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Ćirić-Berinde Fixed Point Theorems for Multi-Valued Mappings on $\alpha-$ Complete Metric-Like Spaces

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Abstract. The concept of Hausdorff metric-like has been initiated in [7]. Using this concept, we introduce Ćirić-Berinde type contractive multi-valued mappings via α-admissible mappings on metric-like spaces and we establish several fixed point results. We show that many known fixed point results in literature are simple consequences of our theorems. Our obtained results are supported by some examples and an application.

1. Introduction and preliminaries

In 2012, Samet *et al.* [31] introduced the notions of $\alpha-\psi-$contractive mappings and $\alpha-$admissible mappings in metric spaces and obtained many nice fixed point results. Since then, several authors investigated fixed point results in this direction, for more details see [2, 3, 5, 11, 15, 19, 20, 30, 31]. The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [26]. Recently, Ali *et al.* [2] generalized and extended the notion of $\alpha-\psi-$contractive mappings by introducing the notion of $(\alpha,\psi,\xi)-$contractive multi-valued mappings and gave fixed point theorems in the setting of complete metric spaces. In 2015, Kutbi and Sintunavarat[23] extended this notion in the class of $\alpha-$complete metric spaces and they established new fixed point results. Very recently, Cho [13] introduced a class of Ćirić-Berinde type contractive multi-valued mappings using α-admissible functions and established some fixed point results on metric spaces. On the other hand, Aydi et al. [9, 10] introduced the notion of a partial Hausdorff metric and provided some (common) fixed point results. Very recently, Aydi et al. [7, 8] introduced the concept of Hausdorff metric-like.

The purpose of this paper is to introduce the notion of Ćirić-Berinde type contractive multi-valued mappings on $\alpha-$complete metric-like spaces via α-admissible mappings and the Hausdorff metric-like concept. We will establish some fixed point theorems involving such contractions on $\alpha-$complete metric-like spaces. Some examples and an application will be provided.

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Mention that metric-like spaces have been rediscovered by Amini-Harandi [18]. Some fixed point results in the setting of metric-like spaces have also been established in [18]. For more other fixed point results on metric-like spaces, see [1, 4, 6, 14, 16, 17, 21, 22, 32–35].

At first, denote $\mathbb{R}^+$ the set of nonnegative reals.

Definition 1.1. *Let X be a nonempty set. A function $\sigma : X \times X \to \mathbb{R}^+$ is said to be a metric-like (dislocated metric) on X if for any $x, y, z \in X$, the following conditions hold:*

(P_1) $\sigma(x, y) = 0 \Longrightarrow x = y$;

(P_2) $\sigma(x, y) = \sigma(y, x)$;

(P_3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is then called a metric-like (or a dislocated metric) space.

It is known that a partial metric [24] is also a metric-like. So a trivial example of a metric-like space is the pair $(\mathbb{R}^+, \sigma)$, where $\sigma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $\sigma(x, y) = \max\{x, y\}$.

In the following example, we give a metric-like which is neither a metric nor a partial metric.

Example 1.1. (see [12]) *Take $X = \{1, 2, 3\}$ and consider the metric-like $\sigma : X^2 \to \mathbb{R}^+$ given by*

$$\sigma(1, 1) = 0, \quad \sigma(2, 2) = 1, \quad \sigma(3, 3) = \frac{2}{3},$$

$$\sigma(1, 2) = \sigma(2, 1) = \frac{9}{10}, \quad \sigma(2, 3) = \sigma(3, 2) = \frac{4}{5},$$

and

$$\sigma(1, 3) = \sigma(3, 1) = \frac{7}{10}.$$

Having $\sigma(2, 2) \neq 0$, so σ is not a metric and due to $\sigma(2, 2) > \sigma(1, 2)$, so σ is not a partial metric [24].

Each metric-like σ on X generates a T_0 topology τ_σ on X which has as a base the family of open σ-balls $\{B_\sigma(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_\sigma(x, \varepsilon) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Observe that a sequence $\{x_n\}$ in a metric-like space (X, σ) converges to a point $x \in X$, with respect to τ_σ, if and only if $\sigma(x, x) = \lim_{n\to\infty} \sigma(x, x_n)$.

Definition 1.2. *Let (X, σ) be a metric-like space and $\alpha : X \times X \to [0, \infty)$ be a given mapping.*

(a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite.

(b) (X, σ) is said to be $\alpha-$complete if every Cauchy sequence $\{x_n\}$ in X verifying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$, converges to a point $x \in X$, that is, $\lim_{n\to\infty} \sigma(x, x_n) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m)$.

Remark 1. *If X is a complete metric-like space, then X is also an $\alpha-$complete metric-like space. But, the converse is not true. The following example asserts this statement.*

Example 1.2. *Let $X = (0, \infty)$ and consider the metric-like $\sigma : X \times X \to [0, \infty)$ defined by $\sigma(x, y) = x + y$ for all $x, y \in X$. Define $\alpha : X \times X \to [0, \infty)$ by*

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [1, 2] \\ 0, & \textit{otherwise}. \end{cases}$$

Note that (X, σ) is not a complete metric-like space. Indeed, we argue by contradiction, that is, we suppose that (X, σ) is a complete metric-like space. Take the sequence $\{x_n\} = \{\frac{1}{n}\}$ in X. We have $\sigma(x_n, x_m) = \frac{1}{n} + \frac{1}{m}$. Then, $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$ and so $\{x_n\}$ is a Cauchy sequence. It follows that, there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0.$$

Then, $x = 0$, which is a contradiction. Hence, (X, σ) is not a complete metric-like space.

