



On $[m, C]$ -Isometric Operators

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Abstract. In this paper we introduce an $[m, C]$ -isometric operator T on a complex Hilbert space \mathcal{H} and study its spectral properties. We show that if T is an $[m, C]$ -isometric operator and N is an n -nilpotent operator, respectively, then $T + N$ is an $[m + 2n - 2, C]$ -isometric operator. Finally we give a short proof of Duggal's result for tensor product of m -isometries and give a similar result for $[m, C]$ -isometric operators.

1. Introduction

Let \mathcal{H} be a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on \mathcal{H} . For an integer $m \in \mathbb{N}$ and an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an m -isometric operator if

$$\sum_{j=0}^m (-1)^j \binom{m}{j} T^{*m-j} T^{m-j} = 0.$$

In 1995, J. Agler and M. Stankus [1] introduced an m -isometric operator and showed nice results. An antilinear operator C on \mathcal{H} is said to be *conjugation* if C satisfies $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if $CTC = T^*$. In [11], S. Jung, E. Ko, M. Lee and J. Lee studied spectral properties of complex symmetric operators. In [4], M. Chō, E. Ko and J. Lee introduced (m, C) -isometric operators with conjugation C as follows; For an operator $T \in \mathcal{L}(\mathcal{H})$ and an integer $m \geq 1$, T is said to be an (m, C) -isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^m (-1)^j \binom{m}{j} T^{*m-j} \cdot CT^{m-j} C = 0.$$

According to definitions of m -isometry, (m, C) -isometry and complex symmetric, we define an $[m, C]$ -isometry T as follows; An operator T is said to be an $[m, C]$ -isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^m (-1)^j \binom{m}{j} CT^{m-j} C \cdot T^{m-j} = 0.$$

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It is easy to see that if T is complex symmetric and an $[m, C]$ -isometry, then T is an m -isometry. Throughout the paper, let I be the identity operator on \mathcal{H} .

2. Example

(i) Let $\mathcal{H} = \mathbb{C}^2$ and let C be a conjugation on \mathcal{H} given by $C \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{y} \\ \bar{x} \end{pmatrix}$.

If $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on \mathbb{C}^2 , then $CTC = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = T^*$. Since $T^{*2} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, it follows that

$$\sum_{j=0}^2 (-1)^j \binom{2}{j} T^{*2-j} \cdot CT^{2-j}C = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + I = 0.$$

Therefore, T is a $(2, C)$ -isometric operator. On the other hand, T is not a $[2, C]$ -isometric operator due to the fact that

$$\sum_{j=0}^2 (-1)^j \binom{2}{j} CT^{2-j}C \cdot T^{2-j} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + I = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq 0.$$

(ii) Under the same space \mathcal{H} and the same conjugation C to (i), let S be an operator given by $S = \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$.

Then $CSC = \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$ and $CSC = S \neq S^*$. Moreover, it holds $CSC \cdot S - I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - I = 0$ and hence S is a

$[1, C]$ -isometry. But $S^* \cdot CSC - I = \begin{pmatrix} 2 & -2\sqrt{2}i \\ 2\sqrt{2}i & 2 \end{pmatrix} \neq 0$ and hence S is not a $(1, C)$ -isometry.

(iii) Let F and J be conjugations on a Hilbert space \mathcal{H} such that $JF \neq I$. Define T and C by $T = \begin{pmatrix} 0 & FJ \\ I & 0 \end{pmatrix}$

and $C = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$. Then it is easy to see that C is a conjugation on $\mathcal{H} \oplus \mathcal{H}$, $CTC \cdot T = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and $T^* \cdot CTC = \begin{pmatrix} JF & 0 \\ 0 & JF \end{pmatrix} \neq \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Hence T is a $[1, C]$ -isometric operator and not a $(1, C)$ -isometric operator.

3. $[m, C]$ -Isometric Operators

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation C , we define the operator $\lambda_m(T; C)$ by

$$\lambda_m(T; C) = \sum_{j=0}^m (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j}.$$

Then T is an $[m, C]$ -isometry if and only if $\lambda_m(T; C) = 0$. Moreover, it holds that

$$CTC \cdot \lambda_m(T; C) \cdot T - \lambda_m(T; C) = \lambda_{m+1}(T; C). \quad (1)$$

Hence if T is an $[m, C]$ -isometry, then T is an $[n, C]$ -isometry for every $n \geq m$.

