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Input-To-State Stability of Multi-Group Stochastic Coupled Systems with Time-Varying Delay

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Abstract. We investigate the issue of p-th moment exponentially input-to-state stability (pMEISS) of multi-group stochastic coupled systems with time-varying delay (MSCSTD) in this paper. By means of results from graph theory, we develop a systematic method that allows one to construct a proper Lyapunov function for MSCSTD. More specifically, two kinds of sufficient criteria, which are called Lyapunov-type and coefficient-type respectively, are derived to ensure pMEISS for MSCSTD by using the graph-theoretic approach. To make results more understandable, we apply them to a typical stochastic coupled oscillators with control inputs.

1. Introduction

Over the past few years, multi-group coupled systems have gained great concern and much progress has been made in promising applications, such as engineering, physics and biology [1–5]. In fact, to model the dynamics for multi-group coupled systems, it is indispensable to take into account control inputs. For example, in [4], Liu et al. studied a large-scale coupled system consisting of l subsystems with control inputs and established sufficient conditions for the convergence successfully.

It is well-known that the stability issue of coupled systems has become a significant topic in many research fields (see [5–14] and the references therein). In today's research front line in terms of stability issue, input-to-state stability (ISS) is becoming of great importance especially in the field of stochastic systems. It was introduced by E.D Sontag for deterministic continuous-time systems in [15]. Inspired by his prominent work, an increasing number of scholars started to investigate this stability and many achievements can be further gained (see [16–20]). However, because of complex structures of coupled systems, most researchers studied ISS of coupled systems without delay according to the aforementioned papers. Hence, relevant literature about ISS of stochastic coupled systems with time-varying delay is few and we can just refer to the following several papers to the authors' knowledge. For example, in [21], Zhu et al. discussed mean-square exponential input-to-state stability for stochastic delayed neural networks using the Lyapunov-Krasovskii functional, stochastic analysis techniques, and Itô's formula. In [22], Liu et al. made the research of ISS of impulsive and switching hybrid systems with time-delay and obtained several sufficient conditions successfully.

In general, in the analysis of stability of coupled systems, Lyapunov function method has received great attention due to its usefulness. However, because of the complexity for coupled systems, there is no doubt

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that constructing a proper Lyapunov function for a given coupled system is rather difficult. Some scholars established several Lyapunov functions for some certain coupled systems successfully by using the linear matrix inequality and obtained many achievements (see [28, 29]). In recent few years, Li et al. utilized the graph-theoretic approach to derive the construction of Lyapunov function for some multi-group coupled systems and further made a study of global stability [30]. Since the method was pretty effective, many researchers, inspired by the prominent work, took advantage of this way to study the stability of many multi-group models such as delayed multi-group models [24, 27], stochastic multi-group models [31–34], and impulsive multi-group models [35].

The above observation motivates our research. First, we use graph theory and the vertex-Lyapunov function of the subsystem to construct the appropriate global Lyapunov function for multi-group stochastic coupled systems with time-varying delay (MSCSTD). Then, two kinds of sufficient criteria, which are in the forms of Lyapunov-type and coefficient-type, are gained. To make the results more understandable, we present a concrete example, that is, stochastic coupled oscillators with control inputs and sufficient conditions are obtained successfully. Finally, a numerical example is provided to justify accuracy and effectiveness of our results. In this paper, our chief contribution is to apply graph theory to MSCSTD for the first time and to present novel sufficient conditions of ISS.

The paper is outlined as follows. In the next section, we introduce basic notations, preliminaries, and the formulation of our model. In Section 2, sufficient criteria of the stability for MSCSTD are obtained. Next, an application of stochastic coupled oscillators with control inputs is presented in Section 3. In the last section, a numerical simulation is proposed.

1.1. *Mathematical notations and preliminaries*

For the sake of simplicity, we present several notations. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $B(\cdot)$ be a *one*-dimensional Brownian motion defined on the space. Denote by $\mathbb{E}(\cdot)$ mathematical expectation with respect to the given probability measure $\mathbb{P}$. Write $\mathbb{R}$ and $\mathbb{R}^n$ for the set of real numbers and n-dimensional Euclidean space, respectively. Set $|\cdot|$ to be the Euclidean norm for vectors or the trace norm for matrices. Let $C([-\tau, 0]; \mathbb{R}^n)$ represent the family of continuous functions ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Denote by $L^2_{\mathcal{F}_0}(\Omega; C([-\tau, 0]; \mathbb{R}^n))$ the family of all $\mathcal{F}_0$ measurable, $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic variables $\phi = \{\phi(s) : -\tau \leq s \leq 0\}$ such that $\mathbb{E}\|\phi\|^2 < \infty$. Denote $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$, $\mathrm{L} = \{1, 2, \cdots, l\}$ and $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ for the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}^+$ that are continuously twice differentiable in x and once in t. Let l_∞ be the class of essentially bounded functions u from $[0, \infty)$ to $\mathbb{R}^n$ with $\|u\|_\infty = \mathrm{esssup}_{t\geq 0}|u(t)| < \infty$.

Now, Let us introduce some important concepts of graph theory based on the literature [36].

First, for a digraph $\mathcal{G} = (\mathcal{U}, E)$, it possesses a set $\mathcal{U} = \{1, 2, \cdots, u\}$ of vertices and a set E of arcs (v, w) leading from initial vertex v to terminal vertex w. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is called to be spanning when they have the same vertex set. In addition, a digraph $\mathcal{G}$ is weighted if each arc (w, v) is assigned a positive weight a_{vw}. Here we define the weight matrix $A = (a_{vw})_{u\times u}$ whose entry a_{vw} equals the weight of arc (w, v) if it exists, and 0 otherwise. Accordingly, the weight $W(\mathcal{G})$ of $\mathcal{G}$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{v_1, v_2, \cdots, v_s\}$ such that its set of arcs is $\{(v_i, v_{i+1}) : i = 1, 2, \cdots, s-1\}$. If $v_s = v_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertex v, called the root, if v is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exact one arc. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. A digraph $\mathcal{G}$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. Denote the digraph with weight matrix A as $(\mathcal{G}, A)$. A weighted digraph $(\mathcal{G}, A)$ is said to be balanced if $W(C) = W(-C)$ for all directed cycles C. Here, $-C$ denotes the reverse of C and is constructed by reversing the direction of all arcs in C. For a unicyclic graph $\mathcal{Q}$ with cycle $C_{\mathcal{Q}}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $C_{\mathcal{Q}}$ with $-C_{\mathcal{Q}}$. Suppose that the digraph $(\mathcal{G}, A)$ is balanced, then $W(\mathcal{Q}) = W(\tilde{\mathcal{Q}})$. The Laplacian matrix of the digraph $(\mathcal{G}, A)$ is defined as $\mathbf{P} = (p_{vw})_{u\times u}$, where $p_{vw} = -a_{vw}$ for $v \neq w$ and $p_{vw} = \sum_{j\neq v} a_{vj}$ for $v = w$.

