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On the Pseudo Drazin Inverse of the Sum of Two Elements in a Banach Algebra

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Abstract. In this paper, some additive properties of the pseudo Drazin inverse are obtained in a Banach algebra. In addition, we find some new conditions under which the pseudo Drazin inverse of the sum $a+b$ can be explicitly expressed in terms of $a, a^{\ddagger}, b, b^{\ddagger}$. In particular, necessary and sufficient conditions for the existence as well as the expression for the pseudo Drazin inverse of the sum $a+b$ are obtained under certain conditions. Also, a result of Wang and Chen [Pseudo Drazin inverses in associative rings and Banach algebras, LAA 437(2012) 1332-1345] is extended.

1. Introduction

Throughout this paper, $\mathscr{A}$ denotes a complex Banach algebra with unity 1. For $a \in \mathscr{A}$, we use $\sigma(a)$ to denote the spectrum of a. $\mathscr{A}^{-1}$, $\mathscr{A}^{nil}$, $\mathscr{A}^{qnil}$ stand for the sets of all invertible, nilpotent and quasi-nilpotent elements ($\sigma(a)=\{0\}$) in $\mathscr{A}$, respectively. The Jacobson radical of $\mathscr{A}$ is defined by

$$J(\mathscr{A}) = \{a \in \mathscr{A} | 1 + ax \in \mathscr{A}^{-1} \text{ for any } x \in \mathscr{A}\}.$$

Let $\sqrt{J(\mathscr{A})} = \{a \in \mathscr{A} | a^n \in J(\mathscr{A}) \text{ for some } n \geq 1\}$.

Let us recall that the Drazin inverse [10] of $a \in \mathscr{A}$ is the element $x \in \mathscr{A}$ which satisfies

$$xax = x, \ \ ax = xa, \ \ a - a^2x \in \mathscr{A}^{nil}. \tag{1}$$

The element x above is unique if it exists and is denoted by a^D. The set of all Drazin invertible elements of $\mathscr{A}$ will be denoted by $\mathscr{A}^D$.

The generalized Drazin inverse [12] of $a \in \mathscr{A}$ (or Koliha-Drazin inverse of a) is the element $x \in \mathscr{A}$ which satisfies

$$xax = x, \ \ ax = xa, \ \ a - a^2x \in \mathscr{A}^{qnil}. \tag{2}$$

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Such x, if it exists, is unique and will be denoted by a^d. Let $\mathscr{A}^d$ denote the set of all generalized Drazin invertible elements of $\mathscr{A}$.

In 2012, Wang and Chen [17] introduced the notion of the pseudo Drazin inverse (or p-Drazin inverse for short) in associative rings and Banach algebras. An element $a \in \mathscr{A}$ is called p-Drazin invertible if there exists $x \in \mathscr{A}$ such that

$$xax = x, \;\; ax = xa, \;\; a^k - a^{k+1}x \in J(\mathscr{A}) \tag{3}$$

for some integer $k \geq 1$. Any element $x \in \mathscr{A}$ satisfying (3) is called a p-Drazin inverse of a, such element is unique if it exists, and will be denoted by $a^{\ddagger}$. The set of all p-Drazin invertible elements of $\mathscr{A}$ will be denoted by $\mathscr{A}^{pD}$. In [17], Wang and Chen proved that $\mathscr{A}^D \subsetneqq \mathscr{A}^{pD} \subsetneqq \mathscr{A}^d$.

In 1958, Drazin [10] gave the representation of $(a+b)^D$ under the condition $ab = ba = 0$ in a ring. In 2001, for $P, Q \in \mathbb{C}^{n\times n}$, Hartwig, Wang and Wei [11] gave a formula for $(P+Q)^D$ under the condition $PQ = 0$. Later, Djordjević and Wei [9] generalized the result of [11] to bounded linear operators on an arbitrary complex Banach space. In [4], the expression for $(a + b)^D$ was given under the assumption $ab = 0$ in the context of the additive category. In 2004, Castro-González and Koliha [2] gave a formula for $(a + b)^d$ under the conditions $a^\pi b = b, ab^\pi = a, b^\pi aba^\pi = 0$ which are weaker than $ab = 0$ in Banach algebras. In 2010, Deng and Wei [8] derived a result under the condition $PQ = QP$, where P, Q are bounded linear operators. In 2011, Cvetković-Ilić, Liu and Wei [7] extended the result of [8] to Banach algebras. In 2014, Zhu, Chen and Patrício [19] obtained a result about the p-Drazin inverse of $a + b$ under the conditions $a^2b = aba$ and $b^2a = bab$ which are weaker than $ab = ba$ in Banach algebras. More results on (generalized, pseudo) Drazin inverse can be found in [1, 5, 7, 9, 15, 18].