Now, we shall prove that (X, σ) is an $\alpha-$complete metric-like space. In fact, if $\{x_n\}$ is a Cauchy sequence in (X, σ) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$, then $x_n \in [1, 2]$ for all $n \geq 1$. Moreover, it is easy to see $[1, 2]$ is a closed subset of $(\mathbb{R}^+, \sigma)$ and since $(\mathbb{R}^+, \sigma)$ is a complete metric-like space, it follows that $([1, 2], \sigma)$ is a complete metric-like space. Hence, there exists $x^\star \in [1, 2]$ such that $\lim_{n\to\infty} x_n = x^\star$ in (X, σ). Then, (X, σ) is an $\alpha-$complete metric-like space.

We need in the sequel the following trivial inequality:

$$\sigma(x, x) \leq 2\sigma(x, y) \quad \text{for all } x, y \in X. \tag{1}$$

Following [7, 8], let $CB^\sigma(X)$ be the family of all nonempty, closed and bounded subsets of the metric-like space (X, σ), induced by the metric-like σ. Note that the boundedness is given as follows: A is a bounded subset in (X, σ) if there exist $x_0 \in X$ and $M > 0$ such that for all $a \in A$, we have $a \in B_\sigma(x_0, M)$, that is,

$$|\sigma(x_0, a) - \sigma(a, a)| < M.$$

The closedness is taken in (X, τ_σ) (where τ_σ is the topology induced by σ). Let $\bar{A}$ be the closure of A with respect to the metric-like σ. We have

Definition 1.3.

$$\begin{aligned} a \in \bar{A} &\Longleftrightarrow B_\sigma(a, \varepsilon) \cap A \neq \emptyset \quad \textit{for all } \varepsilon > 0 \\ &\Longleftrightarrow \textit{ there exists } \{x_n\} \subset A, \quad x_n \to a \textit{ in } (X, \sigma). \end{aligned}$$

If $A \in CB^\sigma(X)$, then $\bar{A} = A$.

For $A, B \in CB^\sigma(X)$ and $x \in X$, define

$$\begin{aligned} \sigma(x, A) &= \inf\{\sigma(x, a),\ a \in A\},\ \delta_\sigma(A, B) = \sup\{\sigma(a, B) : a \in A\} \quad \text{and} \\ \delta_\sigma(B, A) &= \sup\{\sigma(b, A) : b \in B\}. \end{aligned}$$

Lemma 1.1. (see [7, 8]) *Let (X, σ) be a metric-like space and A any nonempty set in (X, σ), then*

$$\textit{if } \sigma(a, A) = 0, \quad \textit{then } a \in \bar{A}. \tag{2}$$

Also, if $\{x_n\}$ is a sequence in $(X; \sigma)$ that is τ_σ-convergent to $x \in X$, then

$$\lim_{n\to\infty} |\sigma(x_n, A) - \sigma(x, A)| = \sigma(x, x). \tag{3}$$

Let (X, σ) be a metric-like space. For $A, B \in CB^\sigma(X)$, define $H_\sigma : CB^\sigma(X) \times CB^\sigma(X) \to [0, \infty)$ by

$$H_\sigma(A, B) = \max\{\delta_\sigma(A, B), \delta_\sigma(B, A)\}.$$

In the following, we present some properties of H_σ.

Proposition 1.1. (see [7, 8]) *Let (X, σ) be a metric-like space. For any $A, B, C \in CB^\sigma(X)$, we have the following:*

(i) : $H_\sigma(A, A) = \delta_\sigma(A, A) = \sup\{\sigma(a, A) : a \in A\}$;

(ii) : $H_\sigma(A, B) = H_\sigma(B, A)$;

(iii) : $H_\sigma(A, B) = H_\sigma(B, A) = 0$ *implies that* $A = B$;

(iv) : $H_\sigma(A, B) \leq H_\sigma(A, C) + H_\sigma(C, B)$.

Remark 2. *The converse of Proposition 1.1, (ii) is not true in general as it is clear from the following example.*

Example 1.3. (see [7, 8]) *Let $X = \{0, 1\}$ be endowed with the metric-like $\sigma : X \times X \to \mathbb{R}^+$ defined by*

$$\sigma(1,1) = 2 \quad \textit{and } \sigma(0,0) = \sigma(0,1) = \sigma(1,0) = 1.$$

Note that σ is not a partial metric since $\sigma(1,1) > \sigma(1,0)$. From (i) of Proposition 1.1, we have

$$\begin{aligned} H_\sigma(X,X) = \delta_\sigma(X,X) &= \sup\{\sigma(x,X),\ x \in \{0,1\}\} \\ &= max\{\sigma(0,\{0,1\}), \sigma(1,\{0,1\})\} = 1 \neq 0. \end{aligned}$$

Remark 3. *Mention that a Hausdorff metric is a Hausdorff metric-like. The converse is not true (see Example 1.3).*

In view of Proposition 1.1, the mapping $H_\sigma : CB^\sigma(X) \times CB^\sigma(X) \to [0, +\infty)$ is called a Hausdorff metric-like induced by σ. We also call it a dislocated Hausdorff metric.

