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I-Statistical Limit Superior and I- Statistical Limit Inferior

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Abstract. In this paper we have extended the concepts of I-limit superior and I-limit inferior to I-statistical limit superior and I-statistical limit inferior and studied some of their properties for sequence of real numbers.

1. Introduction

The idea of convergence of real sequences had been extended to statistical convergence by Fast [6]. Later on it was further investigated from sequence space point of view and linked with summability theory by Fridy [7] and Salat [21] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [3, 13, 18, 19]. The idea is based on the notion of natural density of subsets of N, the set of all positive integers which is defined as follows: The natural density of a subset A of N denoted as $d(A)$ is defined by $d(A) = lim_{n\to\infty}\frac{1}{n}|\{k < n : k \in A\}|$. As a natural consequence, statistical limit superior and limit inferior came up for considerations which was studied extensively by Fridy and Orhan [9].

The notion of ideal convergence was introduced by Kostyrko et al. [17] which generalizes and unifies different notion of convergence of sequences including usual convergence and statistical convergence. They used the notion of an ideal I of subsets of the set N to define such a concept. For an extensive view of this article one may refer [11, 16]. In 2001, Demirci [10] introduced the definition of I-limit superior and inferior of a real sequence and proved several basic properties. Later on it was further investigated by Lahiri and Das [2].

The idea of $I-$statistical convergence was introduced by Savas and Das [4] as an extension of ideal convergence. Later on it was further investigated by Savas and Das [5], Debnath and Debnath [20], Et et al. [12] and many others.

In this paper, we will introduce the concepts of I-statistical limit superior and I-statistical limit inferior .

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2. Definitions and Preliminaries

Definition 2.1. [17] Let X be a non-empty set. A family of subsets $I \subset P(X)$ is called an ideal in X if
(i) $\emptyset \in I$;
(ii) for each $A, B \in I$ implies $A \cup B \in I$;
(iii) for each $A \in I$ and $B \subset A$ implies $B \in I$.

Definition 2.2. [17] Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called an filter in X if
(i) $\emptyset \notin \mathcal{F}$;
(ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;
(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. The filter $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$ is called the filter associated with the ideal I. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if $I \supset \{\{x\} : x \in X\}$.

Definition 2.3. [17] Let I be an ideal on N. A sequence $x = \{x_n\}$ of real numbers is said to be I-convergent to $l \in R$ where R is the set of all real numbers if for every $\varepsilon > 0$, $A(\varepsilon) = \{n : |x_n - l| \geq \varepsilon\} \in I$. In this case we write I-lim $x = l$.

Definition 2.4. [10] Let I be an admissible ideal in N and let $x = \{x_n\}$ be a real sequence. Let $B_x = \{b \in R : \{k : x_k > b\} \notin I\}$ and $A_x = \{a \in R : \{k : x_k < a\} \notin I\}$.
Then the I- limit superior of x is given by
$$I\text{- }\limsup x = \begin{cases} sup\ B_x, & if\ B_x \neq \emptyset \\ -\infty, & if\ B_x = \emptyset \end{cases}.$$
and the I- limit inferior of x is given by
$$I\text{- }\liminf x = \begin{cases} inf\ A_x, & if\ A_x \neq \emptyset \\ \infty, & if\ A_x = \emptyset \end{cases}.$$

Definition 2.5. [10] A real sequence $x = \{x_n\}$ is said to be I- bounded if there is a number $B > 0$ such that $\{k : |x_k| > B\} \in I$.

Definition 2.6. [4] A sequence $\{x_n\}$ is said to be $I-$statistically convergent to L if for each $\varepsilon > 0$ and every $\delta > 0$,
$\left\{n \in N : \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in I.$
L is called $I-$statistical limit of the sequence $\{x_n\}$ and we write, $I - st\,lim\,x_n = L$.
Throughout the paper we consider I as an admissible ideal.

3. Main Results

In this section we study the concepts of I-statistical limit superior and I-statistical limit inferior for a real number sequence. For a real sequence $x = (x_n)$ let B_x denote the set
$B_x = \left\{b \in R : \left\{n \in N : \frac{1}{n}|\{k \leq n : x_k > b\}| > \delta\right\} \notin I\right\}$.
Similarly, $A_x = \left\{a \in R : \left\{n \in N : \frac{1}{n}|\{k \leq n : x_k < a\}| > \delta\right\} \notin I\right\}$.

