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Uniquely Remotal Sets in Banach Spaces

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Abstract. The study of the continuity of the farthest point mapping for uniquely remotal sets has been used extensively in the literature to prove the singletoness of such sets. In this article, we show that the farthest point mapping is not continuous even if the set is remotal, rather than being uniquely remotal. Consequently, we obtain some generalizations of results concerning the singletoness of remotal sets. In particular, it is proved that a compact set admitting a unique farthest point to its center is a singleton, generalizing the well known result of Klee.

1. Introduction

Let X be a normed space, and let E be a closed bounded subset of X. It has been a longstanding question of whether or not the unique remotality of E implies the singletoness of this set. The main purpose of this article is to show that unique remotality of the set implies its singletoness in a widely existing case. However, we will see that the singletoness property follows from another criterion; not only unique remotality.
In this section, we need to go through the setup needed to fully understand the notations and terminologies presented in the paper.

For X and E as stated above, we define the real valued function $D(., E) : X \to \mathbb{R}$ by

$$D(x, E) = \sup\{\|x - e\| : e \in E\},$$

the farthest distance function. If for every $x \in X$, there exists $e \in E$ such that $D(x, E) = \|x - e\|$, we say that E is remotal. In this case, we denote the set $\{e \in E : D(x, E) = \|x - e\|\}$ by $F(x, E)$. It is clear that $F(., E) : X \to E$ is a multi-valued function. However, if $F(., E) : X \to E$ is a single-valued function, then E is called uniquely remotal. In this case, we denote $F(x, E)$ by $F(x)$, if no confusion arises.

The study of remotal and uniquely remotal sets has attracted many researchers in the last few decades, due to its connection to the geometry of Banach spaces. We refer the reader to [1], [3], [6], [7] and [10] as a sample of these studies. However, uniquely remotal sets are of special interest. The most challenging question regarding uniquely remotal sets has been: If E is uniquely remotal, does it follow that E is a singleton?
The importance of this question grew more when klee [5] proved that: singletoness of uniquely remotal sets is equivalent to convexity of a Chybechev set.

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Since then, a considerable work has been done to answer this question, and many partial results have been obtained, in the positive direction. We refer the reader to [1], [3], [7] and [10] for some related work on uniquely remotal sets.

Centers of sets have played a major role in the study of uniquely remotal sets, see [1], [2] and [3]. Recall that a center c of a subset E of a normed space X is an element $c \in X$ such that

$$D(c, E) = \inf_{x \in X} D(x, E).$$

Whether a set has a center or not is another question. However, in inner product spaces, any closed bounded set does have a center [1].

In [8] it was proved that if E is a uniquely remotal subset of a normed space, admitting a center c, and if F, restricted to the line segment $[c, F(c)]$ is continuous at c, then E is a singleton.
One of our purposes in this article is to show the singletoness of uniquely remotal sets if the farthest point mapping F restricted to $[c, F(c)]$ is partially continuous at c. This is defined as follows.

Definition 1.1. *Let $F : A \subset X \to X$ be a function, and let $a \in A$. We say that F is partially continuous at a if there exists a non constant sequence $(a_n) \subset A$, such that $a_n \to a$ and $F(a_n) \to F(a)$.*

Observe that partial continuity of F is a way weaker than the continuity condition, as functions can be easily partially continuous but not continuous.
Our second goal is to show that in non-singleton sets, $F(c)$ is isolated in the sense that $\|F(x) - F(c)\| > \delta$ for some $\delta > 0$, for all x in a neighborhood of c on $(c, F(c)]$. Here, F is any single valued function extracted from the multi-valued function $F(., E)$.
This isolation result will help prove the singletoness of uniquely remotal sets under some light restrictions. We should remark that Astaneh [1] proved this isolation result in inner product spaces, and used this result to prove the singletoness of uniquely remotal sets in inner product spaces, under the restriction that the farthest point map, restricted to $[c, F(c)]$, is continuous at c. In this article, we modify this result and generalize it to any normed space, without the use of the continuity condition.

2. Main Results

Now we present our main results related to isolating farthest points, and compare it with known results.

Theorem 2.1. *Let E be a closed bounded subset of the normed space X, admitting a center c. If E is uniquely remotal, and if $F : [c, F(c)] \to E$ is partially continuous at c, then E is a singleton.*

