



A Class of Generalized Operator Equilibrium Problems

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Abstract. In this paper, we consider a class of generalized operator equilibrium problems and derive a Minty type lemma for this class of problems. Further, we establish some existence theorems for the generalized operator equilibrium problems. The theorems presented in this paper generalize and unify many well-known results in the literature.

1. Introduction

The study of vector variational inequality was initiated by Giannessi [9] in 1980 with application in finite dimensional Euclidean spaces. Since then it has been extended and generalized in various directions due to its applicative nature, see Giannessi [10] and the references cited therein.

In 2002, Domokos and Kolumbán [6] introduced and studied a class of operator variational inequalities. These operator variational inequalities include not only scalar and vector variational inequalities as special cases ([2],[12],[14]), but also have sufficient evidence for their importance to study [6]. Motivated by work of Domokos and Kolumbán [6], Kazmi and Raouf [14] introduced and studied a class of operator equilibrium problems.

In this paper, We consider a class of generalized operator equilibrium problems which is more general than many problems considered in [1],[2],[6], and [11]-[17]. Using KKM-Fan Lemma [7] and Fixed point theorem [19], we establish some existence theorems for the generalized operator equilibrium problems. The theorems presented in this paper are generalizations and unifications of many well-known results in the literature, see for examples ([2],[6],[15]-[17]).

2. Preliminaries

Throughout the paper unless otherwise stated, let X and Y be Hausdorff topological vector spaces, $L(X, Y)$ be a space of all continuous linear operators from X to Y , and $K \subset L(X, Y)$ be a nonempty convex set.

Let $C : K \rightarrow 2^Y$ be a set-valued mapping, where 2^A denotes the set of all nonempty subsets of a set A , and for each $f \in K$, $C(f)$ be a solid convex open cone and $0 \notin C(f)$. Let $F : K \times K \rightarrow 2^Y$ be a bi-operator such that

$$F(g, g) \not\subseteq -C(g),$$

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for each $g \in K$.

The *generalized operator equilibrium problem (GOEP)* is to find $f \in K$ such that

$$F(f, g) \not\subseteq -C(f), \quad \forall g \in K. \tag{1.1}$$

Some special cases of GOEP (1.1)

1. If F is a single-valued bi-operator, then GOEP (1.1) reduces to the operator equilibrium problem of finding $f \in K$ such that

$$F(f, g) \not\subseteq -C(f), \quad \forall g \in K,$$

considered and studied by Kazmi and Raouf [14].

2. If F is a single-valued bi-operator and $F(f, g) \equiv \langle \eta(f, g), T(f) \rangle, \quad \forall g \in K$, where $T : K \rightarrow X$ and $\eta : K \times K \rightarrow K$, then GOEP (1.1) reduces to find $f \in K$ such that

$$\langle \eta(f, g), T(f) \rangle \not\subseteq -C(f), \quad \forall g \in K,$$

which appears to be new. And we call it the operator variational-like inequality problem.

3. If F is a single-valued bi-operator and $F(f, g) \equiv \langle f - g, T(f) \rangle, \quad \forall g \in K$, where $T : K \rightarrow X$, then GOEP (1.1) reduces to the operator variational inequality problem considered by Domokos and Kolumbán [6].
4. If F is a single-valued bi-operator and $F(f, g) \equiv \phi(f) - \phi(g)$, where $\phi : K \rightarrow Y$, then GOEP (1.1) reduces to the problem of finding $f \in K$ such that

$$\phi(f) - \phi(g) \not\subseteq -C(f), \quad \forall g \in K,$$

which appears to be a new. We call it operator minimization problem.

5. If $K \subset X$, then GOEP (1.1) reduces to generalized vector equilibrium problem studied by Khaliq [15] and Konnov and Yao [17].
6. If F is a single-valued bi-operator and $K \subset X$, a topological vector space, then GOEP(1.1) reduces to vector equilibrium problem studied by Kazmi ([12],[13]) and references therein ([8],[18]).
7. If F is a single-valued bi-operator, $K \subset X$ and $Y = R$, then GOEP (1.1) reduces to equilibrium problem of finding $x \in K$, such that

$$F(x, y) \geq 0, \quad \forall y \in K.$$

This problem was considered and studied by Blum and Oettli [1].

First, we recall the following definitions for our main results.