Definition 3.1. Let, x be a real number sequence. Then I-statistical limit superior of x is given by,
$$I\text{-st }\limsup x = \begin{cases} sup\ B_x, & if\ B_x \neq \emptyset \\ -\infty & if\ B_x = \emptyset \end{cases}.$$
Also, I-statistical limit inferior of x is given by,

I-st lim inf x=$\begin{cases} inf\, A_x, & if\ A_x \neq \emptyset \\ \infty & if\ A_x = \emptyset \end{cases}$.

Theorem 3.1. If $\beta = I$-st lim sup x is finite, then for every positive number ε,
$\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k > \beta - \varepsilon\}| > \delta\right\} \notin I$ and $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k > \beta + \varepsilon\}| > \delta\right\} \in I$.
Similarly, If $\alpha = I$-st lim inf x is finite, then for every positive number ε,
$\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k < \alpha + \varepsilon\}| > \delta\right\} \notin I$ and $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k < \alpha - \varepsilon\}| > \delta\right\} \in I$.

Proof. It follows from the definition.

Theorem 3.2. For any real number sequence x, I-st lim inf $x \leq I$-st lim sup x.

Proof.
Case-I: If I-lim sup $x = -\infty$, then we have $B_x = \emptyset$. So for every $b \in R$,
$\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k > b\}| > \delta\right\} \in I$
which implies, $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k > b\}| < \delta\right\} \in \mathcal{F}(I)$
i.e, $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k < b\}| > \delta\right\} \in \mathcal{F}(I)$
so for every $a \in R$, $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k < a\}| > \delta\right\} \notin I$.
Hence, I-st lim inf $x = -\infty$ (since $A_x = R$) .
Case-II: If I-lim sup $x = \infty$, then we need no proof.
Case-III: Let $\beta = I$-st lim sup x is finite and $\alpha = I$-st lim inf x.
So for $\varepsilon > 0$, $\delta > 0$, $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k > \beta + \varepsilon\}| > \delta\right\} \in I$
this implies, $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k < \beta + \varepsilon\}| > \delta\right\} \in \mathcal{F}(I)$
i.e, $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k < \beta + \varepsilon\}| > \delta\right\} \notin I$.
So, $\beta + \varepsilon \in A_x$. Since ε was arbitrary and by definition α =inf A_x. Therefore, $\alpha < \beta + \varepsilon$. This proves that $\alpha \leq \beta$.

Definition 3.2. The real number sequence $x = (x_n)$ is said to be I-st bounded if there is a number G such that $\left\{n \in N : \frac{1}{n}|\{k \leq n : |x_k| > G\}| > \delta\right\} \in I$.

Remark 3.1. If a sequence is I-st bounded then I-st lim sup and I-st lim inf of that sequence are finite.

Definition 3.3. An element ξ is said to be an $I-$statistical cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$
$\left\{n \in N : \frac{1}{n}|\{k \leq n : |x_k - \xi| \geq \varepsilon\}| < \delta\right\} \notin I$.

Theorem 3.3. If a I-statistically bounded sequence has one cluster point then it is I-statistically convergent.

Proof. Let (x_n) be a I-statistically bounded sequence which has one cluster point.
Then $M = \left\{n \in N : \frac{1}{n}|\{k \leq n : |x_k| > G\}| > \delta\right\} \in I$.
So, there exist a set $M' = \{n_1 < n_2 < ....\} \subset N$ such that $M' \notin I$ and (x_{n_k}) is a statistically bounded sequence.
Now , since (x_n) has only one cluster point and (x_{n_k}) is a statistically bounded subsequence of (x_n), So (x_{n_k}) also has only one cluster point. Hence (x_{n_k}) is statistically convergent.
Let, St-lim$x_{n_k} = \xi$, then for any $\varepsilon > 0$ and $\delta > 0$ we have the inclusion,
$\left\{n \in N : \frac{1}{n}|\{k \leq n : |x_k - \xi| \geq \varepsilon\}| \geq \delta\right\} \subseteq M \cup A \in I$ where A is a finite set.
i.e, (x_n) is I-statistically convergent to ξ.

Theorem 3.4. A sequence x is I-st convergent if and only if I-st lim inf x = I-st lim sup x, provided x is I-st bounded.

Proof. Let $\alpha = I\text{-st}\liminf x$ and $\beta = I\text{-st}\limsup x$. Let $I\text{-st}\lim x = L$ so, $\left\{n \in N : \frac{1}{n}|\{k \le n : |x_k - L| \ge \varepsilon\}| \ge \delta\right\} \in I$.

i.e, $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k > L + \varepsilon\}| \ge \delta\right\} \in I$ which implies $\beta \le L$.