From now on, we denote by

$$M(x,y) := \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{1}{4}\{\sigma(x,Ty) + \sigma(y,Tx)\}\}$$

for a multi-valued map $T : X \to CB^\sigma(X)$ and $x, y \in X$.

We denote by Γ the class of all functions $\xi : [0,\infty) \to [0,\infty)$ such that
(Γ_1) ξ is continuous at 0 and $\xi^{-1}(\{0\}) = \{0\}$;
(Γ_2) ξ is nondecreasing on $[0,\infty)$;
(Γ_3) ξ is subadditive (i.e, $\xi(a+b) \leq \xi(a) + \xi(b)$ for all $a, b \geq 0$).

Remark 4. *If $\xi \in \Gamma$, we get $\xi(t) > 0$ for all $t > 0$.*

Example 1.4. *Let $\xi : [0,\infty) \to [0,\infty)$ be defined by $\xi(t) = \dfrac{t}{1+t}$ for all $t \geq 0$. It is easy to see that $\xi \in \Gamma$.*

Example 1.5. (see [23]) *Let $\xi : [0,\infty) \to [0,\infty)$ be defined by*

$$\xi(t) = \int_0^t \phi(s)ds$$

for any $t \geq 0$, where $\phi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping, summable on each compact subset of $[0,\infty)$ and satisfies the following conditions:
(i) for each $\epsilon > 0$, we have $\int_0^\epsilon \phi(s)ds > 0$;
(ii) for each $a, b > 0$, we have

$$\int_0^{a+b} \phi(s)ds \leq \int_0^a \phi(s)ds + \int_0^b \phi(s)ds.$$

Then, $\xi \in \Gamma$.

Now, let $L \geq 0$ be a real number and we denote by Ψ_L the family of increasing functions $\psi : [0,\infty) \to [-2L,\infty)$ such that $\sum_n (\psi + 2L\mathbb{I}d)^n(t) < \infty$ for each $t > 0$, where ψ^n is the $n-$th iterate of ψ and $\mathbb{I}d(t) = t$ for all $t \geq 0$.

A simple example of $\psi \in \Psi_L$ is $\psi(t) = (k - 2L)t$ where $k \in (0,1)$. We have the following useful lemma.

Lemma 1.2. *If $\psi \in \Psi_L$, the following properties hold:*
(i) $0 < \psi(t) + 2Lt < t$ for any $t > 0$,
(ii) ψ is continuous at 0 and $\psi(0) = 0$

Let (X,σ) be a metric-like space and let $\alpha : X \times X \to [0,\infty)$ be a given function.

Definition 1.4. *A function $f : X \to [0,\infty)$ is called $\alpha-$lower semi-continuous if for any $x \in X$ and $\{x_n\} \subset X$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = x$ in (X,σ), we have*

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

For a multi-valued map $T : X \to CB^\sigma(X)$, consider $f_T : X \to [0,\infty)$ defined by

$$f_T(x) = \sigma(x, Tx).$$

Lemma 1.3. *Let (X,σ) be a metric-like space. If $\xi \in \Gamma$, then $(X, \xi\ o\ \sigma)$ is a metric-like space too.*

Definition 1.5. *Let (X,σ) be a metric-like space and $T : X \to CB^\sigma(X)$ be a multi-valued mapping. Then, we say that*
(1) T is called $\alpha_\star-$admissible [5] if

$$\alpha(x,y) \geq 1 \quad \textit{implies } \alpha_\star(Tx,Ty) \geq 1,$$

where $\alpha_\star(Tx,Ty) := inf\{\alpha(a,b) :\ a \in Tx, b \in Ty\}$;
(2) T is called $\alpha-$admissible [25] if for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, we have $\alpha(x,z) \geq 1$ for all $z \in Ty$.

We have the following analog lemma as in [13].

Lemma 1.4. *Let (X,σ) be a metric-like space and $T : X \to CB^\sigma(X)$ be a multi-valued mapping. If T is $\alpha_\star-$admissible, then it is $\alpha-$admissible.*

We consider the following condition:
(H): for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

In this paper, we introduce the concept of Ćirić-Berinde type contractive multi-valued mappings on metric-like spaces via α-admissible mappings and Hausdorff metric-like concept. We establish some fixed point results for multi-valued mappings involving the above contractions. We will present some concrete examples and an illustrated application on fixed point results in metric-like spaces endowed with a graph.

2. Fixed point of multi-valued contraction mappings

We start with the following simple useful lemma. One may find its analogous for the metric case in [13].

Lemma 2.1. *Let (X,σ) be a metric-like space, and let $\xi \in \Gamma$ and $B \in CB^\sigma(X)$. If $a \in X$ and $\xi(\sigma(a,B)) < c$ where $c > 0$, then there exists $b \in B$ such that $\xi(\sigma(a,b)) < c$.*

Now, we state and prove our main result.

