Filomat 31:9 (2017), 2779–2785
https://doi.org/10.2298/FIL1709779H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Pointwise Topological Convergence and Topological Graph Convergence of Set-Valued Maps

Ľ. Holá[a], G. Kwiecińska[b]

[a]Academy of Sciences, Institute of Mathematics Štefánikova 49, 81473 Bratislava, Slovakia
[b]Pomeranian Akademy in Słupsk, Institute of Mathematics Arciszewskego 22 d, 76 200 Slupsk, Poland

Abstract. Let X, Y be topological spaces and $\{F_n : n \in \omega\}$ be a sequence of set-valued maps from X to Y with the pointwise topological limit G and with the topological graph limit F. We give an answer to the question from ([19]): which conditions on X, Y and/or $\{F, G, F_n : n \in \omega\}$ are needed to $F = G$.

1. Introduction

The topological (Painlevè-Kuratowski) convergence of graphs of set-valued maps was studied in many books and papers (see for example ([1]), ([2]), ([5]), ([8]), ([9]), ([19]), ([26]). In the books ([1]), ([2]), ([26]) we can find many applications of this convergence to variational and optimization problems, differential equations and approximation theory. We will call this convergence topological graph convergence of set-valued maps. Topological graph convergence of preference relations is used also in mathematical economics ([3]).

In our paper we will be interested in pointwise topological convergence and in topological graph convergence of set-valued maps. Our paper is motivated by the question of S. Kowalczyk in ([19]):

Let X, Y be topological spaces and $\{F_n : n \in \omega\}$ be a sequence of set-valued maps from X to Y with the pointwise topological limit G and with the topological graph limit F. Which conditions on X, Y and/or $\{F, G, F_n : n \in \omega\}$ are needed to ensure $F = G$. The main result of our paper is the following one:

Theorem 1.1. *Let X be a Baire topological space and let Y be a regular T_1 locally countably compact space. Let $\{F, F_n : n \in \omega\}$ be lower quasicontinuous set-valued maps from X to Y. Suppose $\{F_n : n \in \omega\}$ is topologically graph convergent to F and $\{F_n : n \in \omega\}$ is pointwise topologically convergent to a second set-valued function G with closed graph. Then $F = G$.*

Our Theorem 1.1 generalizes Theorem 5 from ([19]) which is stated for locally compact Hausdorff spaces X and Y and for lower semicontinuous set-valued maps.

Notice that the pointwise and graph upper (Painlevè-Kuratowski) limits of a sequence of lower quasicontinuous set-valued maps were also studied by M. Matejdes in ([22]).

2010 *Mathematics Subject Classification*. Primary 54C05; Secondary 54C08, 54C60
Keywords. quasicontinuity, lower quasicontinuous set-valued map, pointwise topological convergence, topological graph convergence, usco map, minimal usco map.
Received: 19 October 2015; Revised: 08 April 2016; Accepted: 01 June 2016
Communicated by Ljubiša D.R. Kočinac
Ľ. Holá would like to thank to grant Vega 2/0018/13 and APVV-0269-11
Email addresses: lubica.hola@mat.savba.sk (Ľ. Holá), grazyna.kwiecinska@apsl.edu.pl (G. Kwiecińska)

2. Definitions and Preliminaries

Let Z be a topological space. Let $\{C_n : n \in \omega\}$ be a sequence of nonempty subsets of Z. The lower limit LiC_n and the upper limit LsC_n of $\{C_n : n \in \omega\}$ are defined as follows (see ([21]): LiC_n (resp. LsC_n) is the set of all points $z \in Z$ each neighbourhood of which meets all but finitely (resp. infinitely many) sets C_n. We say that $\{C_n : n \in \omega\}$ topologically converges to a set C if $LiC_n = LsC_n = C$ and we denote it by $LtC_n = C$.

In what follows let X, Y be T_1 topological spaces. By a set-valued map from X to Y we mean a map which assigns to every point of X a nonempty subset of Y. If F is a set-valued map from X to Y, we denote it by $F : X \rightsquigarrow Y$.

