}{2}}x^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(x-\eta(t))}-1\right|,$$

we get $\gamma^*(u,\eta(t),\eta''(t))\downarrow 0$ as $u\to\infty$, where $\eta(t)$ and $\eta''(t)$ are fixed. Since $\eta(t)\in[\sigma-\varepsilon,\sigma]$ and $|\eta''(t)|\in[\kappa,c]$ we can take $\gamma(u):=\gamma^*(u,\sigma,-\kappa)$.

Now let us consider the point process of local maxima $\{(t,\eta(t),\eta''(t)),\ t\in\mathcal{M}(\varepsilon(u))\}$ as a point process in $[-\delta(u),T+\delta(u)]\times[\sigma-\varepsilon(u),\sigma]\times[-c,-\kappa]$. Its intensity is

$$\nu(t,x,z)=|z|\,\mathbf{1}_{\{z<0\}}\,f_{\mathbf{X}_t}(x,0,z)$$

(see Chapter 3 in [2] for more details).

For any bounded function $F(t,x,z)$ we have (Campbell's Formula, see for instance Theorem 2.2 in [3])

$$\mathsf{E}\left(\sum_{\mathcal{M}(\varepsilon(u))\cap[0,T]}F(t,\eta(t),\eta''(t))\right)=\int_{-\delta(u)}^{T+\delta(u)}\int_{\sigma-\varepsilon(u)}^{\sigma}\int_{-c}^{-\kappa}F(t,x,z)\,\nu(t,x,z)dt\,dx\,dz.$$

It follows

$$\mathbb{P}\left(\max_{t\in[0,T]}(\xi(t)+\eta(t))>u\right)$$

$$\leqslant (1+\gamma(u))\sqrt{2\pi}\,H_\alpha\,u^{\frac{2}{\alpha}-\frac{1}{2}}\left(1-\frac{\omega(h^*)}{\kappa}\right)^{-\frac{1}{2}}\int_{-\delta(u)}^{T+\delta(u)}\int_{\sigma-\varepsilon(u)}^{\sigma}\int_{-c}^{-\kappa}|z|^{\frac{1}{2}}\Psi(u-x)\,f_{\mathbf{X}_t}(x,0,z)dtdxdz$$
$$+\,O\left(u^{-\ell}\Psi(u-\sigma)\right)$$
$$\leqslant (1+\gamma(u))\sqrt{2\pi}\,H_\alpha\,u^{\frac{2}{\alpha}-\frac{1}{2}}\left(1-\frac{\omega(h^*)}{\kappa}\right)^{-\frac{1}{2}}\int_{-\delta(u)}^{T+\delta(u)}\int_{\sigma-\varepsilon}^{\sigma}\int_{-c}^{-\kappa}|z|^{\frac{1}{2}}\Psi(u-x)\,f_{\mathbf{X}_t}(x,0,z)dtdxdz$$
$$+\,O\left(u^{-\ell}\Psi(u-\sigma)\right).$$

By using equality

$$f_{\mathbf{X}_t}(x,0,z)=f_{\eta(t)}(x)f_{\eta'(t),\eta''(t)\,|\,\eta(t)=x}(0,z)$$

and Lemma 2.2 ($g_1(x)=m(t,x)$ and $g_2(x)=f_{\eta(t)}(x)$) we derive the bound

$$\mathbb{P}\left(\max_{t\in[0,T]}(\xi(t)+\eta(t))>u\right)$$
$$\leqslant (1+\gamma(u)+\gamma_1(u))\sqrt{2\pi}\,H_\alpha\,u^{\frac{2}{\alpha}-\frac{3}{2}-r}\left(1-\frac{\omega(h^*)}{\kappa}\right)^{-\frac{1}{2}}\Psi(u-\sigma)\int_{-\delta(u)}^{T+\delta(u)}(-1)^r f_{\eta(t)}^{(r)}(\sigma)\,m(t,\sigma)dt$$
$$+\,O\left(u^{-\ell}\Psi(u-\sigma)\right),$$

where $\gamma_1(u)\to 0$ as $u\to\infty$.

Finally we have

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left(\max_{t\in[0,T]}(\xi(t)+\eta(t))>u\right)}{u^{\frac{2}{\alpha}-\frac{3}{2}-r}\Psi(u-\sigma)}\to\sqrt{2\pi}\,H_\alpha\int_0^T(-1)^r f_{\eta(t)}^{(r)}(\sigma)\,m(t,\sigma)dt,$$

as $h^*\to 0$.

`Lower bound.` If $(s,\eta(s)),(t,\eta(t))\in D_u(\delta)$ and t and s are points of local maximum of η, then $|t-s|\geqslant 2h^*$. It implies that there are at most $\lfloor\frac{T}{2h^*}\rfloor$ points of such local maximum in the $[0,T]$. By setting $\mathcal{M}_1:=\mathcal{M}(\varepsilon(u))\cap[\delta(u),T-\delta(u)]$ we have

$$\mathsf{E}\left(\mathbb{P}\left(\max_{t\in[0,T]}(\xi(t)+\eta(t))>u\,|\,\eta\right)\right)\geqslant\mathsf{E}\left(\mathbb{P}\left(\bigcup_{t\in\mathcal{M}_1}\left\{\max_{s\in[t-h^*/2,t+h^*/2]}(\xi(s)+\eta(s))>u\right\}\,\Big|\,\eta\right)\right)$$
$$\geqslant\mathsf{E}\left(\sum_{t\in\mathcal{M}_1}\mathbb{P}\left(\max_{s\in[t-h^*/2,t+h^*/2]}(\xi(s)+\eta(s))>u\,|\,\eta\right)\right)$$
$$-\,\mathsf{E}\left(\sum_{\substack{s,t\in\mathcal{M}_1\\ s\neq t}}\mathbb{P}\left(\max_{v\in[t-h^*/2,t+h^*/2]}(\xi(v)+\eta(v))>u,\ \max_{v\in[s-h^*/2,s+h^*/2]}(\xi(v)+\eta(v))>u\,|\,\eta\right)\right).\qquad(16)$$