Theorem 2.1. *Let (X,σ) be a metric-like space and $\alpha : X \times X \to [0,+\infty)$ be a function. Suppose that $T : X \to CB^\sigma(X)$ is an $\alpha-$admissible multi-valued mapping such that for all $x,\ y\ \in X$, with $\alpha(x,y) \geq 1$, we have*

$$\xi(H_\sigma(Tx,Ty)) \leq \psi(\xi(M(x,y))) + L\xi(\sigma(y,Tx)) \tag{4}$$

where $L \geq 0$, $\xi \in \Gamma$ and $\psi \in \Psi_L$.
Suppose also that the following conditions are satisfied:

1. (X,σ) *is an $\alpha-$complete metric-like space;*

2. *there exit $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;*

3. (H) *is verified or f_T is $\alpha-$lower semi- continuous.*

Then, T has a fixed point in X.

Proof. By condition (2), there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) \geq 1.$$

Let $c = 1 + \xi(\sigma(x_0, x_1))$. Take $\psi_L(t) = \psi(t) + 2Lt$ for all $t \geq 0$ where $\psi \in \Psi_L$.

Clearly, if $x_0 = x_1$ or $x_1 \in Tx_1$, we deduce that x_1 is a fixed point of T and so this completes the proof. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$. So, $\sigma(x_1, Tx_1) > 0$. Since $\sigma(x_1, Tx_1) \leq H_\sigma(Tx_0, Tx_1)$, then by (4) and triangular inequality, we have

$$\begin{aligned}
0 < \xi(\sigma(x_1, Tx_1)) &\leq \xi(H_\sigma(Tx_0, Tx_1))) \\
&\leq \psi[\xi(\max\{\sigma(x_0, x_1), \sigma(x_0, Tx_0), \sigma(x_1, Tx_1), \\
&\quad \frac{1}{4}(\sigma(x_0, Tx_1) + \sigma(x_1, Tx_0))\})] + L\xi(\sigma(x_1, Tx_0)) \\
&\leq \psi[\xi(\max\{\sigma(x_0, x_1), \sigma(x_0, x_1), \sigma(x_1, Tx_1), \\
&\quad \frac{1}{4}(\sigma(x_0, Tx_1) + \sigma(x_1, x_1))\})] + L\xi(\sigma(x_1, x_1)) \\
&\leq \psi[\xi(\max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1), \\
&\quad \frac{1}{4}(\sigma(x_1, Tx_1) + 3\sigma(x_0, x_1))\})] + 2L\xi(\sigma(x_0, x_1)) \\
&\leq \psi[\xi(\max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1)\})] + 2L\xi(M(x_0, x_1)) \\
&= \psi_L(\xi(\max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1)\})).
\end{aligned}$$

If $\sigma(x_1, Tx_1) > \sigma(x_0, x_1)$, then we have

$$0 < \xi(\sigma(x_1, Tx_1)) \leq \psi_L[\xi(\sigma(x_1, Tx_1))] < \xi(\sigma(x_1, Tx_1)),$$

which is a contradiction. Since ψ_L is increasing, we have

$$0 < \xi(\sigma(x_1, Tx_1)) \leq \psi_L[\xi(\sigma(x_0, x_1))] < \psi_L(c).$$

Hence by lemma 2.1, there exists $x_2 \in Tx_1$ such that

$$\xi(\sigma(x_1, x_2)) < \psi_L(c).$$

Since T is $\alpha-$admissible and $x_2 \in Tx_1$, we have

$$\alpha(x_1, x_2) \geq 1.$$

If $x_2 \in Tx_2$ then x_2 is a fixed point. Let $x_2 \notin Tx_2$, then $\sigma(x_2, Tx_2) > 0$.
Since $\sigma(x_2, Tx_2) \leq H_\sigma(Tx_1, Tx_2)$ and $\alpha(x_1, x_2) \geq 1$, then by (4), we have

$$\begin{aligned}
0 < \xi(\sigma(x_2, Tx_2)) &\leq \xi(H_\sigma(Tx_1, Tx_2))) \leq \psi[\xi(\max\{\sigma(x_1, x_2), \sigma(x_2, Tx_2), \sigma(x_1, Tx_1), \\
&\quad \frac{1}{4}(\sigma(x_1, Tx_2) + \sigma(x_2, Tx_1))\})] + L\xi(\sigma(x_2, Tx_1)) \\
&\leq \psi_L[\xi(\max\{\sigma(x_1, x_2), \sigma(x_2, Tx_2), \\
&\quad \frac{3}{4}\sigma(x_1, x_2) + \frac{1}{4}\sigma(x_2, Tx_2))\}] + 2L\xi(\sigma(x_1, x_2)) \\
&\leq \psi_L[\xi(\max\{\sigma(x_1, x_2), \sigma(x_2, Tx_2)\})].
\end{aligned}$$

If $\sigma(x_1, x_2) < \sigma(x_2, Tx_2)$, we have

$$0 < \xi(\sigma(x_2, Tx_2)) \leq \psi_L[\xi(\sigma(x_2, Tx_2))] < \xi(\sigma(x_2, Tx_2)),$$

which is a contradiction. Since ψ_L is increasing, we have

$$0 < \xi(\sigma(x_2, Tx_2)) \leq \psi_L[\xi(\sigma(x_1, x_2))] < \psi_L^2(c).$$

By lemma 2.1, there exists $x_3 \in Tx_2$ such that

$$\xi(\sigma(x_2, x_3)) < \psi_L^2(c).$$

Since T is $\alpha-$admissible and $x_3 \in Tx_2$, we have $\alpha(x_2, x_3) \geq 1$.
By induction, we construct a sequence $\{x_n\} \subset X$ such that, for all $n \in \mathbb{N}$,
$\alpha(x_n, x_{n+1}) \geq 1$, $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$, and

$$\xi(\sigma(x_n, x_{n+1})) < \psi_L^n(c).$$

In view of $\sum_n \psi_L^n(c) < \infty$, we have for all $p \geq 0$

$$\xi(\sigma(x_n, x_{n+p})) \leq \sum_{k=n}^{n+p-1} \xi(\sigma(x_k, x_{k+1})) \leq \sum_{k=n}^{n+p-1} \psi_L^k(c) \leq \sum_{k=n}^{\infty} \psi_L^k(c) \to 0 \text{ as } n \to \infty.$$