1.2. Model formulation

In this subsection, given a digraph $\mathcal{G}$ consisting of l ($l \geq 2$) vertices, each vertex of the digraph $\mathcal{G}$ is regarded as a group, each of which has a m_k-dimensional individual subsystem, then connect them together according to the digraph $\mathcal{G}$.

Assume that the k-th subsystem is affected by a given control input. Taking into account wide existence of stochastic perturbation, we now use the Brownian motion $B(t)$ as the source of noise to affect the given system, then the k-th subsystem can be described as follows:

$$\mathrm{d}x_k(t) = [f_k(t, x_k(t), x_k(t-\tau(t))) + u_k(t)]\,\mathrm{d}t + [g_k(t, x_k(t), x_k(t-\tau(t)))]\,\mathrm{d}B(t),$$

where $x_k \in \mathbb{R}^{m_k}$ manifests the state of the k-th subsystem, $f_k, g_k : \mathbb{R}^+ \times \mathbb{R}^{m_k} \times \mathbb{R}^{m_k} \to \mathbb{R}^{m_k}$ are continuous functions, and $u_k \in \mathbb{R}^{m_k}$ stands for the control input in the k-th subsystem for $k \in \mathrm{L}$. Moreover, $\tau(t) : \mathbb{R}^+ \to [0, \tau]$ is a differentiable time-varying delay, whose derivative is bounded by a constant $\bar{\tau} \in [0, 1)$. Furthermore, we define $n = \sum_{k=1}^{l} m_k$.

Subsequently, we further use functions $M_{kh}(x_h(t-\tau(t)))$ and $N_{kh}(x_h(t-\tau(t)))$ as the coupling forms from the h-th subsystem to the k-th one. Also, both b_{kh} and e_{kh} stand for the coupling strength from the h-th subsystem to the k-th one. Here, $b_{kh} = e_{kh} = 0$ iff there is no influence from the h-th subsystem to the k-th one for any $k, h \in \mathrm{L}$. Hence, MSCSTD can be depicted as

$$\begin{aligned}\mathrm{d}x_k(t) = &\left[f_k(t, x_k(t), x_k(t-\tau(t))) + \sum_{h=1}^{l} b_{kh} N_{kh}(x_h(t-\tau(t))) + u_k(t)\right]\mathrm{d}t \\ &+ \left[g_k(t, x_k(t), x_k(t-\tau(t))) + \sum_{h=1}^{l} e_{kh} M_{kh}(x_h(t-\tau(t)))\right]\mathrm{d}B(t),\ k \in \mathrm{L}.\end{aligned} \tag{1}$$

Additionally, the initial condition for the system (1) is given by $\phi(t) \in L^2_{\mathcal{F}_0}(\Omega; C([-\tau, 0]; \mathbb{R}^n))$ for $t \in [-\tau, 0]$, where $\phi(t)$ is a continuous vector function on $[-\tau, 0]$. Similarly, $x(t)$ and $u(t)$ are continuous vector functions, which are shown as $x(t) = (x_1(t)^T, x_2(t)^T, \cdots, x_l(t)^T)^T$ and $u(t) = (u_1(t)^T, u_2(t)^T, \cdots, u_l(t)^T)^T$, respectively.

Moreover, we make notations as follows: $f_k = f_k(t, x_k(t), x_k(t-\tau(t)))$, $g_k = g_k(t, x_k(t), x_k(t-\tau(t)))$, $N_{kh} = N_{kh}(x_h(t-\tau(t)))$, and $M_{kh} = M_{kh}(x_h(t-\tau(t)))$. For the goal of the paper, functions f_k, g_k, u_k, N_{kh} and M_{kh} need to guarantee system (1) has a unique global solution. The unique global solution is denoted by $x(t; \phi, u)$, which is the solution at time t with the initial value ϕ and control input u.

Definition 1.1. *[16] The system (1) is said to be p-th moment exponentially input-to-state stable (pMEISS) if for every $\phi \in L^2_{\mathcal{F}_0}(\Omega; C([-\tau, 0]; \mathbb{R}^n))$ and $u \in l_\infty$, there exist scalars $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ such that the following inequality holds:*

$$\mathbb{E}|x(t; \phi, u)|^p \leq \alpha \mathbb{E}\|\phi\|^p e^{-\beta t} + \gamma \|u\|_\infty^p.$$

When $p = 2$, it is called mean-square exponentially input-to-state stable (MEISS).

Definition 1.2. *For each function $V_k(x_k, t) \in C^{2,1}(\mathbb{R}^{m_k} \times \mathbb{R}^+; \mathbb{R}^+)$, if differential operator $\mathcal{L}$ associated with the system (1) acts on a function V_k, then $\mathcal{L}V_k(x_k, t)$ is defined as*

$$\begin{aligned}&\mathcal{L}V_k(x_k, t)\\ &= \frac{\partial V_k(x_k, t)}{\partial t} + \frac{\partial V_k(x_k, t)}{\partial x_k}\left[f_k + \sum_{h=1}^{l} b_{kh} N_{kh} + u_k(t)\right] + \frac{1}{2}\mathrm{trace}\left\{\left[g_k + \sum_{h=1}^{l} e_{kh} M_{kh}\right]^T \frac{\partial^2 V_k(x_k, t)}{\partial x_k^2}\left[g_k + \sum_{h=1}^{l} e_{kh} M_{kh}\right]\right\}.\end{aligned} \tag{2}$$

where $\frac{\partial V_k}{\partial x_k} = \left(\frac{\partial V_k}{\partial x_k^{(1)}}, \frac{\partial V_k}{\partial x_k^{(2)}}, \cdots \frac{\partial V_k}{\partial x_k^{(m_k)}}\right)$, $\frac{\partial^2 V_k}{\partial x_k^2} = \left(\frac{\partial^2 V_k}{\partial x_k^{(m_i)} \partial x_k^{(m_j)}}\right)_{m_k * m_k}$.

For convenience for the later derivations, we give a helpful lemma.