Let C be a conjugation on \mathcal{H} . Then C satisfies $\|Cx\| = \|x\|$ and $C(\alpha x) = \bar{\alpha}Cx$ for all $x \in \mathcal{H}$ and all $\alpha \in \mathbb{C}$. Moreover, since $C^2 = I$, it follows that $(CTC)^n = CT^nC$ and $(CTC)^* = CT^*C$ for every positive integer n (see [10] for more details). For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_p(T)$ and $\sigma_a(T)$ be the point spectrum and the approximate point spectrum of T , respectively. We denote the range of T by $R(T)$. Then we have

Theorem 3.1. Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$ -isometric operator. Then the following statements hold:

- (i) T is bounded below.
- (ii) $0 \notin \sigma_a(T)$.
- (iii) T is injective and $R(T)$ is closed.

Proof. If $0 \in \sigma_a(T)$, then there exists a sequence of unit vectors $\{x_n\}$ of \mathcal{H} such that $\lim_{n \rightarrow \infty} Tx_n = 0$. Since T is an $[m, C]$ -isometric operator, it follows that

$$\sum_{j=0}^{m-1} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j} = (-1)^{m+1}I. \quad (2)$$

Moreover, since $\lim_{n \rightarrow \infty} \left(\sum_{j=0}^{m-1} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j} \right) x_n = 0$, it follows from (2) that $\lim_{n \rightarrow \infty} x_n = 0$, which is a contradiction. Hence $0 \notin \sigma_a(T)$. Since (i), (ii), and (iii) are equivalent, this completes the proof. \square

Theorem 3.2. Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$ -isometric operator. If $\alpha \in \sigma_a(T)$, then $\bar{\alpha}^{-1} \in \sigma_a(T)$. In particular, if α is an eigenvalue of T , then $\bar{\alpha}^{-1}$ is an eigenvalue of T .

Proof. Let $\{x_n\}$ be a sequence of unit vectors such that $\lim_{n \rightarrow \infty} (T - \alpha)x_n = 0$. Since T is an $[m, C]$ -isometric operator, C is bounded, and $\lim_{n \rightarrow \infty} (T^k - \alpha^k)x_n = 0$ for all $k \in \mathbb{N}$, it holds that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left(\sum_{j=0}^m (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j} x_n \right) \\ &= C \lim_{n \rightarrow \infty} \left(\sum_{j=0}^m (-1)^j \binom{m}{j} T^{m-j} \bar{\alpha}^{m-j} \right) Cx_n = C \lim_{n \rightarrow \infty} (\bar{\alpha} T - 1)^m Cx_n. \end{aligned}$$

Moreover, since $C^2 = I$, it holds $\lim_{n \rightarrow \infty} (\bar{\alpha} T - 1)^m Cx_n = 0$. Since $\|Cx_n\| = 1$ and $\alpha \neq 0$ by Theorem 3.1, it follows that $\lim_{n \rightarrow \infty} (T - \bar{\alpha}^{-1})^m Cx_n = 0$ and hence $\bar{\alpha}^{-1} \in \sigma_a(T)$. \square

Corollary 3.3. Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$ -isometric operator. Then $\|T\| \geq 1$.

Proof. If $0 < \|T\| < 1$, then there exists $\alpha \in \sigma(T)$ and a sequence $\{x_n\}$ of unit vectors such that $0 < |\alpha| < 1$ and $\|(T - \alpha)x_n\| \rightarrow 0$. By Theorem 3.2, it holds $\bar{\alpha}^{-1} \in \sigma(T)$. Since $|\bar{\alpha}^{-1}| > 1$, it is a contradiction. \square

Theorem 3.4. Let C be a conjugation on \mathcal{H} and let $T \in \mathcal{L}(\mathcal{H})$. Then the following assertions hold.

- (i) If T is an invertible, then T is an $[m, C]$ -isometric operator if and only if so is T^{-1} .
- (ii) If T is an $[m, C]$ -isometric operator, then T^n is also an $[m, C]$ -isometric operator for any $n \in \mathbb{N}$.