Definition 2.1. Let $T : B \subset X \rightarrow 2^Y$ be a set-valued mapping. Then

- (i) T is said to be *upper semicontinuous* on B , if for each $x_0 \in B$ and any open set V of Y containing $T(x_0)$, there exists an open neighborhood U of x_0 in B such that $T(x) \subset V$ for all $x \in U$.
- (ii) The *graph* of T , denoted by $\mathcal{G}(T)$, is

$$\mathcal{G}(T) = \{(x, z) \in B \times Y : x \in B, z \in T(x)\}.$$

- (iii) The *inverse* T^{-1} of T is the set-valued mapping from $\mathfrak{R}(T)$, range of T , to B defined by

$$x \in T^{-1}(y) \text{ if and only if } y \in T(x).$$

Definition 2.2.([3],[18]) A set-valued mapping $T : B \subset X \rightarrow 2^B$ is said to be a *KKM-mapping* if, for every finite subset $\{x_1, x_2, \dots, x_n\}$ of B ,

$$\text{conv}\{x_1, x_2, \dots, x_n\} \subseteq \bigcup_{i=1}^n T(x_i),$$

where $\text{conv}A$ is the convex hull of the set A .

Definition 2.3.[6] Let B be a subset of K . A set-valued mapping $C : K \rightarrow 2^Y$ is said to have a closed graph with respect to B if for every net $\{f_\alpha\}_{\alpha \in \Gamma} \subset K$ and $\{g_\alpha\}_{\alpha \in \Gamma} \subset Y$ such that $g_\alpha \in C(f_\alpha)$, $\{f_\alpha\}$ converges to $f \in B$ with respect to the topology of pointwise convergence (w.r.t.p.c.) and $\{g_\alpha\}$ converges to $g \in Y$, then $g \in C(f)$.

Theorem 2.4.[7] Let B be an arbitrary nonempty set in a topological vector space E and $T : B \rightarrow 2^E$ be a KKM-mapping. If $T(x)$ is closed for each $x \in B$ and is compact for some $x \in B$, then $\bigcap_{x \in B} T(x) \neq \emptyset$.

Theorem 2.5.[19] Let B be a nonempty convex subset of a Hausdorff topological vector space E . Let $T : B \rightarrow 2^B$ be a set-valued mapping such that:

- (a) for each $x \in B$, $T(x)$ is nonempty convex subset of B ;
- (b) for each $y \in B$, $T^{-1}(y)$ contains relatively open subset O_y of B (O_y maybe empty subset for some $y \in B$) such that $\bigcup_{y \in B} O_y = B$;
- (c) B contains a nonempty subset B_0 which is contained in a compact convex subset D of B such that the set $\bigcap_{y \in B_0} O_y^c$ is either empty or compact, where O_y^c denotes the complements of O_y in B .

Then there exists a point $x_0 \in B$ such that $x_0 \in T(x_0)$.

Now, we give the following definitions:

Definition 2.6. Let $K \subset L(X, Y)$ be a convex set, $C : K \rightarrow 2^Y$ be a set-valued mapping and $C(f)$ be an open convex solid cone with $0 \notin C(f)$, for each $f \in K$. Then we define the following ordering relationship on sets, for $A, B \subset Y$,

(i)
$$B - A \subseteq C(f) \iff A \leq_{C(f)} B \iff a \leq_{C(f)} b$$

for all $a \in A, b \in B$;

(ii)
$$B - A \not\subseteq C(f) \iff A \not\leq_{C(f)} B \iff a \not\leq_{C(f)} b$$

for all $a \in A, b \in B$.

Definition 2.7. Let $C : K \rightarrow 2^Y$ be a set-valued mapping and $C(f)$ be an open convex solid cone with $0 \notin C(f)$ in Y , for each $f \in K$. Then, the set-valued mapping $F : K \times K \rightarrow 2^Y$ is said to be

- (i) $C(f)$ -quasiconvex, if for all $f, g_i \in K, \alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1, \forall i = 1, 2, 3, \dots, n$, at least one of the following inclusions exists

$$F(f, g_\alpha) \subseteq F(f, g_i) - C(f),$$

where $g_\alpha = \sum_{i=1}^n \alpha_i g_i$;

- (ii) $C(f)$ -pseudomonotone, if
$$F(f, g) \not\subseteq -C(f) \implies F(g, f) \not\subseteq C(f);$$

- (iii) v -hemicontinuous, if the function $\lambda \rightarrow F(f + \lambda g, h)$ is upper semicontinuous at 0^+ for all $f, g, h \in K$, as a mapping from R_+ into Y .