By using the left inequality (15), and Lemma 2.3, we get

$$\mathbb{P}\left(\max_{s\in[t-h^*/2,t+h^*/2]}(\xi(s)+\eta(s))>u\,|\,\eta\right)$$
$$\geqslant\mathbb{P}\left(\max_{s\in[t-h^*/2,t+h^*/2]}\left(\xi(s)+\eta(t)+\frac{(s-t)^2}{2}(\eta''(t)-\omega(h^*))\right)>u\,|\,\eta\right)$$

$$\geqslant \mathbb{P}\left(\max_{s\in[t-h^*/2,t+h^*/2]}\left(\xi(s)-\frac{(s-t)^2}{2}\,(-\eta''(t))\left(1+\frac{\omega(h^*)}{\kappa}\right)\right)>u-\eta(t)\,|\,\eta\right)$$

$$\geqslant \sqrt{2\pi}\,H_\alpha\left(-\eta''(t)\left(1+\frac{\omega(h^*)}{\kappa}\right)\right)^{-\frac{1}{2}}u^{\frac{2}{\alpha}-\frac{1}{2}}\,\Psi(u-\eta(t))\,(1-\nu(u)),$$

where $\nu(u)$ ($\to 0$ as $u\to\infty$) can be chosen non-randomly. Now, by using the arguments for the upper bound we get

$$\liminf_{u\to\infty}\frac{\mathsf{E}\left(\sum_{t\in\mathcal{M}_1}\mathbb{P}\left(\max_{s\in[t-h^*/2,t+h^*/2]}(\xi(s)+\eta(s))>u\,|\,\eta\right)\right)}{u^{\frac{2}{\alpha}-\frac{3}{2}-r}\Psi(u-\sigma)}\to\sqrt{2\pi}\,H_\alpha\int_0^T(-1)^r f^{(r)}_{\eta(t)}(\sigma)\,m(t,\sigma)dt,$$

as $h^*\to 0$.

The double sum in (16) can be estimated from above by using Borel's theorem (see Theorem D.1 in [10]). The distances between the considered intervals are at least h^*.

$$\mathbb{P}\left(\max_{v\in[t-h^*/2,t+h^*/2]}(\xi(v)+\eta(v))>u,\ \max_{v\in[s-h^*/2,s+h^*/2]}(\xi(v)+\eta(v))>u\,|\,\eta\right)$$

$$\leqslant \mathbb{P}\left(\max_{(v_1,v_2)\in[t-h^*/2,t+h^*/2]\times[s-h^*/2,s+h^*/2]}(\xi(v_1)+\xi(v_2))>2(u-\sigma)\right).$$

For $(v_1,v_2)\in[t-h^*/2,t+h^*/2]\times[s-h^*/2,s+h^*/2]$ we have

$$\mathrm{Var}(\xi(v_1)+\xi(v_2))=2+2r(|v_1-v_2|)\leqslant 4-2\min_{|v_1-v_2|\geqslant h^*}(1-r(|v_1-v_2|))<4.$$

There exists constant $a>0$ satisfying

$$\mathbb{P}\left(\max_{(v_1,v_2)\in[t-h^*/2,t+h^*/2]\times[s-h^*/2,s+h^*/2]}(\xi(v_1)+\xi(v_2))>a\right)\leqslant\mathbb{P}\left(\max_{(v_1,v_2)\in[0,T]^2}(\xi(v_1)+\xi(v_2))>a\right)$$

$$=\mathbb{P}\left(\max_{v\in[0,T]}\xi(v)>\frac{a}{2}\right)\leqslant\frac{1}{2}.$$

Therefore by Borel's theorem

$$\mathbb{P}\left(\max_{(v_1,v_2)\in[t-h^*/2,t+h^*/2]\times[s-h^*/2,s+h^*/2]}(\xi(v_1)+\xi(v_2))>2(u-\sigma)\right)$$

$$\leqslant 2\Psi\left(\frac{2(u-\sigma)-a}{\sqrt{4-b}}\right)$$

$$=o\left(u^{\frac{2}{\alpha}-\frac{3}{2}-r}\Psi(u-\sigma)\right)\text{ as }u\to\infty,$$

where $b:=2\min_{|v_1-v_2|\geqslant h^*}(1-r(|v_1-v_2|))>0$.

Thus, we get

$$\liminf_{u\to\infty}\frac{\mathbb{P}\left(\max_{t\in[0,T]}(\xi(t)+\eta(t))>u\right)}{u^{\frac{2}{\alpha}-\frac{3}{2}-r}\Psi(u-\sigma)}\to\sqrt{2\pi}\,H_\alpha\int_0^T(-1)^r f^{(r)}_{\eta(t)}(\sigma)\,m(t,\sigma)dt,$$

as $h^*\to 0$. □

Acknowledgements

We are very grateful to the Reviewers of the first version of this paper for their valuable comments which substantially helped to improve the text.

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