The motivation for this paper is the paper of Cvetković-Ilić et al. [6] and the paper of Castro-González and Koliha [2]. In both of these papers the conditions were considered such that the generalized Drazin inverse $(a + b)^d$ could be expressed in terms of a, a^d, b, b^d .

In this paper we investigate the representation for p-Drazin inverse of the sum of two elements in a Banach algebra under various conditions. In particular, necessary and sufficient conditions for the existence as well as the expression for the p-Drazin inverse of the sum $a + b$ are obtained under certain conditions. In addition, we generalized Theorem 5.4 in [17].

Let $e^2 = e \in \mathscr{A}$ be an idempotent. Then we can represent element $a \in \mathscr{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_e,$$

where $a_{11} = eae, \;\; a_{12} = ea(1 - e), \;\; a_{21} = (1 - e)ae, \;\; a_{22} = (1 - e)a(1 - e)$.

2. Preliminary Results

To prove the main results, we need some lemmas.

Lemma 2.1. [13, Exercise 1.6] *Let $a, b \in \mathscr{A}$. If $1 + ab \in \mathscr{A}^{-1}$, then $1 + ba \in \mathscr{A}^{-1}$ and $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.*

Lemma 2.2. [16, Theorem 1(1)] *Let $e^2 = e \in \mathscr{A}$ and let $a \in e\mathscr{A}e$. Then $a \in (e\mathscr{A}e)^{-1}$ if and only if $eae + 1 - e \in \mathscr{A}^{-1}$.*

Lemma 2.3. [13, Corollary 4.2] *Let $a, b \in \mathscr{A}$. Then*
(i) *If $a \in J(\mathscr{A})$ or $b \in J(\mathscr{A})$, then $ab, ba \in J(\mathscr{A})$.*
(ii) *If $a \in J(\mathscr{A})$ and $b \in J(\mathscr{A})$, then $a + b \in J(\mathscr{A})$.*

Lemma 2.4. *Let $a, b \in \sqrt{J(\mathscr{A})}$ with $ab = 0$ or $ab = ba$. Then $a \pm b \in \sqrt{J(\mathscr{A})}$.*

Proof. Let k_1, k_2 be positive integers such that $a^{k_1} \in J(\mathscr{A})$ and $b^{k_2} \in J(\mathscr{A})$. Take $k = max\{k_1, k_2\}$. By Lemma 2.3 (i), we obtain $a^k \in J(\mathscr{A}), b^k \in J(\mathscr{A})$. If $ab = 0$, we have $(a + b)^{2k} = a^{2k} + ba^{2k-1} + \cdots + b^k a^k + b^{k+1}a^{k-1} + \cdots + b^{2k} = (a^k + ba^{k-1} + \cdots + b^k)a^k + b^k(ba^{k-1} + b^2a^{k-2} + \cdots + b^k) \in J(\mathscr{A})$. If $ab = ba$, then $(a + b)^{2k} = a^{2k} + \binom{2k}{1}a^{2k-1}b + \cdots + \binom{2k}{k}a^k b^k + \binom{2k}{k+1}a^{k-1}b^{k+1} + \cdots + b^{2k} = a^k(a^k + \binom{2k}{1}a^{k-1}b + \cdots + \binom{2k}{k}b^k) + (\binom{2k}{k+1}a^{k-1}b + \cdots + b^k)b^k \in J(\mathscr{A})$. Replacing b by $-b$, we can obtain $a - b \in \sqrt{J(\mathscr{A})}$. $\square$

By $M_2(\mathscr{A})$ we denote the set of all 2×2 matrices over $\mathscr{A}$ which is a complex Banach algebra.