Proof. By replacing E with $E - \{c\}$, we may assume, without loss of generality, that $c = 0$.
Assume on the way of contrary that E is not a singleton, so that $F(0) \neq 0$. Since F, restricted to $[0, F(0)]$, is partially continuous at 0, there exists a non constant sequence $(x_n) \subset [0, F(0)]$ such that

$$x_n \to 0 \text{ and } F(x_n) \to F(0).$$

Clearly, $x_n = \lambda_n F(0)$ for some positive sequence (λ_n) with the property $\lambda_n \to 0$.
For each $n \in \mathbb{N}$, let $\varphi_n \in X^*$, the dual of X, be such that

$$\varphi_n(F(x_n) - x_n) = \|F(x_n) - x_n\| \text{ and } \|\varphi_n\| = 1,$$

where such functionals exist by the Hahn-Banach theorem. Now,

$$\begin{aligned} \Re\varphi_n(x_n) &= \Re\varphi_n(F(x_n)) - \varphi_n(F(x_n) - x_n) \\ &\leq \|\varphi_n\| \, \|F(x_n)\| - \|F(x_n) - x_n\| \\ &= \|F(x_n)\| - \|F(x_n) - x_n\| \\ &= \|F(x_n) - 0\| - \|F(x_n) - x_n\| \\ &\leq D(0, E) - D(x_n, E) \\ &\leq 0, \end{aligned}$$

where we have used the fact that 0 is a center of E in the last inequality. Thus, we have shown that $\Re\varphi_n(x_n) \le 0$. But, $x_n = \lambda_n F(0), \lambda_n > 0$, which implies that

$$\Re\varphi_n(\lambda_n F(0)) \le 0 \Rightarrow \lambda_n \Re\varphi_n(F(0)) \le 0.$$

That is

$$\Re\varphi_n(F(0)) \le 0, \text{ for all } n \in \mathbb{N}. \tag{1}$$

Now,

$$\varphi_n(F(x_n) - x_n) = \|F(x_n) - x_n\| \longrightarrow \|F(0)\|,$$

where we have used the fact that $F(x_n) \to F(0)$ and $x_n \to 0$. Moreover,

$$\begin{aligned} \Re\left(\varphi_n(F(x_n) - x_n) - \varphi_n(F(0))\right) &= \Re\left(\varphi_n(F(x_n) - x_n - F(0))\right) \\ &\le \|\varphi_n\| \, \|F(x_n) - x_n - F(0)\| \\ &= \|F(x_n) - x_n - F(0)\| \longrightarrow 0. \end{aligned}$$

Since both $(\varphi_n(F(x_n) - x_n))$ and $\Re(\varphi_n(F(x_n) - x_n) - \varphi_n(F(0)))$ converge, we conclude the convergence of $\Re(\varphi_n(F(0)))$, and

$$\begin{aligned} \lim_{n\to\infty} \Re\varphi_n(F(0)) &= \lim_{n\to\infty} \varphi_n(F(x_n) - x_n) \\ &= \lim_{n\to\infty} \|F(x_n) - x_n\| \\ &= \|F(0)\|. \end{aligned}$$

But, with aid of equation (1), this can happen only if $F(0) = 0$, which contradicts the assumption that $F(0) \neq 0$. Consequently, E must be the singleton $\{0\}$. This completes the proof. □

In fact, in the proof of Theorem 2.1, we didn't use the fact that E is uniquely remotal. What we have used is the fact that F is a well defined function. Thus, we have the following stronger version of Theorem 2.1.

Theorem 2.2. *Let E be a closed bounded remotal subset of the normed space X, admitting a center c, and let $F(x, E)$ be the set of farthest elements of x in E. Let $F : X \to E$ be any extracted single-valued function from $F(x, E)$. If F, restricted to $[c, F(c)]$, is partially continuous at c, then E is a singleton.*

The proof of this theorem is clearly identical to that of Theorem 2.1.
Thus, singletoness of E does not follow only from unique remotality! Rather, it follows from the partial continuity of the farthest point map at the center, whether the set is remotal or uniquely remotal. An astonishing result! As a corollary, we get the following easy result.

Corollary 2.3. *Let E be a closed bounded remotal subset of the normed space X, admitting a center c. If E is not a singleton, then no extracted function F from $F(x, E)$ can be partially continuous at c, when restricted to $[c, F(c)]$.*

As mentioned earlier, we aim to prove an isolation result that helps prove the singletoness of uniquely remotal sets in some occasions.

In fact, once we prove this isolation result, all the above results will be so trivial that we don't need to mention. It puts an end to the study of continuity of the farthest point mapping; as there "is no way" for this mapping to be continuous!
This result is motivated by our results in [10], where we proved the singletoness of uniquely remotal sets assuming the space to have this isolation property. Now, we prove the isolation property to be true in any normed space!
Although we can use Corollary 2.3 to prove the following result, we prefer to present another approach aiming at showing more ideas that may help solve the problem.

Theorem 2.4. *Let E be a closed bounded remotal subset of the normed space X, admitting a center c. If E is not a singleton then, for any extracted function F of $F(x, E)$, there exists a $\delta > 0$ such that*

$$\|F(x) - F(c)\| > \delta$$

for all x in a neighborhood of c on $(c, F(c)]$.