A sequence $\{F_n : n \in \omega\}$ ($F_n : X \rightsquigarrow Y, n \in \omega$) pointwise topologically converges to $F : X \rightsquigarrow Y$ iff $LtF_n(x) = F(x)$ for every $x \in X$.

If $F : X \rightsquigarrow Y$, by $Gr(F)$ we denote the graph of F, i.e.

$$Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

A sequence $\{F_n : n \in \omega\}$ ($F_n : X \rightsquigarrow Y, n \in \omega$) topologically graph converges to $F : X \rightsquigarrow Y$ iff $LtGr(F_n) = Gr(F)$.

In the paper ([18]) Kempisty introduced a notion of quasicontinuity for real-valued functions defined in R. For general topological spaces this notion can be given the following equivalent formulation ([23]).

Definition 2.1. A function $f : X \to Y$ is called quasicontinuous at $x \in X$ if for every open set $V \subset Y$, $f(x) \in V$ and open set $U \subset X$, $x \in U$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

Notice that the topological graph convergence of continuous and quasicontinuous functions was studied in ([5]) and ([6]).

Easy examples show that in the context of metric spaces, pointwise (topological) convergence of a sequence of continuous functions does not ensure topological graph convergence, and topological graph convergence does not ensure pointwise convergence. However, if both limits exist for a sequence of functions as single-valued functions themselves, then they must coincide.

The notion of lower quasicontinuity (upper quasicontinuity) for set-valued maps was introduced in ([23]). First we will mention the notion of lower (upper) semicontinuity for set-valued maps.

A set-valued map $F : X \rightsquigarrow Y$ is lower (upper) semicontinuous at a point $x \in X$, if for every open set V such that $F(x) \cap V \neq \emptyset$ ($F(x) \subset V$), there exists an open neighbourhood U of x such that

$$F(z) \cap V \neq \emptyset \text{ for every } z \in U \ (F(U) = \cup\{F(u) : u \in U\} \subset V).$$

F is (lower) upper semicontinuous if it is (lower) upper semicontinuous at each point of X.

A set-valued map $F : X \rightsquigarrow Y$ is lower (upper) quasicontinuous at a point $x \in X$, if for every open set V in Y with $F(x) \cap V \neq \emptyset$ ($F(x) \subset V$) and every neighbourhood U of x there is a nonempty open set $G \subset U$ such that

$$F(z) \cap V \neq \emptyset \ (F(z) \subset V) \text{ for every } z \in G.$$

A set-valued map $F : X \rightsquigarrow Y$ is lower (upper) quasicontinuous if it is lower (upper) quasicontinuous at each point of X.

We will mention some important examples of lower quasicontinuous set-valued maps.

Lemma 2.2. *Let X, Y be topological spaces and $f : X \to Y$ be a quasicontinuous function. Then $\overline{Gr(f)}$ is the graph of a lower quasicontinuous set-valued map.*

The above Lemma in conjuction with Theorem 2.1 below show that every minimal usco map with values in a regular T_1-space is lower quasicontinuous.

Following Christensen ([12]) we say that a set-valued mapping F is usco if it is upper semicontinuous and takes nonempty compact values. Finally, a set-valued mapping F is said to be minimal usco ([10]) if it is a minimal element in the family of all usco maps (with domain X and range Y); that is if it is usco and does not contain properly any other usco map.

A very useful characterization of minimal usco maps using quasicontinuous subcontinuous selections was given in ([15]) and it will be important also for our analysis.

A function $f : X \to Y$ is subcontinuous at $x \in X$ ([11]) if for every net $\{x_i : i \in I\}$ (I is a directed set) convergent to x, there is a convergent subnet of $\{f(x_i) : i \in I\}$. If f is subcontinuous at every $x \in X$, we say that f is subcontinuous.