Lemma 1.3. *[5] Assume $l \geq 2$. Let c_k denote the cofactor of the k-th diagonal element of the Laplacian matrix of the digraph $(\mathcal{G}, A)$. Then the following identity holds:*

$$\sum_{k,h=1}^{l} c_k a_{kh} F_{kh}(x_k, x_h) = \sum_{Q \in \mathbb{Q}} W(Q) \sum_{(s,r) \in E(C_Q)} F_{rs}(x_r, x_s).$$

Here for any $k, h \in \mathrm{L}$, $F_{kh}(x_k, x_h)$ is an arbitrary function, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$ in which $A = (a_{kh})_{l \times l}$, $W(Q)$ is the weight of Q, and C_Q denotes the directed cycle of Q. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_k > 0$ for $k \in \mathrm{L}$.

2. Stability Analysis for MSCSTD

In this section, let us study the stability of the system (1). To address it, we take advantage of graph theory and Lyapunov function method based on the literatures [5, 32]. We can now explore the main results of this section. To explain them explicitly, a definition of vertex-Lyapunov functions is given first.

Definition 2.1. *Functions $V_k(x_k, t)$, $k \in \mathrm{L}$ as*

$$V_k(x_k, t) = e^{\lambda t} U_k(x_k, t) + S_k(t, \omega), \tag{3}$$

where $\lambda > 0$, $S_k(t, \omega) \geq 0$ a.s. are called vertex-Lyapunov functions for the system (1) if the following conditions hold:

V1. *There exist positive constants $p \geq 2$, α_k, and β_k such that*

$$\alpha_k |x_k|^p \leq U_k(x_k, t) \leq \beta_k |x_k|^p.$$

V2. *There exist positive constants ξ_k, δ_k, η_k, functions $F_{kh}(x_k, x_h)$ and a matrix $A = (a_{kh})_{l \times l}$ such that*

$$\mathcal{L}V_k(x_k(t), t) \leq e^{\lambda t} \left[\xi_k |u_k(t)|^2 + \sum_{h=1}^{l} a_{kh} F_{kh}(x_k(t - \tau(t)), x_h(t - \tau(t))) - \delta_k |x_k(t)|^2 + \eta_k |x_k(t - \tau(t))|^2 \right].$$

V3. *Along each directed cycle C of the weighted digraph $(\mathcal{G}, A)$, there is*

$$\sum_{(h,k) \in E(C)} F_{kh}(x_k, x_h) \leq 0 \tag{4}$$

for $x_k \in \mathbb{R}^{m_k}$, $x_h \in \mathbb{R}^{m_h}$.

2.1. Lyapunov-type theorem

Theorem 2.2. *Assume that there exist vertex-Lyapunov functions in the system (1) and the digraph $(\mathcal{G}, A)$ where $A = (a_{kh})_{l \times l}$ is strongly connected. If*

$$\eta_k - (1 - \bar{\tau})\delta_k < 0, \; k \in \mathrm{L}, \tag{5}$$

then the system (1) is pMEISS.

Proof. Define

$$U(x, t) = \sum_{k=1}^{l} c_k U_k(x_k, t),$$

where c_k stands for the cofactor of the k-th diagonal element of the Laplacian matrix of the digraph $(\mathcal{G}, A)$.

Making use of the condition V1, we can derive

$$U(x,t) \le \sum_{k=1}^{l} c_k\beta_k\,|x_k|^p \le \left(\sum_{k=1}^{l} c_k\beta_k\right)|x|^p = \beta|x|^p, \tag{6}$$

in which $\beta = \sum_{k=1}^{l} c_k\beta_k$. Subsequently,

$$\begin{aligned} U(x,t) &\ge \sum_{k=1}^{l} c_k\alpha_k|x_k|^p \\ &= \sum_{j=1}^{l} c_j\alpha_j \sum_{k=1}^{l} \frac{c_k\alpha_k}{\sum_{u=1}^{l} c_u\alpha_u}\left(|x_k|^2\right)^{\frac{p}{2}} \\ &\ge \sum_{j=1}^{l} c_j\alpha_j \left(\sum_{k=1}^{l} \frac{c_k\alpha_k}{\sum_{u=1}^{l} c_u\alpha_u}|x_k|^2\right)^{\frac{p}{2}} \\ &\ge \left(\sum_{j=1}^{l} c_j\alpha_j\right)^{1-\frac{p}{2}} \left(\min_{1\le k\le l}\{c_k\alpha_k\}\right)^{\frac{p}{2}} |x|^p \\ &= \alpha|x|^p, \end{aligned} \tag{7}$$

where $\alpha = \left(\sum_{j=1}^{l} c_j\alpha_j\right)^{1-\frac{p}{2}} (\min_{1\le k\le l}\{c_k\alpha_k\})^{\frac{p}{2}}$. Combining (6) and (7), it is easy to get

$$\alpha|x|^p \le U(x,t) \le \beta|x|^p.$$

Let $V(x,t) = \sum_{k=1}^{l} c_kV_k(x_k,t)$. When λ tends to be sufficiently small, the following inequality holds:

$$\frac{e^{\lambda\tau}}{1-\bar{\tau}}\eta_k - \delta_k < 0. \tag{8}$$

Subsequently, fix any $\phi \in L^2_{\mathcal{F}_0}(\Omega; C([-\tau,0];\mathbb{R}^n))$ and write $x(t;\phi,u) = x(t)$. By using the condition V2, we then have

$$\begin{aligned} \mathcal{L}V(x(t),t) &= \sum_{k=1}^{l} c_k\mathcal{L}V_k(x_k(t),t) \\ &\le \sum_{k=1}^{l} c_ke^{\lambda t}\left[\xi_k\,|u_k(t)|^2 + \sum_{h=1}^{l} a_{kh}F_{kh}(x_k(t-\tau(t)),x_h(t-\tau(t))) - \delta_k|x_k(t)|^2 + \eta_k|x_k(t-\tau(t))|^2\right] \\ &= e^{\lambda t}\sum_{k=1}^{l} c_k\xi_k\,|u_k(t)|^2 + e^{\lambda t}\sum_{k=1}^{l} c_k\eta_k|x_k(t-\tau(t))|^2 + e^{\lambda t}\sum_{k=1}^{l}\sum_{h=1}^{l} c_ka_{kh}F_{kh}(x_k(t-\tau(t)),x_h(t-\tau(t))) - e^{\lambda t}\sum_{k=1}^{l} c_k\delta_k|x_k(t)|^2. \end{aligned}$$

Further by Lemma 1.3, we obtain

$$\sum_{k=1}^{l}\sum_{h=1}^{l} c_ka_{kh}F_{kh}(x_k(t-\tau(t)),x_h(t-\tau(t))) = \sum_{Q\in\mathbb{Q}} W(Q) \sum_{(s,r)\in E(C_Q)} F_{rs}(x_r(t-\tau(t)),x_s(t-\tau(t))), \tag{9}$$

where $\mathbb{Q}$ denotes the set of all spanning unicyclic graphs of $(\mathcal{G},A)$, $W(Q)$ stands for the weight of Q, and C_Q is the directed cycle of Q.