Lemma 2.5. (i) [13, page 57 Example (7)] $J(M_2(\mathscr{A})) = M_2(J(\mathscr{A}))$.

(ii) [13, Theorem 21.10] *Let* $e^2 = e \in \mathscr{A}$. *Then* $J(\mathscr{A}) \bigcap e\mathscr{A}e = J(e\mathscr{A}e)$.

Let $M_2(\mathscr{A}, e) = \left[\begin{smallmatrix} e\mathscr{A}e & e\mathscr{A}(1-e) \\ (1-e)\mathscr{A}e & (1-e)\mathscr{A}(1-e) \end{smallmatrix}\right]$, where $e \in \mathscr{A}$ is an idempotent. Then $M_2(\mathscr{A}, e)$ is a Banach algebra with unity $I = \left[\begin{smallmatrix} e & 0 \\ 0 & 1-e \end{smallmatrix}\right]$ (see[2]).

Now, we establish a crucial auxiliary result.

Lemma 2.6. *Let* $e^2 = e \in \mathscr{A}$. *Then* $J(M_2(\mathscr{A})) \bigcap M_2(\mathscr{A}, e) = J(M_2(\mathscr{A}, e))$.

Proof. According to Lemma 2.5 (i), we have $J(M_2(\mathscr{A})) \bigcap M_2(\mathscr{A}, e) = M_2(J(\mathscr{A})) \bigcap M_2(\mathscr{A}, e)$. Let $G = M_2(J(\mathscr{A})) \bigcap M_2(\mathscr{A}, e)$ and $H = \left[\begin{smallmatrix} J(\mathscr{A}) \bigcap e\mathscr{A}e & J(\mathscr{A}) \bigcap e\mathscr{A}(1-e) \\ J(\mathscr{A}) \bigcap (1-e)\mathscr{A}e & J(\mathscr{A}) \bigcap (1-e)\mathscr{A}(1-e) \end{smallmatrix}\right]$. We will show that $G = H$. Let $s = \left[\begin{smallmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{smallmatrix}\right] \in G$, then $s_{11}, s_{12}, s_{21}, s_{22} \in J(\mathscr{A})$, also, $s_{11} \in e\mathscr{A}e, s_{12} \in e\mathscr{A}(1-e), s_{21} \in (1-e)\mathscr{A}e$ and $s_{22} \in (1-e)\mathscr{A}(1-e)$, which imply $s_{11} \in J(\mathscr{A}) \bigcap e\mathscr{A}e, s_{12} \in J(\mathscr{A}) \bigcap e\mathscr{A}(1-e), s_{21} \in J(\mathscr{A}) \bigcap (1-e)\mathscr{A}e$ and $s_{22} \in J(\mathscr{A}) \bigcap (1-e)\mathscr{A}(1-e)$. Hence $s \in H$. Conversely, let $r = \left[\begin{smallmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{smallmatrix}\right] \in H$, we get $r_{11} \in J(\mathscr{A}) \bigcap e\mathscr{A}e, r_{12} \in J(\mathscr{A}) \bigcap e\mathscr{A}(1-e), r_{21} \in J(\mathscr{A}) \bigcap (1-e)\mathscr{A}e$ and $r_{22} \in J(\mathscr{A}) \bigcap (1-e)\mathscr{A}(1-e)$, which yield $r \in M_2(J(\mathscr{A}))$ and $r \in M_2(\mathscr{A}, e)$, i.e. $r \in M_2(J(\mathscr{A})) \bigcap M_2(\mathscr{A}, e) = G$. Thus, we obtain $G = H$. By Lemma 2.5 (ii), it follows that $H = \left[\begin{smallmatrix} J(e\mathscr{A}e) & J(\mathscr{A}) \bigcap e\mathscr{A}(1-e) \\ J(\mathscr{A}) \bigcap (1-e)\mathscr{A}e & J((1-e)\mathscr{A}(1-e)) \end{smallmatrix}\right]$. Therefore, we can get $J(M_2(\mathscr{A})) \bigcap M_2(\mathscr{A}, e) = \left[\begin{smallmatrix} J(e\mathscr{A}e) & J(\mathscr{A}) \bigcap e\mathscr{A}(1-e) \\ J(\mathscr{A}) \bigcap (1-e)\mathscr{A}e & J((1-e)\mathscr{A}(1-e)) \end{smallmatrix}\right]$.