Theorem 2.3. *Let X, Y be topological spaces and Y be a regular T_1-space. Let $F : X \rightsquigarrow Y$ be a set-valued map. The following are equivalent:*

(1) *F is a minimal usco map;*

(2) *Every selection f of F is quasicontinuous, subcontinuous and $\overline{Gr(f)} = Gr(F)$;*

(3) *There exists a quasicontinuous, subcontinuous selection f of F with $\overline{Gr(f)} = Gr(F)$.*

Minimal usco maps are a very convenient tool in functional analysis, in optimization, in selection theorems, in the study of differentiability of Lipschitz functions ([16]).

3. Main Results

In the main result of our paper we will use Oxtoby's characterization of Baire spaces. In ([13]), ([17]), ([27]) we can find the following definition of the Choquet game and a characterization of Baire spaces using the Choquet game proved by Oxtoby in ([25]).

Definition 3.1. Let X be a nonempty topological space. The Choquet game G_X of X is defined as follows: Players I and II take turns in playing nonempty open subsets of X

$$\text{I} \ldots\ U_0 \ldots U_1$$

$$\text{II} \ldots\ V_0 \ldots V_1$$

so that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1$.... We say that II wins this run of the game if $\bigcap_n V_n (= \bigcap_n U_n) \neq \emptyset$. (Thus I wins if $\bigcap_n U_n (= \bigcap_n V_n) = \emptyset$.)

A strategy for I in this game is a "rule" that tells him how to play, for each n, his nth move U_n, given II's previous moves $V_0, ..., V_{n-1}$. Formally, this is defined as follows: Let T be the tree of legal positions in the Choquet game G_x, i.e. consists of all finite sequences $(W_0, ..., W_n)$, where W_i are nonempty open subsets of X and $W_0 \supseteq W_1 \supseteq ... \supseteq W_n$. A strategy for I in G_X is a subtree $\sigma \subset T$ such that

1) $\sigma \neq \emptyset$;

2) if $(U_0, V_0, ..., U_n) \in \sigma$, then for all open nonempty $V_n \subseteq U_n$, $(U_0, V_0, ..., U_n, V_n) \in \sigma$;

3) if $(U_0, V_0, ...U_{n-1}, V_{n-1}) \in \sigma$, then for a unique U_n, $(U_0, V_0, ..., U_{n-1}, V_{n-1}, U_n) \in \sigma$.

Intuitevely, the strategy σ works as follows: I starts playing U_0 where $(U_0) \in \sigma$ (and this is unique by 3); II then plays any nonempty open $V_0 \subseteq U_0$; by 2) $(U_0, V_0) \in \sigma$. Then I responds by playing the unique nonempty open $U_1 \subseteq V_0$ such that $(U_0, V_0, U_1) \in \sigma$, etc.

A position $(W_0, ..., W_n) \in T$ is compatible with σ if $(W_0, ..., W_n) \in \sigma$. A run of the game $(U_0, V_0, U_1, V_1, ...)$ is compatible with σ if for every $n \in \omega$ we have

$$(U_0, V_0, ..., U_{n-1}, V_{n-1}, U_n) \in \sigma \text{ and } (U_0, V_0, ..., U_n, V_n) \in \sigma.$$

The strategy σ is a winning strategy for I if he wins every compatible with σ run $(U_0, V_0, ...)$(i.e., if $(U_0, V_0, ...)$ is a run compatible with σ, then $\bigcap_n U_n = \bigcap_n V_n = \emptyset$).

The corresponding notions of strategy and winning strategy for II are defined mutatis mutandis.

Theorem 3.2. ([25]) A nonempty topological space X is a Baire space if and only if player I has no winning strategy in the Choquet game G_X.

Proof of Theorem 1.1.