Applying (9) and the condition V3, we can further get that

$$\begin{aligned}\mathcal{L}V(x(t),t) &\le e^{\lambda t}\sum_{k=1}^{l} c_k\xi_k|u_k(t)|^2 - e^{\lambda t}\sum_{k=1}^{l} c_k\delta_k|x_k(t)|^2 + e^{\lambda t}\sum_{k=1}^{l} c_k\eta_k|x_k(t-\tau(t))|^2 \\ &\quad + e^{\lambda t}\sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r)\in E(C_{\mathcal{Q}})} F_{rs}(x_r(t-\tau(t)), x_s(t-\tau(t))) \\ &\le \max_{1\le k\le l}\{c_k\xi_k\} e^{\lambda t}\|u\|_\infty^2 - e^{\lambda t}\sum_{k=1}^{l} c_k\delta_k|x_k(t)|^2 + e^{\lambda t}\sum_{k=1}^{l} c_k\eta_k|x_k(t-\tau(t))|^2.\end{aligned}$$

By (8) and Itô's formula, one can show that

$$\begin{aligned}\mathbb{E}V(x(t),t) &\le \mathbb{E}V(x(0),0) + \max_{1\le k\le l}\{c_k\xi_k\}\|u\|_\infty^2 \int_0^t e^{\lambda s}\mathrm{d}s - \sum_{k=1}^{l} c_k\delta_k\mathbb{E}\int_0^t e^{\lambda s}|x_k(s)|^2\mathrm{d}s \\ &\quad + \sum_{k=1}^{l} c_k\eta_k\mathbb{E}\int_0^t e^{\lambda s}|x_k(s-\tau(s))|^2\mathrm{d}s \\ &\le \sum_{k=1}^{l} c_k\beta_k\mathbb{E}|x_k(0)|^p + \sum_{k=1}^{l} c_k\mathbb{E}S_k(0) + \max_{1\le k\le l}\{c_k\xi_k\}\|u\|_\infty^2\int_0^t e^{\lambda s}\mathrm{d}s \\ &\quad - \sum_{k=1}^{l} c_k\delta_k\mathbb{E}\int_0^t e^{\lambda s}|x_k(s)|^2\mathrm{d}s + \frac{e^{\lambda\tau}}{1-\bar{\tau}}\sum_{k=1}^{l} c_k\eta_k\mathbb{E}\int_{-\tau}^t e^{\lambda s}|x_k(s)|^2\mathrm{d}s \\ &\le \sum_{k=1}^{l} c_k\beta_k\mathbb{E}|x_k(0)|^p + \sum_{k=1}^{l} c_k\mathbb{E}S_k(0) + \max_{1\le k\le l}\{c_k\xi_k\}\|u\|_\infty^2\int_0^t e^{\lambda s}\mathrm{d}s \\ &\quad + \left(\frac{e^{\lambda\tau}}{1-\bar{\tau}}\sum_{k=1}^{l} c_k\eta_k - \sum_{k=1}^{l} c_k\delta_k\right)\mathbb{E}\int_0^t e^{\lambda s}|x_k(s)|^2\mathrm{d}s + \frac{e^{\lambda\tau}}{1-\bar{\tau}}\sum_{k=1}^{l} c_k\eta_k\mathbb{E}\int_{-\tau}^0 e^{\lambda s}\max_{1\le k\le l}|x_k(s)|^2\mathrm{d}s \\ &\le \max_{1\le k\le l}\{c_k\beta_k\}\sum_{k=1}^{l}\mathbb{E}|x_k(0)|^p + \max_{1\le k\le l}\{c_k\}\sum_{k=1}^{l}\mathbb{E}S_k(0) + \frac{\max_{1\le k\le l}\{c_k\xi_k\}}{\lambda}\|u\|_\infty^2\left(e^{\lambda t}-1\right) \\ &\quad + \left(\frac{e^{\lambda\tau}}{1-\bar{\tau}}\sum_{k=1}^{l} c_k\eta_k - \sum_{k=1}^{l} c_k\delta_k\right)\mathbb{E}\int_0^t e^{\lambda s}|x_k(s)|^2\mathrm{d}s + \frac{e^{\lambda\tau}}{1-\bar{\tau}}\sum_{k=1}^{l} c_k\eta_k\mathbb{E}\int_{-\tau}^0 e^{\lambda s}\max_{1\le k\le l}|x_k(s)|^2\mathrm{d}s \\ &< \max_{1\le k\le l}\{c_k\beta_k\}\sum_{k=1}^{l}\mathbb{E}|x_k(0)|^p + \max_{1\le k\le l}\{c_k\}\sum_{k=1}^{l}\mathbb{E}S_k(0) + \frac{\max_{1\le k\le l}\{c_k\xi_k\}}{\lambda}\|u\|_\infty^2\left(e^{\lambda t}-1\right) \\ &\quad + \frac{e^{\lambda\tau}}{1-\bar{\tau}}\sum_{k=1}^{l} c_k\eta_k\mathbb{E}\int_{-\tau}^0 e^{\lambda s}\max_{1\le k\le l}|x_k(s)|^2\mathrm{d}s.\end{aligned} \tag{10}$$

It is easy to get that there exists a constant σ such that

$$\mathbb{E}S_k(0) \le \sigma\mathbb{E}\|\phi\|^p.$$

Then, we can further derive that

$$\begin{aligned}\mathbb{E}V(x(t),t) &\le \max_{1\le k\le l}\{c_k\beta_k\}\sum_{k=1}^{l}\mathbb{E}|x_k(0)|^p + \sigma\max_{1\le k\le l}\{c_k\}\sum_{k=1}^{l}\mathbb{E}\|\phi\|^p + \frac{\max_{1\le k\le l}\{c_k\xi_k\}}{\lambda}\|u\|_\infty^2\left(e^{\lambda t}-1\right) \\ &\quad + \frac{e^{\lambda\tau}-1}{\lambda(1-\bar{\tau})}\sum_{k=1}^{l} c_k\eta_k\mathbb{E}\|\phi\|^p\end{aligned} \tag{11}$$

$$\leq \left(\max_{1\leq k\leq l}\{c_k\beta_k\} + \sigma \max_{1\leq k\leq l}\{c_k\} + \epsilon \sum_{k=1}^{l} c_k\eta_k \right) \mathbb{E}\|\phi\|^p + \frac{\max_{1\leq k\leq l}\{c_k\xi_k\}}{\lambda}\|u\|_\infty^p \left(e^{\lambda t} - 1\right),$$

where $\epsilon = \frac{e^{\lambda\tau}-1}{\lambda(1-\bar{\tau})}$.