We also have $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k < L - \varepsilon\}| \ge \delta\right\} \in I$ which implies $L \le \alpha$. Therefore, $\beta \le \alpha$. But we know that, $\alpha \le \beta$. i.e, $\alpha = \beta$.

Now let $\alpha = \beta$ and define $L = \alpha$.

for $\varepsilon > 0$, $\delta > 0$ $\left\{n \in N : \frac{1}{n}|\left\{k \le n : x_k > L + \frac{\varepsilon}{2}\right\}| > \delta\right\} \in I$

and $\left\{n \in N : \frac{1}{n}|\left\{k \le n : x_k < L - \frac{\varepsilon}{2}\right\}| > \delta\right\} \in I$

i.e, $\left\{n \in N : \frac{1}{n}|\{k \le n : |x_k - L| > \varepsilon\}| > \delta\right\} \in I$. So x is I-statistical convergent.

Theorem 3.5. If x, y are two I-st bounded sequences, then

(i) I-st lim sup $(x + y) \le I$-st lim sup x + I-st lim sup y.

(ii) I-st lim inf $(x + y) \ge I$-st lim inf x + I-st lim inf y.

Proof. (i) Let, $l_1 = I$-st lim sup x and $l_2 = I$-st lim sup y.

So, $\left\{n \in N : \frac{1}{n}|\left\{k \le n : x_k > l_1 + \frac{\varepsilon}{2}\right\}| > \delta\right\} \in I$

and $\left\{n \in N : \frac{1}{n}|\left\{k \le n : y_k > l_2 + \frac{\varepsilon}{2}\right\}| > \delta\right\} \in I$

Now, $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k + y_k > l_1 + l_2 + \varepsilon\}| > \delta\right\} \subset \left\{n \in N : \frac{1}{n}|\left\{k \le n : x_k > l_1 + \frac{\varepsilon}{2}\right\}| > \delta\right\} \cup \left\{n \in N : \frac{1}{n}|\left\{k \le n : y_k > l_2 + \frac{\varepsilon}{2}\right\}|\right.$

so, $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k + y_k > l_1 + l_2 + \varepsilon\}| > \delta\right\} \in I$.

If $c \in B_{(x+y)}$, then by definition $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k + y_k > c\}| > \delta\right\} \notin I$. We show that $c < l_1 + l_2 + \varepsilon$.

If $c \ge l_1 + l_2 + \varepsilon$ then

$\left\{n \in N : \frac{1}{n}|\{k \le n : x_k + y_k > c\}| > \delta\right\} \subseteq \left\{n \in N : \frac{1}{n}|\{k \le n : x_k + y_k > l_1 + l_2 + \varepsilon\}| > \delta\right\}$

Therefore $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k + y_k > c\}| > \delta\right\} \in I$ which is a contradiction.

Hence, $c < l_1 + l_2 + \varepsilon$. As this is true for all $c \in B_{(x+y)}$,

so, I-st lim sup $(x + y)$ =sup $B_{(x+y)} < l_1 + l_2 + \varepsilon$.

Since, $\varepsilon > 0$ is arbitrary so,

I-st lim sup $(x + y) \le I$-st lim sup x + I-st lim sup y.

Definition 3.4. A sequence x is said to be I-st convergent to $+\infty$ (or$-\infty$) if for every real number $G > 0$, $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k \le G\}| > \delta\right\} \in I$ (or, $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k \ge -G\}| > \delta\right\} \in I$).

Theorem 3.6. If I-st lim sup $x = l$, then there exists a subsequence of x that is I-st convergent to l.

Proof.

Case-I: If $l = -\infty$ then $B_x = \emptyset$.

So for any real number $G > 0$, $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k \ge -G\}| > \delta\right\} \in I$

i.e, I-st lim $x = -\infty$.

Case-II: If $l = +\infty$, then $B_x = R$. So for any $b \in R$, $\left\{n \in N : \frac{1}{n}|\{k \le n : x_k > b\}| > \delta\right\} \notin I$. Let, x_{n_1} be arbitrary member of x and so,

$A_{n_1} = \left\{n \in N : \frac{1}{n}|\{k \le n : x_k > x_{n_1} + 1\}| > \delta\right\} \notin I$. Since, I is an admissible ideal, so A_{n_1} must be an infinite set.

i.e, $d(\{k \le n : x_k > x_{n_1} + 1\}) \ne 0$. We claim that there is atleast $k \in \{k \le n : x_k > x_{n_1} + 1\}$ such that $k > n_1 + 1$, for otherwise $\{k \le n : x_k > x_{n_1} + 1\} \subseteq \{1, 2, ...n_1, n_1 + 1\}$ i.e, $d(\{k \le n : x_k > x_{n_1} + 1\}) \le d(\{1, 2, ...n_1, n_1 + 1\}) = 0$, which is a contradiction.