Proof. Clearly $Gr(G) \subset Gr(F)$. Let us assume that $Gr(F) \not\subseteq Gr(G)$. Let $(x, y) \in Gr(F) \setminus Gr(G)$. There are open sets $U \subset X$, $V \subset Y$, such that $x \in U, y \in V$, $\overline{V}$ is countably compact and

$$(U \times \overline{V}) \cap Gr(G) = \emptyset. \tag{1}$$

The lower quasicontinuity of F at x implies that there is a nonempty open set $H \subset U$ with

$$F(z) \cap V \neq \emptyset \text{ for every } z \in H. \tag{2}$$

We will define the following strategy σ for the first player I in the Choquet game: Since $LtGr(F_n) = Gr(F)$, there is $n_0 \geq 1$ such that $Gr(F_{n_0}) \cap (H \times V) \neq \emptyset$. Let $(x_{n_0}, y_{n_0}) \in Gr(F_{n_0}) \cap (H \times V)$. The lower quasicontinuity of F_{n_0} at x_{n_0} implies that there is a nonempty open set $H_{n_0} \subset H$ such that $F_{n_0}(z) \cap V \neq \emptyset$ for every $z \in H_{n_0}$.

Define the first move U_0 of I as follows: $U_0 = H_{n_0}$.

If $(U_0, V_0) \in \sigma$, we will define U_1. Since $V_0 \subset U_0 \subset H$, for every $z \in V_0$ we have $F(z) \cap V \neq \emptyset$. There is $n_1 > max\{1, n_0\}$ such that

$$Gr(F_{n_1}) \cap (V_0 \times V) \neq \emptyset.$$

Let $(x_{n_1}, y_{n_1}) \in Gr(F_{n_1}) \cap (V_0 \times V)$. There is a nonempty open set $H_{n_1} \subset V_0$ such that $F_{n_1}(z) \cap V \neq \emptyset$ for every $z \in H_{n_1}$. Define the second move U_1 of I as follows: $U_1 = H_{n_1}$.

Suppose now that $(U_0, V_0, U_1, V_1, ...U_{k-1}, V_{k-1}) \in \sigma$, where $U_i = H_{n_i}$, $n_0 < n_1 < ...n_{k-1}$ and $n_i > i$ for every $i \leq k-1$. We will define U_k. Since $V_{k-1} \subset H$, $F(z) \cap V \neq \emptyset$ for every $z \in V_{k-1}$, by (2). There is $n_k > \max\{n_{k-1}, k\}$ such that

$$Gr(F_{n_k}) \cap (V_{k-1} \times V) \neq \emptyset.$$

Let $(x_{n_k}, y_{n_k}) \in Gr(F_{n_k}) \cap (V_{k-1} \times V)$. There is a nonempty open set $H_{n_k} \subset V_{k-1}$ such that $F_{n_k}(z) \cap V \neq \emptyset$ for every $z \in H_{n_k}$. Define $U_k = H_{n_k}$.

Since X is a Baire space, there is no winning strategy for the first player I. Thus, for an appropriate choice of $V_0, V_1, ..., V_n, ..., \bigcap_n U_n \neq \emptyset$. Let $p \in \bigcap_n U_n$.

For every $k \in \omega$ let $s_{n_k} \in F_{n_k}(p) \cap V$. The countable compactness of $\overline{V}$ implies that there is a cluster point y_0 of the sequence $\{s_{n_k} : k \in \omega\}$. Then $y_0 \in LtF_n(p) = G(p)$, thus $(p, y_0) \in Gr(G)$, which contradicts (1). □

Notice that the above theorem generalizes Theorem 5 from ([19]).

The following Theorem shows that the Baireness of X in Theorem 1.1 is necessary.

Theorem 3.3. *If X is not a Baire space, then for every T_1 topological space Y with at least two different points, there are lower quasicontinuous set-valued maps $\{F, G, F_n : n \in \omega\}$ from X to Y such that $LtF_n(x) = G(x)$ for every $x \in X$, G has a closed graph, $LtGr(F_n) = Gr(F)$ and $F \neq G$.*

Proof. There is a nonempty open set O in X which is of the first Baire category. Let $\{K_n : n \in \omega\}$ be a sequence of subsets of O such that $\overline{K_n} \cap O$ is nowhere dense in O for every $n \in \omega$ and $O = \bigcup_{n\in\omega} \overline{K_n} \cap O$. For every $n \in \omega$ we put

$$U_n = O \setminus \bigcup_{i\leq n} \overline{K_i}.$$

Then each set U_n is open and dense in O. For every $n \in \omega$ let $F_n : X \rightsquigarrow Y$ be a lower semicontinuous set-valued map defined as follows:

$$F_n(x) = \begin{cases} A & \text{if } x \in U_n, \\ B & \text{if } x \notin U_n, \end{cases}$$

where A and B are two closed and different subsets of Y such that $B \subset A$. Then $LtF_n(x) = G(x)$ for every $x \in X$, where G is a set-valued map identically equal to B.