Note that the weighted digraph $(\mathcal{G}, A)$ is strongly connected, which implies $c_k > 0$ for any $k \in \mathrm{L}$. On the other hand, by (7), it is easy to obtain

$$\mathbb{E}V(x(t), t) \geq \alpha e^{\lambda t}\mathbb{E}|x(t)|^p. \tag{12}$$

In view of the fact that $c_k > 0$, we further get that $\alpha > 0$. By (11) and (12), the following holds:

$$\mathbb{E}|x(t)|^p \leq \frac{\max_{1\leq k\leq l}\{c_k\beta_k\} + \sigma \max_{1\leq k\leq l}\{c_k\} + \epsilon\sum_{k=1}^{l} c_k\eta_k}{\alpha}\mathbb{E}\|\phi\|^p e^{-\lambda t} + \frac{\max_{1\leq k\leq l}\{c_k\xi_k\}}{\lambda\alpha}\|u\|_\infty^p, \tag{13}$$

which verifies that the system (1) is pMEISS together with Definition 1.1. The proof is complete. □

Remark 2.3. *As we know, it is easy to see that the construction of a global Lyapunov function is rather difficult. However, from Theorem 2.2, it provides an effective way to construct the global Lyapunov function for the system (1) by using vertex-Lyapunov functions $V_k(x_k, t)$. Obviously, this method can be applied in the study of stability of many coupled systems.*

Remark 2.4. *Multi-group coupled systems have received great attention in [24–27]. But, it is noting that both control inputs and stochastic perturbation play an important role in stability analysis of coupled systems. Moreover, Li et al. applied graph theory in the stability of coupled system (see [27, 30]). Therefore, enlightened by their method, we use graph theory to make a study of pMEISS of MSCSTD for the first time.*

Furthermore, considering some properties in graph theory, it is not hard to acquire some other simple conditions. It is noted that if the digraph $(\mathcal{G}, A)$ is balanced, then

$$\sum_{k,h=1}^{l} c_k a_{kh} F_{kh}(x_k, x_h) = \frac{1}{2}\sum_{Q\in\mathbb{Q}} W(Q) \sum_{(h,k)\in E(\mathcal{C}_Q)} \left[F_{kh}(x_k, x_h) + F_{hk}(x_h, x_k)\right].$$

In this case, (4) can be replaced by

$$\sum_{(h,k)\in E(\mathcal{C}_Q)} \left[F_{kh}(x_k, x_h) + F_{hk}(x_h, x_k)\right] \leq 0. \tag{14}$$

And then, a corollary is derived on the basis of the above inequality.

Corollary 2.5. *Suppose that the digraph $(\mathcal{G}, A)$ is balanced. Then the conclusion of Theorem 2.2 holds if (4) is replaced by (14).*

Next, it is noted that if for every $F_{kh}(x_k, x_h)$ there exist functions $P_k(x_k)$ and $P_h(x_h)$, such that

$$F_{kh}(x_k, x_h) \leq P_k(x_k) - P_h(x_h), \ k, h \in \mathrm{L}, \tag{15}$$

then (4) can be obtained easily. Hence, another corollary can be proposed based on the above consideration.

Corollary 2.6. *The conclusion of Theorem 2.2 holds if (4) is replaced by (15).*

2.2. Coefficient-type theorem

Theorem 2.7. *Let $p \geq 2$. The system (1) is pMEISS provided that the following conditions hold for any $k, h \in$ L.*

A1. There are positive constants $\hat{\alpha}_k$, $\hat{\beta}_k$, ϑ_k, and θ_k such that

$$x_k^T f_k(t, x_k, y_k) \leq -\hat{\alpha}_k |x_k|^2 + \hat{\beta}_k |y_k|^2$$

and

$$|g_k(t, x_k, y_k)|^2 \leq \vartheta_k |x_k|^2 + \theta_k |y_k|^2.$$

A2. There exist a constant D_{kh} such that

$$|N_{kh}(x_h)| \vee |M_{kh}(x_h)| \leq D_{kh}|x_h|.$$

A3. There is

$$\hat{\eta}_k - (1-\bar{\tau})\hat{\delta}_k < 0,$$

where $\hat{\eta}_k = 2\hat{\beta}_k - (1-\bar{\tau}) + \theta_k + \sum_{h=1}^l b_{kh} D_{kh} + l \sum_{h=1}^l e_{kh}^2 D_{kh}^2$ and $\hat{\delta}_k = 2\hat{\alpha}_k - \vartheta_k - \sum_{h=1}^l b_{kh} D_{kh} - 2$.

Proof. Define

$$U_k(x_k, t) = |x_k|^2$$

and

$$S_k(t) = \int_{t-\tau(t)}^t e^{\lambda s}|x_k(s)|^2 \mathrm{d}s.$$

Then, from the equation (3), it follows that

$$V_k(x_k(t), t) = e^{\lambda t}|x_k|^2 + \int_{t-\tau(t)}^t e^{\lambda s}|x_k(s)|^2 \mathrm{d}s.$$

It is easy to get that the condition V1 can be satisfied and $S_k(t) \geq 0$.

In the subsequence, compute

$$\begin{aligned}
\mathcal{L}V_k(x_k(t), t) = {} & \lambda e^{\lambda t}|x_k(t)|^2 + e^{\lambda t}|x_k(t)|^2 - (1-\dot{\tau}(t))e^{\lambda(t-\tau(t))}|x_k(t-\tau(t))|^2 \\
& + 2e^{\lambda t}x_k^T(t)\left(f_k + \sum_{h=1}^l b_{kh} N_{kh}(x_h(t-\tau(t))) + u_k(t)\right) \\
& + \frac{1}{2}\text{trace}\left\{\left(g_k + \sum_{h=1}^l e_{kh} M_{kh}(x_h(t-\tau(t)))\right)^T \left(2e^{\lambda t} I\right)\left(g_k + \sum_{h=1}^l e_{kh} M_{kh}(x_h(t-\tau(t)))\right)\right\}.
\end{aligned}$$