Definition 2.8. Let B be a convex compact (w.r.t.p.c.) subset of K . Then the mapping $F : K \times K \rightarrow 2^Y$ is said to be coercive with respect to B , if there exists $g_0 \in B$ such that for all $f \in K \setminus B$,

$$F(f, g_0) \subseteq -C(f).$$

3. Existence Theorems for GOEP (1.1)

First, we prove the following lemma:

Lemma 3.1. *Let X, Y be Hausdorff topological vector spaces and let $K \subset L(X, Y)$ be a nonempty convex set. Let $(Y, C(f))$ be an ordered topological vector space with open convex solid cone $C(f)$, $0 \notin C(f)$, for each $f \in K$. Then for all $f, g \in K$, we have*

- (a) $g - f \subseteq C(f)$ and $g \not\subseteq C(f)$ implies $f \not\subseteq C(f)$;
- (b) $g - f \subseteq -C(f)$ and $g \not\subseteq -C(f)$ implies $f \not\subseteq -C(f)$.

Proof. (a) Let $g - f \subseteq C(f)$ and $g \not\subseteq C(f)$ then

$$\begin{aligned} f &\subseteq -C(f) + g \\ &\subseteq -C(f) + Y \setminus C(f) \\ &\subseteq Y \setminus C(f), \end{aligned}$$

that is, $f \not\subseteq C(f)$.

(b) Similarly, we can prove part (b). This completes the proof. □

Remark 3.2. Since $C(f)$ is open for each $f \in K$, we know that both $Y \setminus C(f)$ and $Y \setminus \{-C(f)\}$ are closed.

Next, we prove the following Minty-type Lemma for GOEP(1.1).

Lemma 3.3. *Let X, Y be Hausdorff topological vector spaces and let $K \subset L(X, Y)$ be a nonempty convex set. Let the set-valued mapping $F : K \times K \rightarrow 2^Y$ be $C(f)$ -pseudo monotone and ν -hemicontinuous in the first argument and $C(f)$ -quasi convex in the second argument and let $B \subseteq K$. Then the following two GOEP's are equivalent:*

- (a) $\exists f \in B$ such that $F(f, g) \not\subseteq -C(f)$, $\forall g \in K$;
- (b) $\exists f \in B$ such that $F(g, f) \not\subseteq C(f)$, $\forall g \in K$.

Proof. (a) \implies (b) It is a direct consequence of the $C(f)$ -pseudomonotonicity of F .

(b) \implies (a) Suppose that there exists a $f \in B$ such that

$$F(g, f) \not\subseteq C(f), \quad \forall g \in K.$$

Since K is convex, for all $f, g \in K$ and $\lambda \in [0, 1]$, $h_\lambda := \lambda g + (1 - \lambda)f \in K$. Hence, we have

$$F(h_\lambda, f) \not\subseteq C(f).$$

Since F is $C(f)$ -quasiconvex, we have

$$F(h_\lambda, h_\lambda) \subseteq F(h_\lambda, f) - C(h_\lambda),$$

it implies that

$$F(h_\lambda, h_\lambda) - F(h_\lambda, f) \subseteq -C(h_\lambda).$$

Hence we have

$$F(h_\lambda, f) \not\subseteq -C(h_\lambda).$$

This implies that

$$F(f, f) \subseteq Y \setminus \{-C(f)\}.$$

And also, we have for $g \in K$,

$$F(h_\lambda, g) \not\subseteq -C(h_\lambda).$$

This means that

$$F(h_\lambda, g) \subseteq Y \setminus \{-C(h_\lambda)\}.$$

Since F is hemicontinuous and the set $Y \setminus \{-C(f)\}$ is closed, preceding inclusion implies that $F(f, g) \not\subseteq -C(f)$. This completes the proof. \square

Now we are in a position to introduce and prove the following existence theorem for GOEP(1.1):

Theorem 3.4. *Let X, Y be Hausdorff topological vector spaces and let $K \subset L(X, Y)$ be a nonempty convex set. Let the set-valued mapping $F : K \times K \rightarrow 2^Y$ be $C(f)$ -pseudomonotone and hemicontinuous in the first argument and $C(f)$ -quasiconvex in the second argument, and coercive with respect to the compact convex set $B \subset K$. If for each $g \in K$, $F(g, \cdot)$ is upper semicontinuous with compact values on B and for each $f \in K$, the graph of $Y \setminus \{-C(f)\}$ is closed with respect to B , then GOEP(1.1) has a solution.*