Proof. (i) Suppose that T is invertible and an $[m, C]$ -isometry. Since $C^2 = I$, it follows that

$$\begin{aligned} 0 &= (CT^{-m}C) \left[\sum_{j=0}^m (-1)^j \binom{m}{j} (CT^{m-j}C) T^{m-j} \right] T^{-m} \\ &= \sum_{j=0}^m (-1)^j \binom{m}{j} (C(T^{-1})^j C) (T^{-1})^j. \end{aligned}$$

Since

$$\sum_{j=0}^m (-1)^j \binom{m}{j} (C(T^{-1})^{m-j} C) (T^{-1})^{m-j} = 0$$

is equivalent to

$$\sum_{j=0}^m (-1)^j \binom{m}{j} (C(T^{-1})^j C)(T^{-1})^j = 0,$$

T^{-1} is an $[m, C]$ -isometry. Hence the statement (i) holds.

(ii) Since

$$\begin{aligned} a^n - 1)^m &= (a - 1)^m (a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a + 1)^m \\ &= (a - 1)^m (\xi_0 a^{m(n-1)} + \xi_1 a^{m(n-1)-1} + \xi_2 a^{m(n-1)-2} + \cdots + \xi_{m(n-1)}) \end{aligned}$$

where ξ_i are coefficients ($i = 0, \dots, m(n-1)$), it follows that

$$\lambda_m(T^n; C) = \sum_{i=0}^{m(n-1)} \xi_i C T^{m(n-1)-i} C \cdot \lambda_m(T; C) \cdot T^{m(n-1)-i}. \quad (3)$$

From (3), if $\lambda_m(T; C) = 0$, then $\lambda_m(T^n; C) = 0$. Hence T^n is an $[m, C]$ -isometric operator for any $n \in \mathbb{N}$. So this completes the proof. \square

An operator $N \in \mathcal{L}(\mathcal{H})$ is said to be n -nilpotent if $N^n = 0$ ($n \in \mathbb{N}$). In [2] T. Bermúdez, A. Martínón, V. Müller and A.J. Noda proved the following.

Proposition 3.5. (Theorem 3.1, [2]) *Let T be an m -isometry on \mathcal{H} and N be an n -nilpotent operator such that $TN = NT$. Then $T + N$ is an $(m + 2n - 2)$ -isometry.*

We have following similar result.

Theorem 3.6. *Let T be an $[m, C]$ -isometric operator on \mathcal{H} and N be an n -nilpotent operator such that $TN = NT$. Then $T + N$ is an $[m + 2n - 2, C]$ -isometry.*

Proof. In the proof, we denote $\lambda_m(T; C)$ by $\lambda_m(T)$ simply. First we show

$$\lambda_m(T + N) = \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^j C \cdot \lambda_k(T) \cdot T^j \cdot N^i, \quad (4)$$

where $\binom{m}{i, j, k} = \frac{m!}{i! \cdot j! \cdot k!}$ and $\lambda_0(*) = I$. It is easy to see that (4) holds for $m = 1$. Assume that (4) holds for m . Then by (1) we have

$$\begin{aligned} \lambda_{m+1}(T + N) &= C(T + N)C \cdot \lambda_m(T + N) \cdot (T + N) - \lambda_m(T + N) \\ &= (C(T + N)C) \left[\sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C C N^j C \lambda_k(T) T^j N^i \right] (T + N) \\ &\quad - \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C C N^j C \lambda_k(T) T^j N^i \\ &= \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C C N^j C \left(C T C \lambda_k(T) T - \lambda_k(T) \right) T^j N^i \\ &\quad + \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C C N^{j+1} C \lambda_k(T) T^{j+1} N^i \\ &\quad + \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^{i+1} C C N^j C \lambda_k(T) T^j N^{i+1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i+j+k=m} \binom{m}{i, j, k} C(T+N)^i C C N^j C \lambda_{k+1}(T) T^j N^i \\
&\quad + \sum_{i+j+k=m} \binom{m}{i, j, k} C(T+N)^i C C N^{j+1} C \lambda_k(T) T^{j+1} N^i \\
&\quad + \sum_{i+j+k=m} \binom{m}{i, j, k} C(T+N)^{i+1} C C N^j C \lambda_k(T) T^j N^{i+1} \\
&= \sum_{i+j+k=m+1} \binom{m+1}{i, j, k} C(T+N)^i C C N^j C \lambda_k(T) T^j N^i.
\end{aligned}$$

Hence (4) holds for $m+1$ and holds for any $m \in \mathbb{N}$. By (4) it holds

$$\lambda_{m+2n-2}(T+N) = \sum_{i+j+k=m+2n-2} \binom{m+2n-2}{i, j, k} C(T+N)^i C C N^j C \lambda_k(T) T^j N^i.$$

(i) If $\max\{i, j\} \geq n$, then $C N^j C = 0$ or $N^i = 0$.