By conditions A1 and A2, the $\mathcal{L}V_k$ becomes

$$\begin{aligned}
\mathcal{L}V_k(x_k(t), t) \leq {} & \lambda e^{\lambda t}|x_k(t)|^2 - 2\hat{\alpha}_k e^{\lambda t}|x_k(t)|^2 + 2\hat{\beta}_k e^{\lambda t}|x_k(t-\tau(t))|^2 + e^{\lambda t}\left|\sum_{h=1}^l e_{kh} M_{kh}(x_h(t-\tau(t)))\right|^2 \\
& + 2x_k^T(t)e^{\lambda t}\sum_{h=1}^l b_{kh} N_{kh}(x_h(t-\tau(t))) + e^{\lambda t}|x_k(t)|^2 - (1-\bar{\tau})\, e^{\lambda(t-\tau)}|x_k(t-\tau(t))|^2 \\
& + 2e^{\lambda t}|x_k(t)||u_k(t)| + e^{\lambda t}|g_k|^2
\end{aligned}$$

$$\leq \lambda e^{\lambda t}|x_k(t)|^2 - 2\hat{\alpha}_k e^{\lambda t}|x_k(t)|^2 + 2\hat{\beta}_k e^{\lambda t}|x_k(t-\tau(t))|^2 + e^{\lambda t}(\vartheta_k|x_k(t)|^2 + \theta_k|x_k(t-\tau(t))|^2)$$
$$+ 2x_k^T(t)e^{\lambda t}\sum_{h=1}^{l} b_{kh}N_{kh}(x_h(t-\tau(t))) + e^{\lambda t}|x_k(t)|^2 - (1-\bar{\tau})\, e^{\lambda(t-\tau)}|x_k(t-\tau(t))|^2$$
$$+ e^{\lambda t}\left|\sum_{h=1}^{l} e_{kh}M_{kh}(x_h(t-\tau(t)))\right|^2 + 2e^{\lambda t}|x_k(t)||u_k(t)|$$

Using the inequality

$$|a|^p|b|^q \leq \varepsilon|a|^{p+q} + \frac{q}{p+q}\left[\frac{p}{\varepsilon(p+q)}\right]^{\frac{p}{q}}|b|^{p+q},$$

in which $a, b \in \mathbb{R}$, $p, q, \varepsilon > 0$ we further obtain

$$\begin{aligned} 2x_k^T(t)\sum_{h=1}^{l} b_{kh}N_{kh}(x_h(t-\tau(t))) &\leq 2|x_k(t)|\sum_{h=1}^{l} b_{kh}D_{kh}|x_h(t-\tau(t))| \\ &= 2\sum_{h=1}^{l} b_{kh}D_{kh}|x_k(t)||x_h(t-\tau(t))| \\ &\leq \sum_{h=1}^{l} b_{kh}D_{kh}|x_k(t)|^2 + \sum_{h=1}^{l} b_{kh}D_{kh}|x_h(t-\tau(t))|^2 \end{aligned} \tag{16}$$

and

$$\left|\sum_{h=1}^{l} e_{kh}M_{kh}\right| \leq l\sum_{h=1}^{l} e_{kh}^2D_{kh}^2|x_h(t-\tau(t))|^2. \tag{17}$$

It then follows from (16) and (17) that

$$\begin{aligned} \mathcal{L}V_k(x_k(t),t) &\leq \lambda e^{\lambda t}|x_k(t)|^2 - 2\hat{\alpha}_k e^{\lambda t}|x_k(t)|^2 + 2\hat{\beta}_k e^{\lambda t}|x_k(t-\tau(t))|^2 + 2e^{\lambda t}|x_k(t)||u_k(t)| + \vartheta_k e^{\lambda t}|x_k(t)|^2 \\ &+ e^{\lambda t}\sum_{h=1}^{l} b_{kh}D_{kh}|x_k(t)|^2 + e^{\lambda t}\sum_{h=1}^{l} b_{kh}D_{kh}|x_h(t-\tau(t))|^2 + e^{\lambda t}|x_k(t)|^2 - (1-\bar{\tau})\, e^{\lambda(t-\tau)}|x_k(t-\tau(t))|^2 \\ &+ e^{\lambda t}l\sum_{h=1}^{l} e_{kh}^2D_{kh}^2\,|x_h(t-\tau(t))|^2 + \theta_k e^{\lambda t}|x_k(t-\tau(t))|^2 \\ &\leq \lambda e^{\lambda t}|x_k(t)|^2 - 2\hat{\alpha}_k e^{\lambda t}|x_k(t)|^2 + 2\hat{\beta}_k e^{\lambda t}|x_k(t-\tau(t))|^2 + e^{\lambda t}(|x_k(t)|^2 + |u_k(t)|^2) + \vartheta_k e^{\lambda t}|x_k(t)|^2 \\ &+ e^{\lambda t}\sum_{h=1}^{l} b_{kh}D_{kh}|x_k(t)|^2 + e^{\lambda t}\sum_{h=1}^{l} b_{kh}D_{kh}|x_h(t-\tau(t))|^2 + e^{\lambda t}|x_k(t)|^2 - (1-\bar{\tau})\, e^{\lambda(t-\tau)}|x_k(t-\tau(t))|^2 \\ &+ e^{\lambda t}l\sum_{h=1}^{l} e_{kh}^2D_{kh}^2\,|x_h(t-\tau(t))|^2 + \theta_k e^{\lambda t}|x_k(t-\tau(t))|^2 \\ &= \left(\lambda - 2\hat{\alpha}_k + 2 + \vartheta_k + \sum_{h=1}^{l} b_{kh}D_{kh}\right)e^{\lambda t}|x_k(t)|^2 + \left(2\hat{\beta}_k - (1-\bar{\tau})\, e^{-\lambda\tau} + \theta_k\right)e^{\lambda t}|x_k(t-\tau(t))|^2 \\ &+ \left(\sum_{h=1}^{l} b_{kh}D_{kh} + l\sum_{h=1}^{l} e_{kh}^2D_{kh}^2\right)e^{\lambda t}|x_h(t-\tau(t))|^2 + e^{\lambda t}|u_k(t)|^2 \end{aligned}$$

$$= \left(\lambda - 2\hat{\alpha}_k + 2 + \vartheta_k + \sum_{h=1}^{l} b_{kh}D_{kh}\right) e^{\lambda t}|x_k(t)|^2 + e^{\lambda t}|u_k(t)|^2$$
$$+ \left(2\hat{\beta}_k - (1-\bar{\tau})\, e^{-\lambda\tau} + \theta_k + \sum_{h=1}^{l} b_{kh}D_{kh} + l\sum_{h=1}^{l} e_{kh}^2 D_{kh}^2\right) e^{\lambda t}|x_k(t-\tau(t))|^2$$
$$+ \left(\sum_{h=1}^{l} b_{kh}D_{kh} + l\sum_{h=1}^{l} e_{kh}^2 D_{kh}^2\right) e^{\lambda t} F_{kh}(x_k(t-\tau(t)), x_h(t-\tau(t))),$$

where $F_{kh}(x_k(t-\tau(t)), x_h(t-\tau(t))) = |x_h(t-\tau(t))|^2 - |x_k(t-\tau(t))|^2$.