Proof. Let F be any extracted function from $F(., E)$ and $x \in (c, F(c)]$, then

$$\begin{aligned} \|x - F(x)\| &= \|tc + (1-t)F(c) - F(x)\|; t \in (0,1) \\ &\leq t\|c - F(x)\| + (1-t)\|F(c) - F(x)\| \\ &\leq t\|c - F(c)\| + (1-t)\|F(c) - F(x)\| \\ &\leq t\|x - F(x)\| + (1-t)\|F(c) - F(x)\|, \end{aligned}$$

which implies

$$\|F(x) - F(c)\| \geq \|x - F(x)\| \geq \|c - F(c)\| := r.$$

Thus, δ can be chosen so that $\delta \leq r$; the Chybechev center of E. □

Observe the extensive use of the fact that $\|c - F(c)\| \leq \|x - F(x)\|; \forall x \in X$. Thus, for the farthest point mapping to be continuous at c, the set E must be a singleton. This isolation result generalizes all known results relating the continuity of the farthest point mapping and singletoness of the uniquely remotal sets. See [2], [3], [4], [8] and [9].

We should remark that in [1], it was proved that if c is a center of a closed bounded subset E in an inner product space, and if E is non-singleton and uniquely remotal, then a $\delta > 0$ exists with the above isolation property.
However, our result generalizes this result in many ways; where our space can be any normed space (not inner product), and our E need not be uniquely remotal, but remotal!

We emphasize that this isolation results isolates the farthest points of a certain selection of the multivalued mapping $F(x, E)$. However, it doesn't imply that a neighborhoof $[c, a]$ of c, on $[c, F(c)]$, and a neighborhood $B(F(c), \delta)$ of $F(c)$ exist such that if $x \in (c, a)$ and $F(x) \in F(x, E)$, then $\|F(x) - F(c)\| > \delta$. We can see this in any normed space if E is a ball.

Now, having proved this isolation result, we follow the ideas in [10] to prove the singletoness of certain uniquely remotal sets.

Theorem 2.5. *Let E be a non-singleton closed bounded subset of the normed space X, admitting a center c, and let δ be as in Theorem 2.4. If the distance $D(c, E\backslash B(F(c), \delta))$ is attained, then E cannot be uniquely remotal.*

Proof. Suppose on the way of contrary that E is uniquely remotal, and assume $c = 0$. Let $a \in (0, F(0)]$ be such that

$$\|F(x) - F(0)\| > \delta, \ \forall x \in (0, a].$$

Let $(x_n) \subset (0, a]$ be such that $x_n \to 0$. By the isolation result, $F(x_n) \in E\backslash B(F(c), \delta)$. Now,

$$\begin{aligned} D(0, E) &= \lim_{n\to\infty} D(x_n, E) \\ &= \lim_{n\to\infty} D(x_n, E\backslash B(F(0), \delta)) \\ &= D(0, E\backslash B(F(0), \delta)). \end{aligned}$$

Consequently, since $D(0, E\backslash B(F(0), \delta))$ is attained by assumption, there exists $e \in E\backslash B(F(0), \delta))$ such that $D(0, E) = \|0 - e\|$. Since $e \in E\backslash B(F(0), \delta))$, $e \neq F(0)$; contradicting the assumption that E is uniquely remotal. □

Thus, unique remotality of E is implied by the behavior with regard to c. The following is an immediate corollary of this.

Corollary 2.6. *Let E be a closed bounded subset of the normed space X, admitting a center c, and let F be any extracted single valued mapping of $F(.,E)$. If a sequence $(x_n) \in (c, F(c)]$ exists such that $x_n \to c$ and $(F(x_n))$ converges, then E cannot be uniquely remotal.*

Proof. The proof is immediate once the fact

$$F(x_n) \to e \Rightarrow e \in F(x, E)$$

has been observed. □

As a corollary of this result, we have the following [5].

Corollary 2.7. *If a compact subset E, with a center c, is uniquely remotal in a normed space X, then E must be a singleton.*

Proof. Assume E is not a singleton. Let δ be as above. Since E is compact, $E \backslash B(F(c), \delta))$ is compact, hence is remotal. That is, $D(c, E \backslash B(F(c), \delta))$ is attained. Consequently, E is not uniquely remotal by Theorem 2.5; contradicting our assumption. This completes the proof. □

Observe that the proof of this result uses unique remotality of E with respect to its center only. That is, for E to be a singleton, E need not be uniquely remotal. In other words:

Theorem 2.8. *Let E be a compact subset of the normed space X, with a center. If E is uniquely remotal with respect to its center, then E is a singleton.*

We conclude by emphasizing that this article puts an end to the study of the continuity behavior of the farthest point mapping of remotal (and uniquely remotal) sets!

Remark 2.9. *We easily see that the above results are still valid if c is almost a center, rather than being a center. This is defined as: Let E be a closed remotal subset of the normed space X, and let $c \in X$ and $F(c) \in F(c, E)$ be such that*

$$D(c, E) \leq D(x, E); \forall x \in (c, F(c)],$$

then c is said to be almost a center of E.

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