(ii) If $\max\{i, j\} \leq n-1$, then $k \geq m$ and hence $\lambda_k(T) = 0$.

By (i) and (ii), we have $\lambda_{m+2n-2}(T+N) = 0$. Therefore, $T+N$ is an $[m+2n-2, C]$ -isometric operator. \square

Remark 3.7. Let $T \in \mathcal{L}(\mathcal{H})$. If $\beta_m(T)$ is defined by

$$\beta_m(T) = \sum_{j=0}^m (-1)^j \binom{m}{j} T^{*m-j} T^{m-j},$$

then T is an m -isometric operator if and only if $\beta_m(T) = 0$. Since, for any commuting pair (T, S) , it follows that

$$\beta_m(T+S) = \sum_{i+j+k=m} \binom{m}{i, j, k} (T+S)^{*i} \cdot S^{*j} \cdot \beta_k(T) \cdot T^j \cdot S^i.$$

So we have other proof of Proposition 3.5.

From Theorem 3.6, we get the following corollary.

Corollary 3.8. If T is a $[1, C]$ -isometric operator on \mathcal{H} and N is an n -nilpotent operator such that $TN = NT$, then $T+N$ is an $[2n-1, C]$ -isometry.

Example 3.9. Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$ on \mathbb{C}^3 . If $T = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on \mathbb{C}^3 , then

$$\begin{aligned}
T &= I + N \text{ where } N = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \text{ Thus we have } T^2 = \begin{pmatrix} 1 & 0 & 2a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T^3 = \begin{pmatrix} 1 & 0 & 3a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, CTC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \bar{a} & 0 & 1 \end{pmatrix}, \\
CT^2C &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\bar{a} & 0 & 1 \end{pmatrix}, \text{ and } CT^3C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3\bar{a} & 0 & 1 \end{pmatrix}. \text{ Then we have}
\end{aligned}$$

$$\lambda_3(T; C) = CT^3CT^3 - 3CT^2CT^2 + 3CTCT - I = 0.$$

On the other hand, since $N^2 = 0$, it follows from Theorem 3.6 that T is a $[3, C]$ -isometric operator.

For an operator $T \in \mathcal{L}(\mathcal{H})$, the numerical range $W(T)$ of T is $W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *convexoid* if $\overline{W(T)} = \text{co } \sigma(T)$, that is, the closure of $W(T)$ is equal to the convex hull of $\sigma(T)$. An operator T is called *power bounded* if there exists a positive number M such that $\|T^n\| \leq M$ for all $n \in \mathbb{N}$.

Theorem 3.10. *Let T be a $[2, C]$ -isometric operator. If T is power bounded and $CTC \cdot T - I$ is convexoid, then T is a $[1, C]$ -isometric operator.*

Proof. For the proof, we will show that $W(CTC \cdot T - I) = \{0\}$. Assume that $W(CTC \cdot T - I) \neq \{0\}$. Since $CTC \cdot T - I$ is convexoid, it holds $\overline{W(CTC \cdot T - I)} = \text{co } \sigma(CTC \cdot T - I)$. Then there exist a non-zero $a \in \mathbb{C}$ and a sequence $\{x_n\}$ of unit vectors in \mathcal{H} such that $\lim_{n \rightarrow \infty} (CTC \cdot T - I - a)x_n = 0$. Since T is a $[2, C]$ -isometric operator, it holds $\lim_{n \rightarrow \infty} (CT^2C \cdot T^2 - (1 + 2a))x_n = 0$. Inductively, we have

$$\lim_{n \rightarrow \infty} (CT^n C \cdot T^n - (1 + na))x_n = 0.$$

Therefore, it holds that $\|CT^n C \cdot T^n\| \geq |1 + na|$. Since $a \neq 0$, it follows that $\lim_{n \rightarrow \infty} |1 + na| = \infty$. Since T is power bounded, so is $CT^n C \cdot T^n$ and hence it is a contradiction. \square

4. Tensor Products of $[m, C]$ -Isometric Operators

For a complex Hilbert space \mathcal{H} , let $\mathcal{H} \otimes \mathcal{H}$ denote the completion of the algebraic tensor product of \mathcal{H} and \mathcal{H} endowed a reasonable uniform cross-norm. For operators $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$, $T \otimes S \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ denote the *tensor product* operator defined by T and S . Note that $T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$. Then B. Duggal in [9] proved the following result.