First, we prove $J(M_2(\mathscr{A}, e)) \subseteq J(M_2(\mathscr{A})) \bigcap M_2(\mathscr{A}, e)$.

Let $x = \left[\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right] \in J(M_2(\mathscr{A}, e))$, then for any $y = \left[\begin{smallmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{smallmatrix}\right] \in M_2(\mathscr{A}, e)$, we have $I + xy \in [M_2(\mathscr{A}, e)]^{-1}$. Thus,

(a) For any $a \in e\mathscr{A}e$, let $y_1 = \left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Since $I + xy_1 \in [M_2(\mathscr{A}, e)]^{-1}$, we can obtain $e + x_{11}a \in (e\mathscr{A}e)^{-1}$, which implies $x_{11} \in J(e\mathscr{A}e)$.

(b) For any $b \in \mathscr{A}$, let $y_2 = \left[\begin{smallmatrix} 0 & 0 \\ (1-e)be & 0 \end{smallmatrix}\right]$. From $I + xy_2 \in [M_2(\mathscr{A}, e)]^{-1}$, we can conclude $e + x_{12}(1-e)be \in (e\mathscr{A}e)^{-1}$. By Lemma 2.2, $1 + x_{12}be = e[e + x_{12}(1-e)be]e + 1 - e \in \mathscr{A}^{-1}$. Using Lemma 2.1, $1 + x_{12}b = 1 + ex_{12}b \in \mathscr{A}^{-1}$, which implies $x_{12} \in J(\mathscr{A})$. Hence, $x_{12} \in J(\mathscr{A}) \bigcap e\mathscr{A}(1-e)$.

It is analogous to prove $x_{21} \in J(\mathscr{A}) \bigcap (1-e)\mathscr{A}e, x_{22} \in J((1-e)\mathscr{A}(1-e))$.

Next, we prove $J(M_2(\mathscr{A})) \bigcap M_2(\mathscr{A}, e) \subseteq J(M_2(\mathscr{A}, e))$.

Let $u = \left[\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right] \in J(M_2(\mathscr{A})) \bigcap M_2(\mathscr{A}, e), a = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] \in M_2(\mathscr{A}, e)$. In order to prove $u \in J(M_2(\mathscr{A}, e))$, we need to prove

$$I + ua = \begin{bmatrix} e + u_{11}a_{11} + u_{12}a_{21} & u_{11}a_{12} + u_{12}a_{22} \\ u_{21}a_{11} + u_{22}a_{21} & (1-e) + u_{21}a_{12} + u_{22}a_{22} \end{bmatrix} \in [M_2(\mathscr{A}, e)]^{-1}.$$

Denote $b_{11} = e + u_{11}a_{11} + u_{12}a_{21}$, $b_{12} = u_{11}a_{12} + u_{12}a_{22}$, $b_{21} = u_{21}a_{11} + u_{22}a_{21}$, $b_{22} = (1-e) + u_{21}a_{12} + u_{22}a_{22}$. Since $u_{11}, u_{12} \in J(\mathscr{A})$, by Lemma 2.3 and Lemma 2.5 (ii) we have $u_{11}a_{11} + u_{12}a_{21} \in J(\mathscr{A}) \bigcap e\mathscr{A}e = J(e\mathscr{A}e)$. Thus $b_{11} \in (e\mathscr{A}e)^{-1}$. Similarly, $b_{22} \in ((1-e)\mathscr{A}(1-e))^{-1}$. Note that $b_{21} = u_{21}a_{11} + u_{22}a_{21} \in J(\mathscr{A})$, then $b_{21}b_{11}^{-1}b_{12} \in J(\mathscr{A}) \bigcap (1-e)\mathscr{A}(1-e) = J((1-e)\mathscr{A}(1-e))$. Thus we get $b_{22} - b_{21}b_{11}^{-1}b_{12} = b_{22}((1-e) - b_{22}^{-1}(b_{21}b_{11}^{-1}b_{12})) \in ((1-e)\mathscr{A}(1-e))^{-1}$. So,