Proof. For each $g \in K$, define the set-valued mappings $S, T : K \rightarrow 2^K$ by

$$S(g) := \{f \in K : F(f, g) \not\subseteq -C(f)\}$$

and

$$T(g) := \{f \in B : F(g, f) \not\subseteq C(f)\}.$$

First, we claim that S is a KKM-mapping. Indeed, let $\{g_1, g_2, \dots, g_n\}$ be a finite subset of K and let $g \in \text{conv}\{g_1, g_2, \dots, g_n\}$ be arbitrary. Then $g = \sum_{i=1}^n \lambda_i g_i$, $\lambda_i \geq 0$, and $\sum_{i=1}^n \lambda_i = 1$. Suppose, if possible $g \notin \bigcup_{i=1}^n S(g_i)$, then

$$F(g, g_i) \subseteq -C(g), \quad \forall i = 1, 2, \dots, n. \tag{3.1}$$

Since F is $C(f)$ -quasi convex in the second argument, we have

$$\begin{aligned} F(g, g) &= F(g, \sum_{i=1}^n \lambda_i g_i) \\ &\subseteq \sum_{i=1}^n \lambda_i F(g, g_i) \\ &= F(g, g_i) - C(g), \quad \forall i = 1, 2, \dots, n. \end{aligned}$$

Therefore, using (3.1), we have

$$F(g, g) \subseteq -C(g) - C(g) \subseteq -C(g),$$

which is a contradiction to the assumption $F(g, g) \not\subseteq -C(g)$, for each $g \in K$.

Thus

$$g = \sum_{i=1}^n \lambda_i g_i \in \bigcup_{i=1}^n S(g_i),$$

this means that

$$\text{conv}\{g_1, g_2, \dots, g_n\} \subset \bigcup_{i=1}^n S(g_i).$$

Hence, the mapping $\bar{S} : K \rightarrow 2^K$ defined by $\bar{S}(g) = \overline{S(g)}$, the closure (w.r.t.p.c.) of $S(g)$, is also a KKM-mapping. The coercivity of F with respect to B implies that $\overline{S(g_0)} \subset B$. Hence $\overline{S(g_0)}$ is compact (w.r.t.p.c.). Thus, it follows from Theorem 2.4 that

$$\bigcap_{g \in K} \overline{S(g)} \neq \emptyset.$$

Next, we claim that

$$\bigcap_{g \in K} \overline{S(g)} \subset T(h), \quad \forall h \in K.$$

Indeed, let $f \in \bigcap_{g \in K} \overline{S(g)}$. Since

$$\bigcap_{g \in K} \overline{S(g)} = \bigcap_{g \in K} (\overline{S(g)} \cap \overline{S(g_0)}) \subset \overline{S(g)} \cap B \subset B,$$

we have $f \in \bigcap_{g \in K} \overline{S(g)} \cap B, \forall g \in K$. Let $h \in K$ be arbitrary. Then there exists a net $\{f_\alpha\}_{\alpha \in \Gamma}$ in $S(h)$ such that $\{f_\alpha\}$ converges (w.r.t.p.c.) to $f \in B$. Hence, we have

$$F(f_\alpha, h) \not\subseteq -C(f_\alpha), \text{ for each } h \in K.$$

Since F is $C(f)$ -pseudomonotone,

$$F(h, f_\alpha) \not\subseteq C(f_\alpha). \tag{3.2}$$

That is, there exists $p_\alpha \in F(h, f_\alpha)$ such that $p_\alpha \notin C(f_\alpha)$ for all $\alpha \in \Gamma$. Since the set $A := \{f_\alpha\} \cup \{f\}$ is compact, $p_\alpha \in F(h, A)$ for all $\alpha \in \Gamma$. Since $F(h, A)$ is compact, $\{p_\alpha\}$ has a convergent subnet with limit, say, p . Without loss of generality, we can assume that $\{p_\alpha\}$ converges (w.r.t.p.c.) to p . Further, since the graph of $Y \setminus \{-C(f)\}$ is closed, clearly the graph of $Y \setminus C(f)$ is also closed. Hence, the upper semicontinuity of $F(h, \cdot)$ implies $p \in F(h, f)$. Hence the inclusion (3.2) implies that $F(h, f) \not\subseteq C(f)$, that is, $f \in T(h), \forall h \in K$. Therefore, we have

$$\bigcap_{g \in K} \overline{S(g)} \subset \bigcap_{g \in K} \overline{T(g)} \subset B.$$