Let $\epsilon > 0$. There exists $n \in \mathbb{N}$ such that for all $n \geq N$

$$\xi(\sigma(x_n, x_{n+p})) \leq \sum_{k=n}^{\infty} \psi_L^k(c) < \xi(\epsilon).$$

By symmetry of σ, we get $\sigma(x_n, x_m) < \epsilon$ for all $m, n \geq N$, so

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0.$$

Hence, $\{x_n\}$ is $\sigma-$Cauchy sequence in X. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, it follows from the $\alpha-$completeness of (X, σ) that exists $x^\star \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x^\star) = \sigma(x^\star, x^\star) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0.$$

Now, we should prove that $x^\star$ is a fixed point of T.

Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^\star$, then by hypothesis (H), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^\star) \geq 1$ for all $k \in \mathbb{N}$. Assume that $\sigma(x^\star, Tx^\star) > 0$.

Let $k \in \mathbb{N}$. We have

$$\begin{aligned}\sigma(x^\star, Tx^\star) &\leq \sigma(x^\star, x_{n(k)+1}) + \sigma(x_{n(k)+1}, Tx^\star)\\ &\leq \sigma(x^\star, x_{n(k)+1}) + H_\sigma(Tx_{n(k)}, Tx^\star).\end{aligned}$$

From (4), we get

$$\begin{aligned}\xi(\sigma(x^\star, Tx^\star)) &\leq \xi(\sigma(x^\star, x_{n(k)+1})) + \xi(H_\sigma(Tx_{n(k)}, Tx^\star))\\ &\leq \xi(\sigma(x^\star, x_{n(k)+1})) + \psi_L[\xi(M(x_{n(k)}, x^\star))]\end{aligned}$$

where

$$\begin{aligned}M(x_{n(k)}, x^\star) = \max\{&\sigma(x_{n(k)}, x^\star), \sigma(x^\star, Tx^\star), \sigma(x_{n(k)}, Tx_{n(k)}),\\ &\frac{1}{4}(\sigma(x_{n(k)}, Tx^\star) + \sigma(x^\star, Tx_{n(k)}))\}.\end{aligned}$$

Since $\lim_{n\to\infty} \sigma(x_n, x^\star) = \sigma(x^\star, x^\star) = 0$, all sequences $\{\sigma(x_{n(k)}, x^\star)\}$, $\{\sigma(x_{n(k)}, Tx_{n(k)})\}$, $\{\sigma(x^\star, Tx_{n(k)})\}$ converge to 0 and $\lim_{k\to\infty} \sigma(x_{n(k)}, Tx^\star) = \sigma(x^\star, Tx^\star)$. These facts ensure that there exists $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq N$

$$M(x_{n(k)}, x^\star) = \sigma(x^\star, Tx^\star).$$

Then for all $k \geq N$, we have

$$\xi(\sigma(x^\star, Tx^\star)) \leq \xi(\sigma(x^\star, x_{n(k)+1})) + \psi_L[\xi(\sigma(x^\star, Tx^\star))].$$

Having in mind that ξ is continuous at 0 and $\xi(0) = 0$, so we have

$$\lim_{k\to\infty} \xi(\sigma(x^\star, x_{n(k)+1})) = 0.$$

Thus, by taking $k \to \infty$

$$\xi(\sigma(x^\star, Tx^\star)) \leq \psi_L[\xi(\sigma(x^\star, Tx^\star))] < \xi(\sigma(x^\star, Tx^\star))$$

which is a contradiction. We deduce that $\sigma(x^\star, Tx^\star) = 0$. By lemma 1.1, we have $x^\star \in \overline{Tx^\star} = Tx^\star$. Then, $x^\star$ is a fixed point of T.

Now, passing to the case where f_T is $\alpha-$lower semi-continuous, we obtain

$$f_T(x^\star) \leq \liminf_{n\to\infty} \sigma(x_n, Tx_n) \leq \lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0.$$

Thus $\sigma(x^\star, Tx^\star) = 0$ and so $x^\star \in Tx^\star$. □

2.1. Some consequences

We state the following simple corollaries as consequences of Theorem 2.1.

Corollary 2.1. *Let (X, σ) be a metric-like space and $\alpha : X\times X \to [0, +\infty)$ be a function. Suppose that $T : X \to CB^\sigma(X)$ is an $\alpha-$admissible multi-valued mapping such that for all $x,\ y \in X$, with $\alpha(x, y) \geq 1$, we have*

$$\xi(\alpha(x, y)H_\sigma(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(\sigma(y, Tx)) \tag{5}$$

where $L \geq 0$, $\xi \in \Gamma$ and $\psi \in \Psi_L$.