We call this k as n_2, thus $x_{n_2} > x_{n_1} + 1$. Proceeding in this way we obtain a subsequence $\{x_{n_k}\}$ of x with $x_{n_k} > x_{n_{k-1}} + 1$. Since for any $G > 0$,

$\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k \leq G\}| > \delta\right\} \in I$ so, I-st lim $x_{n_k} = +\infty$.

Case-III: $-\infty < l < +\infty$. So, $\left\{n \in N : \frac{1}{n}|\left\{k \leq n : x_k > l + \frac{1}{2}\right\}| > \delta\right\} \in I$

and $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k > l - 1\}| > \delta\right\} \notin I$

So there must be a m in this set for which

$\frac{1}{m}|\{k \leq m : x_k > l - 1\}| > \delta$ and $\frac{1}{m}|\left\{k \leq m : x_k \leq l + \frac{1}{2}\right\}| > \delta$.

For otherwise $\left\{n \in N : \frac{1}{n}|\{k \leq n : x_k > l - 1\}| > \delta\right\} \subset \left\{n \in N : \frac{1}{n}|\left\{k \leq n : x_k > l + \frac{1}{2}\right\}| > \delta\right\} \in I$, which is a contradiction.

Now for maximum $k \leq m$ will satisfy $x_k > l - 1$ and $x_k \leq l + \frac{1}{2}$ so we must have a n_1 for which $l - 1 < x_{n_1} \leq l + \frac{1}{2} < l + 1$.

Next we proceed to choose an element x_{n_2} from x, $n_2 > n_1$ such that $l - \frac{1}{2} < x_{n_2} < l + \frac{1}{2}$.

Now $\left\{n \in N : \frac{1}{n}|\left\{k \leq n : x_k > l - \frac{1}{2}\right\}| > \delta\right\}$ is an infinite set. so, $d\left(\left\{k \leq n : x_k > l - \frac{1}{2}\right\}\right) \neq 0$. We observe that there is at least one $k > n_1$ for which $x_k > l - \frac{1}{2}$, for otherwise $d\left(\left\{k \leq n : x_k > l - \frac{1}{2}\right\}\right) \leq d(\{1, 2, ...n_1\}) = 0$ which is a contradiction.

Let $E_{n_1} = \left\{k \leq n : k > n_1\ x_k > l - \frac{1}{2}\right\} \neq \emptyset$ if $k \in E_{n_1}$ always implies $x_k \geq l + \frac{1}{2}$ then,

$E_{n_1} \subseteq \left\{k \leq n : x_k > l + \frac{1}{2}\right\}$

i.e, $d(E_{n_1}) \leq d\left(\left\{k \leq n : x_k > l + \frac{1}{2}\right\}\right) = 0$. Since, $\left\{n \in N : \frac{1}{n}|\left\{k \leq n : x_k > l + \frac{1}{2}\right\}| < \delta\right\} \in \mathcal{F}(I)$

Thus, $\left\{k \leq n : x_k > l - \frac{1}{2}\right\} \subseteq \{1, 2, ...n_1\} \cup E_{n_1}$

So, $d\left(\left\{k \leq n : x_k > l - \frac{1}{2}\right\}\right) \leq d(\{1, 2, ...n_1\}) + d(E_{n_1}) \leq 0$, which is a contradiction.

This shows that there is a $n_2 > n_1$ such that $l - \frac{1}{2} < x_{n_2} < l + \frac{1}{2}$. Proceeding in this way we obtain a sub sequence $\{x_{n_k}\}$ of x, $n_k > n_{k-1}$ such that $l - \frac{1}{k} < x_{n_k} < l + \frac{1}{k}$ for each k. This subsequence $\{x_{n_k}\}$ ordinarily converges to l and thus I-st convergent to l.

Theorem 3.7. If I-st lim inf $x = l$, then there exists a subsequence of x that is I-st convergent to l.

Proof. The proof is analogous to Theorem 3.6 and so omitted.

Theorem 3.8. Every I-st bounded sequence x has a subsequence which is I-st convergent to a finite real number.

Proof. The proof follows from Remark 3.1 and Theorem 3.6.

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