Proposition 4.1. (Theorem 2.10, [9]) *Let T and S be an m -isometry and an n -isometry on \mathcal{H} , respectively. Then $T \otimes S$ is an $(m + n - 1)$ -isometry on $\mathcal{H} \otimes \mathcal{H}$.*

Since Duggal's proof is long and difficult, we firstly give a short proof. A pair of operators (T, S) is said to be a *doubly commuting pair* if (T, S) satisfies $TS = ST$ and $T^*S = ST^*$. Then, for a doubly commuting pair (T, S) , it holds

$$\beta_m(TS) = \sum_{k=0}^m \binom{m}{k} T^{*k} \cdot \beta_{m-k}(T) \cdot T^k \cdot \beta_k(S). \quad (5)$$

Equation (5) is a result of Lemma 3.1 of [3]. It comes from the following equation;

$$(ab - 1)^m = ((a - 1) + a(b - 1))^m = \sum_{k=0}^m \binom{m}{k} (a - 1)^{m-k} a^k (b - 1)^k.$$

Proposition 4.2. *Let T and S be an m -isometry and an n -isometry on \mathcal{H} , respectively. If (T, S) is a doubly commuting pair, then TS is an $(m + n - 1)$ -isometry on \mathcal{H} .*

Proof. By Equation (5), we have

$$\beta_{m+n-1}(TS) = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} T^{*k} \cdot \beta_{m+n-1-k}(T) \cdot T^k \cdot \beta_k(S).$$

(i) If $0 \leq k \leq n - 1$, then $m + n - 1 - k \geq m$ and hence $\beta_{m+n-1-k}(T) = 0$.

(ii) If $k \geq n$, then $\beta_k(S) = 0$.

Therefore, $\beta_{m+n-1}(TS) = 0$ and so TS is an $(m + n - 1)$ -isometry. \square

Proof of Proposition 4.1. It is clear that $T \otimes I$ and $I \otimes S$ are an m -isometry and an n -isometry on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $(T \otimes I, I \otimes S)$ is a doubly commuting pair, by Proposition 4.2, $(T \otimes I)(I \otimes S) = T \otimes S$ is an $(m + n - 1)$ -isometry on $\mathcal{H} \otimes \mathcal{H}$.

Next we show following similar result of Proposition 4.1. For $[m, C]$ -operators, let (T, S) be a commuting pair and satisfy $S \cdot CTC = CTC \cdot S$, where C is a conjugation. Then it holds

$$\lambda_m(TS; C) = \sum_{k=0}^m \binom{m}{k} CT^k C \cdot \lambda_{m-k}(T; C) \cdot T^k \cdot \lambda_k(S; C). \quad (6)$$

Then, by a similar proof of Proposition 4.2, we have

Theorem 4.3. *Let T and S be an $[m, C]$ -isometry and an $[n, C]$ -isometry on \mathcal{H} , respectively. If (T, S) is a commuting pair and satisfies $S \cdot CTC = CTC \cdot S$, then TS is an $[m + n - 1, C]$ -isometry on \mathcal{H} .*

Proof. By Equation (6), it holds

$$\lambda_{m+n-1}(TS; C) = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} CT^k C \cdot \lambda_{m+n-1-k}(T; C) \cdot T^k \cdot \lambda_k(S; C).$$

Hence TS is an $[m + n - 1, C]$ -isometry on \mathcal{H} . \square

Theorem 4.4. *Let T and S be an $[m, C]$ -isometry and an $[n, D]$ -isometry on \mathcal{H} , respectively. Then $T \otimes S$ is an $[m + n - 1, C \otimes D]$ -isometry on $\mathcal{H} \otimes \mathcal{H}$.*

For conjugations C and D on \mathcal{H} , we define $C \otimes D$ on $\mathcal{H} \otimes \mathcal{H}$ by

$$(C \otimes D)\left(\sum_{j=1}^n \alpha_j x_j \otimes y_j\right) = \sum_{j=1}^n \overline{\alpha_j} Cx_j \otimes Dy_j.$$

First we prepare the following lemma.