Note that $LtGr(F_n) = Gr(F)$, where F is a set-valued map defined as follows:

$$F(x) = \begin{cases} A & \text{if } x \in \overline{O}, \\ B & \text{if } x \notin \overline{O}. \end{cases}$$

Moreover F is a lower quasi-continuous set-valued map. □

For single-valued functions there was in ([5]) a Baire category result that says that if X is a complete metric space and Y is any metric space and $\{f_n : n \in \omega\}$ topologically graph converges to f, $\{f, f_n : n \in \omega\}$ are continuous functions from X to Y, then there exists a G_δ-set A such that for each $x \in A$, $f(x)$ is a subsequential limit of $\{f_n(x) : n \in \omega\}$. S. Kowalczyk showed in ([19]) that for set-valued maps this is not true even if X and Y are compact. However, if the limit set-valued map is minimal usco, then we have this variant of Beer's result under certain connectivity assumptions.

Theorem 3.4. *Let X be a Baire locally connected space and Y be a locally compact metric space. Let $\{F_n : n \in \omega\}$ be a sequence of set-valued maps from X to Y which preserve connected sets. Let $F : X \rightsquigarrow Y$ be a minimal usco map such that $LtGr(F_n) = Gr(F)$. There is a dense G_δ-set H such that $F(x) = LtF_n(x)$ for every $x \in H$.*

Proof. Since, by assumption, $F : X \rightsquigarrow Y$ is a minimal usco set-valued map, there is a quasicontinuous selection f of F with $\overline{Gr(f)} = Gr(F)$, by Theorem 2.1. By quasicontinuity of f, the set $C(f)$ of all continuity points of f, is a dense G_δ-subset of X. Note that

$$|F(x)| = 1 \text{ for every } x \in C(f). \tag{1}$$

Indeed, if not, then there is $y \in F(x)$ such that $y \neq f(x)$. Then there are open sets $U \subset Y$ and $V \subset Y$ such that

$$y \in U,\ f(x) \in V \text{ and } U \cap V = \emptyset. \tag{2}$$

Since $x \in C(f)$, there is an open set $G \subset X$ such that $x \in G$ and $f(G) \subset V$. Moreover $y \in F(x)$, thus $(x, y) \in Gr(F) = \overline{Gr(f)}$. Since $G \times U$ is an open neighbourhood (x, y), $G \times U \cap Gr(f) \neq \emptyset$, which contradicts (2).Therefore (1) is true.

Let us put $L = \{x \in X : |F(x)| = 1\}$. We will show that for every $x \in L$, $F(x) = LtF_n(x)$. Let $x \in L$. Note that if $z \in LsF_n(x)$, then $(x, z) \in LsGr(F_n) = Gr(F)$. Thus

$$LsF_n(x) \subseteq F(x). \tag{3}$$

If we prove that

$$F(x) \in LiF_n(x), \tag{4}$$

the assertion follows. So, we will prove (4). Suppose that $F(x) \notin LiF_n(x)$. There is an open set U in Y such that $F(x) \in U$ and

$$\forall n \in \omega\, \exists k_n \in \omega, k_n \geq n,\ F_{k_n}(x) \cap U = \emptyset. \tag{5}$$

Let O be an open set in Y such that

$$F(x) \in O \subset \overline{O} \subset U \tag{6}$$

and $\overline{O}$ is compact. Put

$$\mathcal{B}(x) = \{V : x \in V,\ V \text{ is open and connected}\}.$$

For every $V \in \mathcal{B}(x)$ we denote

$$N_V = \{n \in \omega : (x_n, y_n) \in GrF_n, x_n \in V, y_n \in \overline{O} \setminus O\}.$$

We claim that for every $V \in \mathcal{B}(x)$, for every $n \in \omega$ there is $l \geq n$ with $l \in N_V$.