Finally, using Lemma 3.3, we get

$$\bigcap_{g \in K} S(g) = \bigcap_{g \in K} T(g).$$

Thus

$$\bigcap_{g \in K} S(g) \neq \emptyset,$$

that is, there exists $f \in K$ such that

$$F(f, g) \not\subseteq -C(f).$$

This completes the proof. □

Next, we prove the following theorem without using Lemma 3.3.

Theorem 3.5 *Let X, Y be Hausdorff topological vector spaces and let $K \subset L(X, Y)$ be a nonempty convex set. Let $F : K \times K \rightarrow 2^Y$ be a set-valued mapping such that for each $g \in K, F(\cdot, g)$ is upper semicontinuous with compact values on B and let, for each $f \in K, \text{ the graph of } Y \setminus \{-C(f)\} \text{ be closed with respect to } B. \text{ Let the set-valued mapping } M : K \times K \rightarrow 2^Y \text{ be } C(f)\text{-quasiconvex in the second argument, coercive with respect to the compact convex set } B \subset K, \text{ and } M(g, g) \not\subseteq -C(g), \text{ for each } g \in K. \text{ If for each } f, g \in K, M(f, g) \subseteq -C(f) \text{ implies } F(f, g) \subseteq -C(f), \text{ then GOEP (1.1) has a solution.}$*

Proof. For each $g \in K$, define a set-valued mapping $S : K \rightarrow 2^K$ by

$$S(g) := \{f \in B : F(f, g) \not\subseteq -C(f)\}.$$

First, we claim that $S(g)$ is closed. Indeed, for any $g \in K$, there exists a net $\{f_\alpha\}_{\alpha \in \Gamma}$ in $S(g)$ such that $\{f_\alpha\}$ converges (w.r.t.p.c.) to $f \in K$. Then we have

$$F(f_\alpha, g) \not\subseteq -C(f_\alpha), \tag{3.3}$$

for each $g \in K$, that is, there exists $p_\alpha \in F(f_\alpha, g)$ such that $p_\alpha \notin -C(f_\alpha)$ for all $\alpha \in \Gamma$ or $p_\alpha \in Y \setminus \{-C(f_\alpha)\}$ for all $\alpha \in \Gamma$. Since the set $A := \{f_\alpha\} \cup \{f\}$ is compact, $p_\alpha \in F(A, g)$. Since $F(A, g)$ is compact, $\{p_\alpha\}$ has a convergent subnet with limit, say, p . Without loss of generality, we can assume that $\{p_\alpha\}$ converges (w.r.t.p.c.) to p . Then by the upper semicontinuity of $F(\cdot, g)$ implies $p \in F(f, g)$. Therefore, from inclusion (3.3), we have that

$$F(f, g) \not\subseteq -C(f).$$

Hence, $f \in S(g)$ for each $g \in K$.

Our aim is to show that $\bigcap_{g \in K} \overline{S(g)} \neq \emptyset$. Since B is compact, it is sufficient to show that the family $\{S(g)\}_{g \in K}$ has the finite intersection property.

Let $\{g_1, g_2, \dots, g_n\}$ be a finite subset of K . Then $D := \text{conv}\{g_1, g_2, \dots, g_n\}$ is convex and compact subset of K , (see [5]).

Now, for each $g \in D$, define a set-valued mapping $T : D \rightarrow 2^D$ by

$$T(g) := \{f \in D : M(f, g) \not\subseteq -C(f)\}.$$

Obviously, we have that $T(g)$ is nonempty for each $g \in K$. By using similar arguments in the proof of the Theorem 3.4, we can easily prove that T is a KKM-mapping. Hence the mapping $\bar{T} : D \rightarrow 2^D$, defined by $\bar{T}(g) = \overline{T(g)}$, the closure (w.r.t.p.c.) of $T(g)$ in D , is also a KKM-mapping.

Since, for each $g \in D$, the set $\overline{T(g)}$ is closed in D , it is compact. Thus, by Theorem 2.4, it follows that $\bigcap_{g \in K} \overline{T(g)} \neq \emptyset$.