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & 1-e \end{bmatrix} \begin{bmatrix} e & b_{11}^{-1}b_{12} \\ 0 & b_{22} - b_{21}b_{11}^{-1}b_{12} \end{bmatrix} \in [M_2(\mathscr{A}, e)]^{-1}.$$

This completes the proof. □

Lemma 2.7. *Let* $e^2 = e \in \mathscr{A}$ *and let* $a \in e\mathscr{A}e$. *Then* $a \in \mathscr{A}^{pD}$ *if and only if* $a \in (e\mathscr{A}e)^{pD}$. *Moreover,* $a^{\ddagger}_{\mathscr{A}} = a^{\ddagger}_{e\mathscr{A}e}$.

Proof. We assume that $a \in \mathscr{A}^{pD}$ and let $a^{\ddagger}_{\mathscr{A}} = x$. Next, we prove $a^{\ddagger}_{e\mathscr{A}e} = x$. Indeed, $x = ax^3a \in e\mathscr{A}e$. Since $a^{\ddagger}_{\mathscr{A}} = x$, there exists $k \geq 1$ such that $a^k(e - ax) = a^k(1 - ax) \in J(\mathscr{A}) \bigcap e\mathscr{A}e$. By Lemma 2.5 (ii), $a^k(e - ax) \in J(e\mathscr{A}e)$. Also $ax = xa, xax = x$. Thus $a \in (e\mathscr{A}e)^{pD}$ and $a^{\ddagger}_{e\mathscr{A}e} = x$.

Conversely, suppose $a \in (e\mathscr{A}e)^{pD}$ and let $a^{\ddagger}_{e\mathscr{A}e} = y$. We need to prove that $a^{\ddagger}_{\mathscr{A}} = y$. The condition $a^{\ddagger}_{e\mathscr{A}e} = y$ ensures that (a) $yay = y$, (b) $ya = ay$, (c) $a^k(e - ay) \in J(e\mathscr{A}e)$ for some $k \geq 1$. Applying Lemma 2.5(ii), we have $a^k(1 - ay) = a^k(e - ay) \in J(\mathscr{A})$. Hence $a \in \mathscr{A}^{pD}$ and $a^{\ddagger}_{\mathscr{A}} = y$. $\square$

The next result is well-known for the Drazin inverse and the generalized Drazin inverse [14], and it is equally true for the p-Drazin inverse.

Lemma 2.8. *Let $a \in \mathscr{A}$. Then the following conditions are equivalent:*
(i) $a \in \mathscr{A}^{pD}$;
(ii) $a^n \in \mathscr{A}^{pD}$ *for any integer* $n \geq 1$;
(iii) $a^n \in \mathscr{A}^{pD}$ *for some integer* $n \geq 1$.

Proof. (i) $\Rightarrow$ (ii) [18, Theorem 2.3(1)].
(ii) $\Rightarrow$ (iii) It is obvious.
(iii) $\Rightarrow$ (i) First, we prove $a^{n-1} \in \mathscr{A}^{pD}$, and $(a^{n-1})^{\ddagger} = (a^n)^{\ddagger}a = a(a^n)^{\ddagger}$. Let $y = (a^n)^{\ddagger}a = a(a^n)^{\ddagger}$. A direct calculation shows that $ya^{n-1}y = y$, $ya = ay$. Since $a^n \in \mathscr{A}^{pD}$, there exists $k \geq 0$ such that $(a^n)^k[1 - a^n(a^n)^{\ddagger}] \in J(\mathscr{A})$. Take $m = \lfloor \frac{nk}{n-1} \rfloor + 1$, where $\lfloor \frac{nk}{n-1} \rfloor$ denote the integer part of $\frac{nk}{n-1}$. Therefore, by Lemma 2.3, we get $(a^{n-1})^m - (a^{n-1})^{m+1}y = a^{(n-1)m}[1 - a^n(a^n)^{\ddagger}] \in J(\mathscr{A})$. Thus $a^n \in \mathscr{A}^{pD} \Rightarrow a^{n-1} \in \mathscr{A}^{pD} \Rightarrow a^{n-2} \in \mathscr{A}^{pD} \Rightarrow \cdots \Rightarrow a \in \mathscr{A}^{pD}$. $\square$