Suppose also that the following conditions are satisfied:

1. *(X, σ) is an $\alpha-$complete metric-like space;*
2. *there exit $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
3. *(H) is verified or f_T is $\alpha-$lower semi- continuous.*

Then T has a fixed point in X.

Corollary 2.2. (see [7]) *Let (X, σ) be a complete metric-like space. If $T : X \to CB^\sigma(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_\sigma(Tx, Ty) \leq k\, M(x, y) \tag{6}$$

where $k \in [0, 1)$ and

$$M(x, y) = \max\left\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{1}{4}\left(\sigma(x, Ty) + \sigma(y, Tx)\right)\right\}.$$

Then, T has a fixed point in X.

Proof. It suffices to take $\alpha(x, y) = 1$, for all $x, y \in X$ and $\psi(t) = kt$ where $k \in (0, 1)$ and $L = 0$ in Corollary 2.1. □

Corollary 2.3. *Let (X, σ) be a metric-like space and $\alpha : X \times X \to [0, +\infty)$ be a function. Suppose that $T : X \to X$ is an $\alpha-$admissible mapping such that for all $x,\ y \in X$, with $\alpha(x, y) \geq 1$, we have*

$$\xi(H_\sigma(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(\sigma(y, Tx)) \tag{7}$$

where $L \geq 0$, $\xi \in \Gamma$ and $\psi \in \Psi_L$.

Suppose also that the following conditions are satisfied:

1. *(X,σ) is an $\alpha-$complete metric-like space;*
2. *there exits $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *(H) is verified or f_T is $\alpha-$lower semi- continuous.*

Then T has a fixed point in X.

Proof. It suffices to take T as a single-valued mapping in Theorem 2.1. □

Definition 2.1. *Let (X,σ) be a metric-like space, a is given point in X and let $T : X \to CB^{\sigma}(X)$ be a multi-valued mapping.*

(a) (X,σ) is said to be $(a,T)-$complete if every Cauchy sequence $\{x_n\}$ in X, such that $a \in Tx_n \cap Tx_{n+1}$ for all $n \geq 1$, converges to a point $x^\star \in X$, that is, $\lim\limits_{n\to\infty} \sigma(x^\star, x_n) = \sigma(x^\star, x^\star) = \lim\limits_{n,m\to\infty} \sigma(x_n, x_m)$.

(b) A function $f : X \to [0,\infty)$ is called $(a,T)-$lower semi-continuous if, for any $y \in X$ and $\{x_n\} \subset X$ with $a \in Tx_n \cap Tx_{n+1}$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} x_n = y$ in (X,σ), we have

$$f(y) \leq \liminf_{n\to\infty} f(x_n).$$

We also provide the following result.

Theorem 2.2. *Let (X,σ) be a metric-like space and $T : X \to CB^{\sigma}(X)$ be a multi-valued mapping. Take $a \in X$. Assume that, there exist two functions $\xi \in \Gamma$ and $\psi \in \Psi_L$ such that*

$$\xi(H_\sigma(Tx,Ty)) \leq \psi(\xi(M(x,y))) + L\xi(\sigma(y,Tx)) \tag{8}$$

for all $x, y \in X$, with $a \in Tx \cap Ty$. Suppose also that

1. *(X,σ) is $(a,T)-$complete metric-like space;*
2. *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $a \in Tx_0 \cap Tx_1$;*
3. *for each $x \in X$ and $y \in Tx$ with $a \in Tx \cap Ty$, we have $a \in Ty \cap Tz$ for all $z \in Ty$;*
4. *f_T is $(a,T)-$lower semi-continuous or for a sequence $\{x_n\} \subset X$ with $a \in Tx_n \cap Tx_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to x$ in (X,σ), then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_\star \in Tx_{n(k)} \cap Tx$, for all $k \in \mathbb{N}$.*

Then T has a fixed point in X.

Proof. Let the function $\alpha : X \times X \to [0,\infty)$ be such that

$$\alpha(x,y) = \begin{cases} 1 & if\ a \in Tx \cap Ty \\ 0 & otherwise. \end{cases}$$

By condition (1), (X,σ) is an $\alpha-$complete metric-like space. The multi-valued mapping T is $\alpha-$admissible. In fact, if $x \in X$ and $y \in Ty$ with $\alpha(x,y) \geq 1$, then $a \in Tx \cap Ty$. By condition (2), we have $a \in Ty \cap Tz$ for all $z \in Ty$, then $\alpha(y,z) \geq 1$. By (8), T also verifies (4) of theorem 2.1. Finally, by condition (3), the sequence $\{x_n\}$ verifies hypothesis (H). Thus, all hypotheses of Theorem 2.1 are satisfied and hence T has a fixed point. □

Remark 5. *Theorem 2.1 is the analogous of Theorem 2.1 of Cho [13] on metric-like spaces. Corollary 2.2 extends Corollary 2.5 of Aydi et al. [10] to metric-like spaces.*

2.2. Fixed point theory in ordered metric-like spaces

The study of fixed points in partially ordered sets has been developed in [27–29]. In this subsection, we give some results of fixed point for multi-valued mappings in the concept of metric-like space endowed with a partial order. Thus, a metric-like space $(X, \sigma, \preceq)$ may be naturally endowed with a partial ordering, that is, if $(X, \preceq)$ is a partially ordered set, then $(X, \sigma, \preceq)$ is called an ordered metric-like space. Finally, we say that, $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ holds. For $A, B \subseteq X$, we also have $A \preceq B$ whenever for each $a \in A$ there exists $b \in B$ such that $a \preceq b$.