Let $f_0 \in \bigcap_{g \in K} \overline{T(g)}$. Then, the coercivity of M with respect to B implies that $T(g_0) \subset B$. Since $\overline{T(g_0)}$ is contained in K , it is contained in B . Since $f_0 \in \bigcap_{i=1}^n \overline{T(g_i)}$ and, for each $i = 1, 2, \dots, n$,

$$\begin{aligned} \overline{T(g_0)} &= \overline{\{f \in D : M(f, g_i) \not\subseteq -C(f)\}} \\ &\subseteq \overline{\{f \in D : F(f, g_i) \not\subseteq -C(f)\}} \\ &= \{f \in D : F(f, g_i) \not\subseteq -C(f)\} \\ &= S(g), \end{aligned}$$

we have $f_0 \in \bigcap_{i=1}^n S(g_i)$. Hence, the family $\{S(g)\}_{g \in K}$ has the finite intersection property. This complete the proof. \square

Theorem 3.6. Let X, Y be Hausdorff topological vector spaces and let $K \subset L(X, Y)$ be a nonempty convex set. Let $F : K \times K \rightarrow 2^Y$ be a set-valued mapping and for each $g \in K$, $F(\cdot, g)$ be upper semicontinuous with compact values on K and coercive with respect to the compact convex set $B \subset K$, and let for each $f \in K$, the graph of $Y \setminus \{-C(f)\}$ be closed. Let the set-valued mapping $M : K \times K \rightarrow 2^Y$ be $C(f)$ -quasiconvex in the second argument and $M(g, g) \not\subseteq -C(g)$, for each $g \in K$. If for each $f, g \in K$, $F(f, g) \subseteq -C(f)$ implies $M(f, g) \subseteq -C(f)$, then GOEP (1.1) has a solution.

Proof. For each $g \in K$, define a set-valued mapping $S : K \rightarrow 2^K$ by

$$S(g) := \{f \in K : F(f, g) \not\subseteq -C(f)\}.$$

Then, as in the proof of Theorem 3.5, we can show that $S(g)$ is closed for each $g \in K$. Next, we claim that $S(g) \neq \emptyset$ for each $g \in K$. Assume, if possible, $S(g) = \emptyset$ for each $g \in K$. Then for each $f \in K$, the set

$$T(f) := \{g \in K : F(f, g) \subseteq -C(f)\} \neq \emptyset.$$

Now, for each $f \in K$, define a set-valued mapping $Q : K \rightarrow 2^K$ by

$$Q(f) := \{g \in K : M(f, g) \subseteq -C(f)\}.$$

From assumption that $F(f, g) \subseteq -C(f)$ implies $M(f, g) \subseteq -C(f)$, for each $f, g \in K$, we have

$$Q(f) \supseteq T(f).$$

Hence $Q(f) \neq \emptyset$ for each $f \in K$. We now show that $Q(f)$ is convex for each $f \in K$. Indeed, let $g_1, g_2 \in Q(f)$. Then for each $f \in K$, we have $M(f, g_1) \subseteq -C(f)$ and $M(f, g_2) \subseteq -C(f)$. Since M is $C(f)$ -quasiconvex, we have either

$$M(f, g_\alpha) \subseteq M(f, g_1) - C(f) \subseteq -C(f) - C(f) \subseteq -C(f),$$

or

$$M(f, g_\alpha) \subseteq M(f, g_2) - C(f) \subseteq -C(f) - C(f) \subseteq -C(f),$$

where $g_\alpha = \alpha g_1 + (1 - \alpha)g_2$ for all $\alpha \in [0, 1]$. Hence $g_\alpha \in Q(f)$. Thus for each $f \in K$, $Q(f)$ is nonempty and convex.