Indeed, let $V \in \mathcal{B}(x)$ and $n \in \omega$ be fixed. Since, by assumption, $LtGr(F_n) = Gr(F)$, there is $m \geq n$ such that

$$(V \times O) \cap Gr(F_l) \neq \emptyset, \text{for every } l \geq m.$$

By (5), there is $k_m \geq m$ such that $F_{k_m}(x) \cap O = \emptyset$. Since V is connected and F_{k_m} preserves connected sets, $F_{k_m}(V)$ is connected too. Thus there must exists

$$(x_{k_m}, y_{k_m}) \in Gr(F_{k_m}),\ x_{k_m} \in V, \text{ and } y_{k_m} \in \overline{O} \setminus O,$$

i.e. $k_m \in N_V$. Thus every N_V contains an increasing sequence $S(N_V)$ in ω.

The compactness of $\overline{O} \setminus O$ implies that for every $V \in \mathcal{B}(x)$, the sequence $\{y_k : k \in S(N_V)\}$ has a cluster point $y_V \in \overline{O} \setminus O$. The net $\{y_V : V \in \mathcal{B}(x)\}$ has a cluster point $y \in \overline{O} \setminus O$. Note that

$$(x, y) \in LsGr(F_n). \tag{7}$$

Indeed, let $n \in \omega$, $G \in \mathcal{B}(x)$ and L be an open neighbourhood of y. There is $V \in \mathcal{B}(x)$ such that $V \subset G$ and $y_V \in L$. Since y_V is a cluster point of the sequence $\{y_k : k \in S(N_V)\}$, there must exist $k \geq n$ such that $y_k \in L$ and $x_k \in V$. Thus $(x_k, y_k) \in (V \times L) \cap Gr(F_k) \subset (G \times L) \cap Gr(F_k)$, i.e. (7) is true, contrary to (6) .

Now put $H = C(f)$. □

Finishing our paper it is worthwhile to ask whether our main theorem is true for the nets.

Definition 3.5. ([4]), ([20]) Let Z be a topological space and Σ be a directed set. Let $\{G_\sigma : \sigma \in \Sigma\}$ be a net of subsets of Z. The lower limit LiG_σ and the upper limit LsG_σ of $\{G_\sigma : \sigma \in \Sigma\}$ are defined as follows: LiG_σ is the set of all points $z \in Z$ such that for every neighbourhood U of z there is $\sigma_0 \in \Sigma$ such that $G_\sigma \cap U \neq \emptyset$ for each $\sigma \geq \sigma_0$ and, respectively, LsG_σ is the set of all points $z \in Z$ such that for every neighbourhood U of z and for every $\sigma \in \Sigma$ there is $\eta \geq \sigma$ such that $G_\eta \cap U \neq \emptyset$.

Claim 3.6. *Theorem 1.1 does not work for nets as the following example shows.*

Example 3.7. Let $X = Y = [0, 1]$ with the usual Euclidean topology. Let $\mathcal{K}$ be the family of all finite sets in X ordered by the inclusion. Then $\mathcal{K}$ equipped with the set inclusion is a directed set. Define a net $\{F_K : K \in \mathcal{K}\}$ of lower semicontinuous set-valued maps from X to Y as follows:

$$F_K(x) = \begin{cases} \{0\} & \text{if } x \in K, \\ \{0, 1\} & \text{if } x \notin K, \end{cases}$$

Let a set-valued map $G : X \rightsquigarrow Y$ be given by $G(x) = \{0\}$ for every $x \in X$. Then $Lt\{F_K(x) : K \in \mathcal{K}\} = G(x)$ for every $x \in X$ and G has a closed graph. Let $F : X \rightsquigarrow Y$ be a set-valued map given by $F(x) = \{0, 1\}$ for every $x \in X$. It is easy to verify that $Lt\{Gr(F_K) : K \in \mathcal{K}\} = Gr(F)$.