Lemma 2.9. (i) [17, Theorem 5.3] *If $a, b \in \mathscr{A}$ are p-Drazin invertible, then* $M = \begin{bmatrix} a & d \\ 0 & b \end{bmatrix}$ *is p-Drazin invertible in $M_2(\mathscr{A})$ and* $M^{\ddagger} = \begin{bmatrix} a^{\ddagger} & z_1 \\ 0 & b^{\ddagger} \end{bmatrix}$, *where* $z_1 = \sum_{n=0}^{\infty}(a^{\ddagger})^{n+2}db^nb^{\pi} + \sum_{n=0}^{\infty} a^{\pi}a^nd(b^{\ddagger})^{n+2} - a^{\ddagger}db^{\ddagger}$.

(ii) *If $a, b \in \mathscr{A}$ are p-Drazin invertible, then* $M = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$ *is p-Drazin invertible in $M_2(\mathscr{A})$ and* $M^{\ddagger} = \begin{bmatrix} a^{\ddagger} & 0 \\ z_2 & b^{\ddagger} \end{bmatrix}$, *where* $z_2 = \sum_{n=0}^{\infty}(b^{\ddagger})^{n+2}ca^na^{\pi} + \sum_{n=0}^{\infty} b^{\pi}b^nc(a^{\ddagger})^{n+2} - b^{\ddagger}ca^{\ddagger}$.

Lemma 2.10. [17, Theorem 5.4] *If $a, b \in \mathscr{A}$ are p-Drazin invertible and $ab = 0$, then $a + b$ is p-Drazin invertible and* $(a + b)^{\ddagger} = [\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}a^i]a^{\pi} + b^{\pi}\sum_{i=0}^{\infty} b^i(a^{\ddagger})^{i+1}$.

Lemma 2.11. (i) [17, Theorem 3.6] *Let $a, b \in \mathscr{A}$. If ab is p-Drazin invertible, then so is ba and* $(ba)^{\ddagger} = b((ab)^{\ddagger})^2a$.
(ii) [17, Proposition 3.7] *Let $A \in M_{m\times n}(\mathscr{A})$ and $B \in M_{n\times m}(\mathscr{A})$. If AB has a p-Drazin inverse in $M_m(\mathscr{A})$, then so does BA in $M_n(\mathscr{A})$ and* $(BA)^{\ddagger} = B((AB)^{\ddagger})^2A$.

3. Main Results

In what follows, by $\mathscr{A}_1$, $\mathscr{A}_2$ we denote the algebra $e\mathscr{A}e$, $(1 - e)\mathscr{A}(1 - e)$, where $e^2 = e \in \mathscr{A}$, respectively. If $a \in \mathscr{A}^{pD}$, we use a^{π} to denote $1 - aa^{\ddagger}$. We start with a theorem which gives a matrix representation of a p-Drazin invertible element in a Banach algebra.

Theorem 3.1. *$a \in \mathscr{A}$ is p-Drazin invertible if and only if there exists an idempotent $e \in \mathscr{A}$ such that*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_e, \text{ where } a_1 \in \mathscr{A}_1^{-1},\ a_2 \in \sqrt{J(\mathscr{A}_2)}. \tag{4}$$

In which case,

$$a^{\ddagger} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_e \text{ and } e = aa^{\ddagger}. \tag{5}$$

Proof. We suppose that $a \in \mathscr{A}^{pD}$, then let $e = aa^\ddagger$. Obviously, $ea(1-e) = aa^\ddagger a(1-aa^\ddagger) = 0, (1-e)ae = (1-aa^\ddagger)aaa^\ddagger = 0$. Since $a_1(ea^\ddagger e) = e, (ea^\ddagger e)a_1 = e$, so $a_1 \in \mathscr{A}_1^{-1}$.