Lemma 4.5. *Let C and D be conjugations on \mathcal{H} . Then $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$.*

Proof. Let $x = \sum_{i=1}^n \alpha_i x_i^1 \otimes x_i^2$ and $y = \sum_{j=1}^m \beta_j y_j^1 \otimes y_j^2 \in \mathcal{H} \otimes \mathcal{H}$ where $\alpha_i, \beta_j \in \mathbb{C}$. Since C and D are isometric, it follows that

$$\begin{aligned} \langle (C \otimes D)x, (C \otimes D)y \rangle &= \langle (C \otimes D)\left(\sum_{i=1}^n \alpha_i x_i^1 \otimes x_i^2\right), (C \otimes D)\left(\sum_{j=1}^m \beta_j y_j^1 \otimes y_j^2\right) \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \overline{\alpha_i} \langle Cx_i^1, Cy_j^1 \rangle \cdot \overline{\beta_j} \langle Dx_i^2, Dy_j^2 \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \overline{\alpha_i} \langle y_j^1, x_i^1 \rangle \cdot \overline{\beta_j} \langle y_j^2, x_i^2 \rangle \\ &= \left\langle \sum_{j=1}^m \overline{\beta_j} y_j^1 \otimes y_j^2, \sum_{i=1}^n \overline{\alpha_i} x_i^1 \otimes x_i^2 \right\rangle = \langle y, x \rangle. \end{aligned} \quad (7)$$

Moreover, since C and D are involutive, it follows that

$$(C \otimes D)^2 = (C^2 \otimes D^2) = I \otimes I \quad (8)$$

on the algebraic tensor product of $\mathcal{H} \otimes \mathcal{H}$. Since C and D are bounded, it follows from (7) and (8) that $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$. \square

Proof of Theorem 4.4. By Lemma 4.5, $C \otimes D$ is a conjugation. It is clear that $T \otimes I$ and $I \otimes S$ are $[m, C \otimes D]$ -isometry and $[n, C \otimes D]$ -isometry on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $(T \otimes I, I \otimes S)$ is a commuting pair and satisfies

$$(I \otimes S) \cdot ((C \otimes D)(T \otimes I)(C \otimes D)) = ((C \otimes D)(T \otimes I)(C \otimes D)) \cdot (I \otimes S),$$

by Theorem 4.3, $(T \otimes I)(I \otimes S) = T \otimes S$ is an $[m + n - 1, C \otimes D]$ -isometry.

For an (m, C) -isometric operator T , $\Lambda_m(T; C)$ is defined by

$$\Lambda_m(T; C) = \sum_{j=0}^m (-1)^j \binom{m}{j} T^{*m-j} \cdot C T^{m-j} C.$$

Let a commuting pair (T, S) satisfy $S^* \cdot CTC = CTC \cdot S^*$, where C is a conjugation. Then it holds

$$\Lambda_m(TS; C) = \sum_{k=0}^m \binom{m}{k} T^{*k} \cdot \Lambda_{m-k}(T; C) \cdot C T^k C \cdot \Lambda_k(S; C). \quad (9)$$

By similar proofs of Proposition 4.2 and Theorems 4.3 and 4.4, we have following results.

Theorem 4.6. *Let T and S be an (m, C) -isometry and an (n, C) -isometry on \mathcal{H} , respectively. If (T, S) is a commuting pair and satisfies $S^* \cdot CTC = CTC \cdot S^*$, then TS is an $(m + n - 1, C)$ -isometry on \mathcal{H} .*

Proof. The proof follows from Equation (9). \square

Theorem 4.7. *Let T and S be an (m, C) -isometry and an (n, D) -isometry on \mathcal{H} , respectively. Then $T \otimes S$ is an $(m + n - 1, C \otimes D)$ -isometry on $\mathcal{H} \otimes \mathcal{H}$.*

Proof. Operators $T \otimes I$ and $I \otimes S$ are $(m, C \otimes D)$ -isometry and $(n, C \otimes D)$ -isometry on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $(T \otimes I, I \otimes S)$ is a commuting pair and satisfies $(I \otimes S)^* \cdot ((C \otimes D)(T \otimes I)(C \otimes D)) = ((C \otimes D)(T \otimes I)(C \otimes D)) \cdot (I \otimes S)^*$, by Theorem 4.6 $(T \otimes I)(I \otimes S) = T \otimes S$ is an $(m + n - 1, C \otimes D)$ -isometry on $\mathcal{H} \otimes \mathcal{H}$. \square

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