Definition 2.2. *Let $(X, \sigma, \preceq)$ be an ordered metric-like space and $T : X \to CB^\sigma(X)$ be a multi-valued mapping.*

(a) The metric-like space (X, σ) is said to be $(\preceq, T)-$complete if every Cauchy sequence $\{x_n\}$ in X, with $Tx_n \preceq Tx_{n+1}$, for all $n \geq 1$ converges to a point $x^\star \in X$ such that $\lim_{n\to\infty} \sigma(x^\star, x_n) = \sigma(x^\star, x^\star) = \lim_{n,m\to\infty} \sigma(x_n, x_m)$.

(b) A function $f : X \to [0, \infty)$ is called $(\preceq, T)-$lower semi-continuous if, for any $x \in X$ and $\{x_n\} \subset X$ with $Tx_n \preceq Tx_{n+1}$, for all $n \geq 1$ and $\lim_{n\to\infty} x_n = x$ in (X, σ), we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

We have the following theorem.

Theorem 2.3. *Let $(X, \sigma, \preceq)$ be an ordered metric-like space. Suppose that $T : X \to CB^\sigma(X)$ is a multi-valued mapping. Assume that, there exist two functions $\xi \in \Gamma$ and $\psi \in \Psi_L$ such that*

$$\xi(H_\sigma(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(\sigma(y, Tx)) \tag{9}$$

for all $x, y \in X$, with $Tx \preceq Ty$. Suppose also that

1. *$(X, \sigma, \preceq)$ is $(\preceq, T)-$complete metric-like space;*
2. *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $Tx_0 \preceq Tx_1$;*
3. *for each $x \in X$ and $y \in Tx$ with $Tx \preceq Ty$, we have $Ty \preceq Tz$ for all $z \in Ty$;*
4. *f_T is $(\preceq, T)-$lower semi-continuous or for a sequence $\{x_n\} \subset X$ with $Tx_n \preceq Tx_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to x$ in (X, σ), then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $Tx_{n(k)} \preceq Tx$, for all $k \in \mathbb{N}$.*

Then T has a fixed point in X.

Proof. Let the function $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y) = \begin{cases} 1 & if\ \ Tx \preceq Ty \\ 0 & otherwise. \end{cases}$$

By condition (1), $(X, \sigma, \preceq)$ is an $\alpha-$complete metric-like space. The multi-valued mapping T is $\alpha-$admissible. In fact, if $x \in X$ and $y \in Ty$ with $\alpha(x, y) \geq 1$, then $Tx \preceq Ty$. By condition (2), we have $Ty \preceq Tz$ for all $z \in Ty$, then $\alpha(y, z) = 1$. By (9), T also verifies the contraction (4) of Theorem 2.1. Finally, by condition (3), the sequence $\{x_n\}$ verifies hypothesis (H). Thus all hypotheses of Theorem 2.1 are satisfied and hence T has a fixed point. $\square$

3. Examples

We give the following illustrative examples.

Example 3.1. *Let $X = [0, \infty)$ and $\sigma(x, y) = x + y$ for all $x, y \in X$. Mention that (X, σ) is a complete metric-like space and σ is not a partial metric on X. Define a multi-valued mapping $T : X \to CB^{\sigma}(X)$ by*

$$Tx = \begin{cases} \{\frac{1}{4}x, \frac{1}{9}x\} & if\ 0 \leq x \leq 1 \\ \{\frac{3x}{x+1}\} & if\ x > 1. \end{cases}$$

Let $\xi(t) = \sqrt{t}$ and $\psi(t) = \dfrac{1}{2}t$ for all $t \geq 0$. Then, $\xi \in \Gamma$ and $\psi \in \Psi_L$ for $0 \leq L < \frac{1}{4}$.

Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} cosh(x + y) & if\ 0 \leq x, y \leq 1 \\ \frac{1}{x+y+1} & otherwise. \end{cases}$$

Condition (1) of Theorem 2.1 is satisfied with $x_0 = 1$ and $x_1 = \frac{1}{3}$. Obviously, condition (H) is satisfied. We will show that (4) of Theorem 2.1 is satisfied.

For this, let $x, y \in X$ such that $\alpha(x, y) \geq 1$, then $0 \leq x, y \leq 1$. We have

$$\begin{aligned} \xi(H_{\sigma}(Tx, Ty)) &= (\max\{\delta_{\sigma}(Tx, Ty), \delta_{\sigma}(Ty, Tx)\})^{\frac{1}{2}} \\ &= (\max\{\frac{x}{9} + \frac{y}{4}, \frac{x}{4} + \frac{y}{9}\})^{\frac{1}{2}} \\ &\leq \frac{1}{2}(x + y)^{\frac{1}{2}} = \psi(\xi(\sigma(x, y))) \leq \psi(\xi(M(x, y))) \\ &\leq \psi(\xi(M(x, y))) + L\xi(\sigma(y, Tx)). \end{aligned}$$

Thus, (4) is satisfied.