Acknowledgement

The authors would like to thank the referee for his (her) valuable comments.

References

[1] Attouch, H.:*Variational Convergence for Functions and Operators*, Appl. Math. Ser., Pitman, London, 1984.
[2] Aubin, J.P.—Frankowska, H.:*Set-Valued Analysis*, Birhauser, Cambridge, MA, 1990.
[3] Back, K.:*Concepts of similarity for utility functions*, J. Math. Econom. **15**, (1986), 129–142.
[4] Beer, G.:*Topologies on Closed and Closed Convex Sets*, Kluwer Academic Publishers, 1993.
[5] Beer, G.: *More on convergence of continuous functions and topological convergence of sets*, Canad. Math. Bull. **28**, (1985), 52-59.
[6] Borsík, J.—Holá, Ľ.—Holý D.: *Baire spaces and quasicontinuous mappings*, Filomat **25**, (2011), 69-83.
[7] Berge, C.:*Topological Spaces*, Oliver and Boyd, Edinburgh 1963.
[8] Bagh, A.—Wets, R.J.B.: *Convergence of set-valued mappings: Equi-outer semicontinuity*, Set-Valued Analysis 4 (1996), 333-360.
[9] Del Prete, I.—Di Iorio, M.—Holá, Ľ.: *Graph convergence of set-valued maps and its relationship to other convergences*, J. Appl. Anal. **6**, (2000), 213–226.
[10] Drewnowski, L.—Labuda I.: *On minimal upper semicontinuous compact valued maps*, Rocky Mountain J. Math. **20**, (1990), 737–752.
[11] R.V. Fuller, *Sets of points of discontinuity*, Proc. Amer. Math. Soc. **38** (1973), 193–197.
[12] Christensen, J.P.R.: *Theorems of Namioka and R.E. Johnson type for upper semicontinuous and compact valued mappings*, Proc. Amer. Math. Soc. **86**, (1982), 649–655.
[13] Choquet, G.: *Lectures on Analysis Volume 1*, W.A. Benjamin, New York, 1969.
[14] Engelking, R.: *General Topology*, PWN 1977.
[15] Holá, Ľ.—Holý, D.: *Minimal usco maps, densely continuous forms and upper semicontinuous functions*, Rocky Mountain J. Math. **39**, (2009), 545–562.
[16] Holá, Ľ.—Holý, D.:*Relations between minimal usco and minimal cusco maps*, Portugaliae Mathematica **70**, (2013), 211–224.
[17] Kechris, A. S.: *Classical descriptive set theory*, Springer-Verlag, New York, 1995.
[18] Kempisty, S.: *Sur les fonctions quasi-continues*, Fund. Math. **19**, (1932), 184–197.
[19] Kowalczyk, S.: *Topological convergence of multivalued maps and topological convergence of graphs*, Demonstratio Mathematica **XXVII**, (1994), 79–87.
[20] Klein, E.—Thompson, A.C.: *Theory of Correspondences*, Wiley, 1984.
[21] Kuratowski, K.:*Topology*, Academic Press, New York, 1966.
[22] Matejdes M.: *Topological and pointwise upper Kuratowski limits of a sequence of lower quasi continuous multifunctions*, preprint.
[23] Neubrunn, T.: *Quasi-continuity*, Real Anal. Exchange, **14** (1988), 259–306.
[24] McCoy R. A.—Ntantu I.: *Topological properties of spaces of continuous functions*, Lecture Notes in Mathematics, Springer-Verlag, **1315**, (1988).
[25] Oxtoby, J.C.: *The Banach-Mazur game and Banach category Theorem*, Contribution to the theory of games, Vo. III, Annals of Mathematics Studies 39, Princeton University Press.
[26] Rockafellar, R.T.—Wets, R.J.B.: *Variational Analysis*, Springer, Berlin, 1998.
[27] Telgársky, R.: *Topological games: On the 50th anniversary of the Banach Mazur game*, Rocky Mountain J. Math. 17 (1987), 227-276.