Now, for each $g \in K$, the set

$$\begin{aligned} Q^{-1}(g) &= \{f \in K : g \in Q(f)\} \\ &= \{f \in K : M(f, g) \subseteq -C(f)\} \\ &\supseteq \{f \in K : F(f, g) \subseteq -C(f)\} \\ &= \{f \in K : F(f, g) \not\subseteq -C(f)\}^c \\ &= [S(g)]^c \\ &:= O_g \end{aligned}$$

which is a relatively open set in K . From coercivity of F with respect to B , we have $S(g) \subset B$ for each $g \in B$ which implies that

$$D := \bigcap_{g \in B} O_g^c = \bigcap_{g \in B} S(g) \subset B.$$

Since $O_g = \{f \in K : F(f, g) \subseteq -C(f)\} = T^{-1}(g)$, we claim that

$$\bigcup_{g \in K} O_g = \bigcup_{g \in K} T^{-1}(g) = K.$$

Indeed, let $f \in K$. From $T(f) \neq \emptyset$, we can choose $g \in T(f)$. Hence we have,

$$f \in T^{-1}(g) = O_g.$$

By Theorem 2.5, there exists a point $f_0 \in Q(f_0)$, that is, $M(f_0, f_0) \subseteq -C(f_0)$, which is a contradiction of the assumption $M(g, g) \not\subseteq -C(g)$ for each $g \in K$. Hence, the set $\bigcap_{g \in K} S(g)$ is nonempty. This complete the proof. \square

References

- [1] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63**(1994), 123-145.
- [2] G.Y. Chen, Existence of solutions for a vector variational inequalities: An extension of the Hartmann-Stampacchia theorem, *J. Optim. theory Appl.*, **74**(1992), 445-456.
- [3] Y.Q. Chen, Y.J. Cho, J.K. Kim, B.S.Lee, Note on KKM maps and Applications, *Fixed Point Theory and Applications*, Volume 2006, Article ID 53286, (2006), 1-9
- [4] Y.J. Cho, J.K. Kim, B.S. Lee, Remarks on KKM maps and applications, *Jour. of Advanced Research in Applied Mathematics*, **1**(1)(2009), 1-8
- [5] M.S.R. Chowdhury, K.K. Tan, Generalization of Ky Fan's minimax inequality with applications to generalized variational inequalities for pseudomonotone operators and fixed point theorem, *J. Math. Anal. Appl.*, **204**(1996), 910-929.
- [6] A. Domokos, J. Kolumbán, Variational inequalities with operator solutions, *J. Global Optim.*, **23**(2002), 99-110.
- [7] K. Fan, A generalization of Tychonoff's fixed-point theorem, *Math. Ann.*, **142**(1961), 305-310.
- [8] Y.P. Fang, N.J. Huang, J.K. Kim, Existence results for systems of vector equilibrium problems, *J. of Gobal Optimization*, **35**(1)(2006), 71-83
- [9] F. Giannessi, Theorems of alternative, quadratic programs and complementarity problems, pp. 151-186, In: *Variational Inequalities and Complementarity Problems*, (Edited by R. W. Cottle, F. Giannessi and J. L. Lions), John Wiley and sons, Chichester 1980.
- [10] F. Giannessi, *Vector Variational Inequalities and Vector Equilibria: Mathematical Theories*, Kluwer Academic Publishers, Dordrecht, Boston 2000.
- [11] N. Hadjisavas, S. Schaible, From scalar to vector equilibrium problems in the quasimonotone case, *J. Optim. Theory Appl.*, **96** (1998), 297-309.
- [12] K.R. Kazmi, Some remarks on vector optimization problems, *J. Optim. Theory Appl.*, **96** (1998), 133-138.
- [13] K.R. Kazmi, On vector equilibrium problem, *Proc. Indian Acad. Sci., (Math. Sci.)*, **110**(2000), 213-223.
- [14] K. R. Kazmi, A. Raouf, A class of operator equilibrium problems, *J. Math. Anal. Appl.*, **308** (2005), 554-564.
- [15] A. Khaliq, On generalized veator equilibrium problems, *Ganit: J. Bangladesh Math. Soc.*, **19** (1999), 69-83.
- [16] A. Khaliq, A. Raouf, Generalized vector quasi-equilibrium problems, *Adv. Nonl. Var. Inequal.*, **7**(1)(2004), 47-57.
- [17] I.V. Konnov, J.C. Yao, Existence of solutions for generalized vector equilibrium problems, *J. Math. Anal. Appl.*, **233**(1999), 328-335.
- [18] J. Li, N.J. Huang, J.K. Kim, On implicit vector equilibrium problems, *Jour. of Math. Anal. and Appl.*, **283**(2)(2003), 503-514
- [19] E. Tarafdar, A fixed point thorem equivalent to Fan-Knaster-Kuratowski-Muzurkiewicz's theorem, *J. Math. Anal. Appl.*, **128** (1987), 475-479.