According to the condition A3, it is not difficult that there exists a sufficiently small positive constant λ which satisfies

$$2\hat{\beta}_k - (1-\bar{\tau})\, e^{-\lambda\tau} + \theta_k + \sum_{h=1}^{l} b_{kh}D_{kh} + l\sum_{h=1}^{l} e_{kh}^2 D_{kh}^2 - (1-\bar{\tau})\left(2\hat{\alpha}_k - \vartheta_k - \sum_{h=1}^{l} b_{kh}D_{kh} - 2 - \lambda\right) < 0. \tag{18}$$

Therefore, (5) of Theorem 2.2 can be justified. Moreover, the condition V3 can be satisfied easily. That is, all conditions of Theorem 2.2 are fulfilled. Hence, the system (1) is pMEISS. This completes the proof of Theorem 2.7. □

Remark 2.8. *It is not difficult to see that the conditions of Theorem 2.7 can be more convenient to be justified compared with those of Theorem 2.2, because the conditions of Theorem 2.7 are built according to the coefficients and the strong connectedness of the digraph* $(\mathcal{G}, A)$.

3. An Example of Stochastic Coupled Oscillators with Control Inputs

In this section, to explain concrete practicability of the model stated before, we shall discuss a typical application of stochastic coupled oscillators with control inputs, which is shown as follows:

$$\begin{aligned} &\ddot{x}_k(t) + \varphi_k(x_k(t))\dot{x}_k(t) + x_k(t) + \varepsilon_k x_k(t-\tau(t)) + \sum_{h=1}^{l} b_{kh}N_{kh}(x_h(t-\tau(t))) + u_k(t) \\ &= \left[g_k(x_k(t)) + \sum_{h=1}^{l} e_{kh}M_{kh}(x_h(t-\tau(t)))\right]\dot{B}(t), \end{aligned} \tag{19}$$

where ε_k is a positive constant for $k \in \mathrm{L}$. Here N_{kh} and $M_{kh} : \mathbb{R}^1 \to \mathbb{R}^1$ for $k, h \in \mathrm{L}$, are both coupling functions. Also, $g_k : \mathbb{R}^1 \to \mathbb{R}^1$ indicates the random perturbation of the k-th subsystem for $k \in \mathrm{L}$.

Based on the model we present, now we will give a theorem.

Theorem 3.1. *The system (19) is pMEISS if the following conditions hold for any* $k, h \in \mathrm{L}$.

C1. There are positive constants m_k, M_k, and p_k such that $|g_k(x_k)| \le p_k|x_k|$ and $m_k \le \varphi_k(x_k) \le M_k$.

C2. There exists a constant ϖ_{kh} such that

$$|N_{kh}(x_h)| \vee |M_{kh}(x_h)| \le \varpi_{kh}\, |x_h|\,.$$

C3. There exist a positive constant γ_k satisfying that $\gamma_k + 2M_k \le 4$, $4 - 2M_k \ge \gamma_k$ and $\gamma_k \le m_k$ such that

$$\hat{\eta}_k - (1-\bar{\tau})\hat{\delta}_k < 0,$$

where $\hat{\eta}_k = \frac{\varepsilon_k}{2} - (1-\bar{\tau}) + \sum_{h=1}^{l} b_{kh}\varpi_{kh} + l\sum_{h=1}^{l} e_{kh}^2\varpi_{kh}^2$ and $\hat{\delta}_k = 2m_k - 2\gamma_k - \gamma_k M_k - 2\gamma_k^2 - 2\varepsilon_k - p_k^2 - \sum_{h=1}^{l} b_{kh}\varpi_{kh} - 2$.

Proof. Define $z_k(t) = \dot{x}_k(t) + \gamma_k x_k(t)$. Subsequently, the equation (19) is rewritten as

$$\begin{cases} \mathrm{d}x_k(t) = [z_k(t) - \gamma_k x_k(t)]\,\mathrm{d}t, \\ \mathrm{d}z_k(t) = \left[(\gamma_k - \varphi_k(x_k(t)))z_k(t) + (\gamma_k\varphi_k(x_k(t)) - \gamma_k^2)x_k(t) - x_k(t) - \varepsilon_k x_k(t-\tau(t)) - \sum_{h=1}^{l} b_{kh}N_{kh}(x_h(t-\tau(t))) - u_k(t)\right]\mathrm{d}t \\ \qquad + \left[g_k(x_k) + \sum_{h=1}^{l} e_{kh}M_{kh}(x_h(t-\tau(t)))\right]\mathrm{d}B(t). \end{cases}$$

Let

$$X_k(t) = \begin{pmatrix} x_k(t) \\ z_k(t) \end{pmatrix}$$

as a vector function and

$$F_k(X_k(t),t) = \begin{pmatrix} z_k(t) - \gamma_k x_k(t) \\ (\gamma_k - \varphi_k(x_k(t)))z_k(t) + (\gamma_k\varphi_k(x_k(t)) - \gamma_k^2)x_k(t) - x_k(t) - \varepsilon_k x_k(t-\tau(t)) \end{pmatrix}.$$

Hence, it follows that

$$\begin{aligned} X_k^T(t)F_k(X_k(t),t) &\le (\gamma_k\varphi_k(x_k(t)) - \gamma_k^2)|z_k(t)||x_k(t)| - \gamma_k\,|x_k(t)|^2 - (\varphi_k(x_k(t)) - \gamma_k)\,|z_k(t)|^2 - \varepsilon_k x_k(t-\tau(t))z_k(t) \\ &\le \gamma_k M_k|z_k(t)||x_k(t)| - \gamma_k^2|z_k(t)||x_k(t)| - \gamma_k\,|x_k(t)|^2 - m_k\,|z_k(t)|^2 + \gamma_k\,|z_k(t)|^2 - \varepsilon_k x_k(t-\tau(t))z_k(t) \\ &\le \frac{\gamma_k M_k}{2}\left(|z_k(t)|^2 + |x_k(t)|^2\right) - \gamma_k^2|z_k(t)||x_k(t)| - \gamma_k\,|x_k(t)|^2 - m_k\,|z_k(t)|^2 + \gamma_k\,|z_k(t)|^2 - \varepsilon_k x_k(t-\tau(t))z_k(t). \end{aligned}$$