Now, we show that T is α−admissible. Given $x \in X$ and let $y \in Tx$ be such that $\alpha(x, y) \geq 1$. Then $0 \leq x, y \leq 1$. So $Ty = \{\frac{y}{9}, \frac{y}{4}\}$. Thus, for all $z \in Ty$, we have $\alpha(y, z) \geq 1$. Hence T is α−admissible. Thus all hypotheses of Theorem 2.1 are satisfied and T has a fixed point which is $u = 0$.

Example 3.2. *Let $X = [0, \infty)$ and $\sigma(x, y) = \sqrt{x + y}$ for all $x, y \in X$. It's easy to show that that (X, σ) is a complete metric-like space. σ is neither a metric, nor a partial metric on X. Define the multi-valued mapping $T : X \to CB^{\sigma}(X)$ by*

$$Tx = \begin{cases} \{0, \frac{1}{4}x\} & if\ 0 \leq x \leq 2 \\ [1, 3] & if\ x > 2. \end{cases}$$

Let $\xi(t) = \sqrt{t}$ and $\psi(t) = \dfrac{1}{\sqrt{2}}t$ for all $t \geq 0$. Then $\xi \in \Gamma$ and $\psi \in \Psi_L$ for $0 \leq L < \frac{2-\sqrt{2}}{4}$.

Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 2 + cos(x + y) & if\ 0 \leq x, y \leq 2 \\ \frac{1}{2} & otherwise. \end{cases}$$

Condition (1) of Theorem 2.1 is satisfied with $x_0 = 2$ and $x_1 = \frac{1}{2}$. Obviously, condition (H) is satisfied. We will show that (4) of Theorem 2.1 is satisfied.

For this, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. Then, $0 \leq x, y \leq 2$. In this case, we have

$$\begin{aligned} H_\sigma(Tx, Ty) &= \max\{\delta_\sigma(Tx, Ty), \delta_\sigma(Ty, Tx)\} \\ &= \max\{\frac{\sqrt{x}}{2}, \frac{\sqrt{y}}{2}\} \\ &= \frac{1}{2}\max\{\sqrt{x}, \sqrt{y}\} \\ &= \frac{1}{2}\max\{\sigma(x, Tx), \sigma(y, Ty)\}. \end{aligned}$$

Then

$$\begin{aligned} \xi(H_\sigma(Tx, Ty)) &= \psi(\xi(\max\{\sigma(x, Tx), \sigma(y, Ty)\})) \\ &\leq \psi(\xi(M(x, y))) \leq \psi(\xi(M(x, y))) + L\xi(\sigma(y, Tx)). \end{aligned}$$

Thus (4) is satisfied. Given $x \in X$ and let $y \in Tx$ be such that $\alpha(x, y) \geq 1$. Then $0 \leq x, y \leq 2$, so $Ty = \{0, \frac{y}{4}\}$. Thus, for all $z \in Ty$, we have $\alpha(y, z) \geq 1$. Hence T is $\alpha-$admissible. Thus, all hypotheses of Theorem 2.1 are satisfied. Then, T has a fixed point which is $u = 0$.

4. Fixed point results in metric-like spaces endowed with a graph

In this section, we give fixed point results on metric-like spaces endowed with a graph. Before representing our results, we give the following notations and definitions.
First, let (X, σ) be a metric-like space. A set $\{(x, x) : x \in X\}$ is called a diagonal cartesian product $X \times X$ and is denoted by Δ. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, *i.e.*, $\Delta \subset E(G)$. We assume G has no parallel edges, so we can identify G with the $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

Definition 4.1. (see [23]) *Let X be a nonempty set endowed with a graph G and $T : X \to CB^\sigma(X)$ be a multi-valued mapping. We say that T weakly preserves edges if for each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$.*

Definition 4.2. *Let (X, σ) be a metric-like space endowed with a graph G.*

(a) The metric-like space X is said to be $E(G)-$complete if for every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, converges in (X, σ).

(b) A function $f : X \to [0, \infty)$ is called $E(G)-$lower semi-continuous if for any $x \in X$ and $\{x_n\} \subset X$ with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ in (X, σ), we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Definition 4.3. *Let (X, σ) be a metric-like space endowed with a graph G. A mapping $T : X \to CB^\sigma(X)$ is called an $(E(G), \psi, \xi)$- contractive multi-valued mapping if there exist two functions $\xi \in \Gamma$ and $\psi \in \Psi_L$ such that*

$$\forall x, y \in X, \quad (x, y) \in E(G) \Rightarrow \xi(H_\sigma(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(\sigma(y, Tx)) \tag{10}$$

where $L \geq 0$ and $M(x, y) = \max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{1}{4}\{\sigma(x, Ty) + \sigma(y, Tx)\}\}$.

By using Theorem 2.1, we get the following result.

Theorem 4.1. *Let (X, σ) be a metric-like space endowed with a graph G and $T : X \to CB^\sigma(X)$ be a $(E(G), \psi, \xi)$-contractive mapping. Suppose that the following conditions hold:*

1. *(X, σ) is an $E(G)-$complete metric-like space;*
2. *T weakly preserves edges;*
3. *there exit $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*
4. *f_T is $E(G)-$lower semi-continuous or for any sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $(x_{n(k)}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Then T has a fixed point in X.

Proof. Let the mapping $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in E(G), \\ 0, & \textit{otherwise}. \end{cases}$$

It is easy to see that all conditions of Theorem 2.1 are satisfied and so T has a fixed point. □

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