Using the inequality that $(\frac{b}{2})^2 + c^2 \ge -bc$, then one has that

$$\begin{aligned} X_k^T(t)F_k(X_k(t),t) &\le \frac{\gamma_k M_k}{2}\left(|z_k(t)|^2 + |x_k(t)|^2\right) + \gamma_k^2\left(|z_k(t)|^2 + \frac{|x_k(t)|^2}{4}\right) - \gamma_k\,|x_k(t)|^2 - m_k\,|z_k(t)|^2 \\ &\quad + \gamma_k\,|z_k(t)|^2 + \varepsilon_k\left(\frac{|x_k(t-\tau(t))|^2}{4} + |z_k(t)|^2\right) \\ &\le \left(-\gamma_k + \frac{\gamma_k M_k}{2} + \frac{\gamma_k^2}{4}\right)|x_k(t)|^2 + \left(\gamma_k - m_k + \frac{\gamma_k M_k}{2} + \gamma_k^2 + \varepsilon_k\right)|z_k(t)|^2 \\ &\quad + \frac{\varepsilon_k}{4}|x_k(t-\tau(t))|^2 \\ &\le -\left(m_k - \gamma_k - \frac{\gamma_k M_k}{2} - \gamma_k^2 - \varepsilon_k\right)|X_k|^2 + \frac{\varepsilon_k}{4}|x_k(t-\tau(t))|^2. \end{aligned}$$

Therefore, by integrating C2 with C3, all conditions in Theorem 2.7 can be satisfied, that is, the system (19) is *p*MEISS. □

4. Numerical Test

In this section, we give an example to show our results.
Consider a coupled system,

$$\begin{aligned} &\ddot{x}_k(t) + \varphi_k(x_k(t))\dot{x}_k(t) + x_k(t) + \varepsilon_k x_k(t-\tau(t)) + \sum_{h=1}^{6} b_{kh}N_{kh}(x_h(t-\tau(t))) + u_k(t) \\ &= \left[g_k(x_k(t)) + \sum_{h=1}^{6} e_{kh}M_{kh}(x_h(t-\tau(t)))\right]\dot{B}(t),\ k = 1,2,\cdots,6. \end{aligned} \tag{20}$$

Let $\varphi_1(x_1) = 1.5 + 0.2\cos(x_1)$, $\varphi_2(x_2) = 1.5 - 0.2\cos(x_2)$, $\varphi_3(x_3) = 1.5 + 0.2\sin(x_3)$, $\varphi_4(x_4) = 1.5 - 0.2\sin(x_4)$, $\varphi_5(x_5) = 1.5 + 0.1\cos(x_5)$, $\varphi_6(x_6) = 1.5 - 0.1\cos(x_6)$. Set $\varepsilon_k = \frac{1}{20}$ where $k = 1, 2, \cdots, 6$. The initial control input $u_k(t) = 1$ where $k = 1, 2, \cdots, 6$. In addition, $g_k(x_k)$ is assumed as $g_k(x_k) = 0.1x_k$ for $k = 1, 2, \cdots, 6$. Moreover, given $M_{kh}(x_h) = N_{kh}(x_h) = x_h$, let $b_{kh} = \frac{1}{200}$ and $e_{kh} = \frac{1}{100}$ for $k, h = 1, 2, \cdots, 6$. We further set time-varying delay $\tau(t) = 0.1\arctan t$.

We now testify the conditions of Theorem 3.1 for $k, h = 1, 2, \cdots, 6$.

First, let $m_k = 1.3$ and $M_k = 1.7$, then we have $m_k \le \varphi_k(x_k) \le M_k$. What's more, one can get that $|g_k(x_k)| \le 0.1|x_k|$, from which we can know that $p_k = 0.1$. Therefore, the condition C1 of Theorem 3.1 is satisfied.

Then, we can obtain that $|N_{kh}(x_h)| \vee |M_{kh}(x_h)| \le |x_h|$. That is, $\varpi_{kh} = 1$. Hence, the condition C2 of Theorem 3.1 is fulfilled.

Finally, we choose $\gamma_k = 0.1 \le m_k$ satisfying $\gamma_k + 2M_k = 3.5 \le 4$ and $4 - 2M_k = 0.6 \ge \gamma_k$. Moreover, let $\bar{\tau} = 0.941$ for the fact that $\dot{\tau}(t) = \frac{0.1}{1+t^2} \le \bar{\tau}$. Furthermore, we can get $\hat{\eta}_k - (1-\bar{\tau})\hat{\delta}_k = -0.0755 < 0$. Consequently, the condition C3 of Theorem 3.1 is satisfied.

Then all the conditions in Theorem 3.1 are fulfilled. Therefore, under the conditions of Theorem 3.1 the system (20) is pMEISS. On the other hand, we calculate $\hat{\alpha}_k = m_k - \gamma_k - \frac{\gamma_k M_k}{2} - \gamma_k^2 - \varepsilon_k = 1.055$ and $\hat{\beta}_k = \frac{\varepsilon_k}{4} = \frac{1}{80}$. Then, we set $\lambda = 0.01$ and $\tau = 1$ satisfying that the value of the left side of (18) is $-0.0034 < 0$. By calculation, we can solve the result that $\eta_k = 0.0002 > 0$, $\delta_k = 0.06 > 0$ and $\xi_k = 1 > 0$. Now we denote $|x_k(0)| = 0.1414$ for $k = 1, 2, \cdots, 6$ as the initial conditions associated with system (20). Given $c_k = 1$ and $\sigma = 1$, we can get that $\mathbb{E}|x(t;\phi)|^2 \le 2.0204\mathbb{E}||\phi||^2 e^{-0.01t} + 100$ when $u_k = 1$ (see Fig.1) and $\mathbb{E}|x(t;\phi)|^2 \le 2.0204\mathbb{E}||\phi||^2 e^{-0.01t}$ when $u_k = 0$ (see Fig.2). Thereby, according to the Definition 1.1, the system (20) is pMEISS. To show our results more straight, we present numerical simulations which verify the effectiveness and feasibility of the developed results.

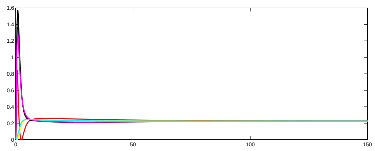

Figure 1: A pMEISS solution of the system (20) with $u = (1, 1, 1, 1, 1, 1)^T$.

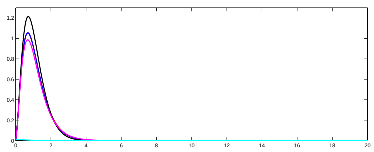

Figure 2: A pMEISS solution of the system (20) with $u = (0, 0, 0, 0, 0, 0)^T$.

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