We know that $a_2^k = [(1-aa^\ddagger)a(1-aa^\ddagger)]^k = a^k(1-aa^\ddagger) \in J(\mathscr{A}) \bigcap \mathscr{A}_2$ for some $k \geq 1$. By Lemma 2.5(ii), we can obtain $a_2^k \in J(\mathscr{A}_2)$, that is $a_2 \in \sqrt{J(\mathscr{A}_2)}$.

Conversely, let

$$x = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_e.$$

A direct calculation shows that $xax = x, ax = xa$. Since $a_2 \in \sqrt{J(\mathscr{A}_2)}$, there exists $k \geq 1$ such that $a_2^k \in J(\mathscr{A}_2)$. Relative to the idempotent e,

$$a^k(1-ax) = \begin{bmatrix} 0 & 0 \\ 0 & a_2^k \end{bmatrix}_e.$$

Thus $a^k(1-ax) = a_2^k \in J(\mathscr{A}_2)$. Using Lemma 2.5(ii), $a^k(1-ax) \in J(\mathscr{A}_2) \subseteq J(\mathscr{A})$. This proves $a \in \mathscr{A}^{pD}$. □

The following result will be very useful in proving our main results.

Theorem 3.2. *Let $e^2 = e, x, y \in \mathscr{A}$ and let x and y have the representation*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_e, \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-e}. \tag{6}$$

(i) *If $a \in \mathscr{A}_1^{pD}$ and $b \in \mathscr{A}_2^{pD}$, then $x, y \in \mathscr{A}^{pD}$ and*

$$x^\ddagger = \begin{bmatrix} a^\ddagger & u \\ 0 & b^\ddagger \end{bmatrix}_e, \quad y^\ddagger = \begin{bmatrix} b^\ddagger & 0 \\ u & a^\ddagger \end{bmatrix}_{1-e}, \tag{7}$$

where

$$u = \sum_{n=0}^{\infty}(a^\ddagger)^{n+2}cb^n b^\pi + \sum_{n=0}^{\infty} a^\pi a^n c(b^\ddagger)^{n+2} - a^\ddagger cb^\ddagger. \tag{8}$$

(ii) *If $x \in \mathscr{A}^{pD}$ [resp. $y \in \mathscr{A}^{pD}$] and $a \in \mathscr{A}_1^{pD}$, then $b \in \mathscr{A}_2^{pD}$, and $x^\ddagger$ [resp. $y^\ddagger$] is given by* (7) *and* (8).

Proof. (i) Applying Lemma 2.9 and Lemma 2.7, we can get

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in [M_2(\mathscr{A})]^{pD}, \quad \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^\ddagger = \begin{bmatrix} a^\ddagger & u \\ 0 & b^\ddagger \end{bmatrix},$$

where $u = \sum_{n=0}^{\infty}(a^\ddagger)^{n+2}cb^n b^\pi + \sum_{n=0}^{\infty} a^\pi a^n c(b^\ddagger)^{n+2} - a^\ddagger cb^\ddagger$. Then there exists $k \geq 1$ such that

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^k - \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^{k+1} \begin{bmatrix} a^\ddagger & u \\ 0 & b^\ddagger \end{bmatrix} \in J(M_2(\mathscr{A})).$$

Lemma 2.6 ensures that

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^k - \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^{k+1} \begin{bmatrix} a^\ddagger & u \\ 0 & b^\ddagger \end{bmatrix} \in J(M_2(\mathscr{A}, e)).$$

Thus, we have that $\left[\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right] \in [M_2(\mathscr{A}, e)]^{pD}$, which implies $x \in \mathscr{A}^{pD}$.

Next, we consider the p-Drazin inverse of y. Since

$$y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-e} = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_{e},$$

from the first part of (i), we obtain $y \in \mathscr{A}^{pD}$ and

$$y^{\ddagger} = \begin{bmatrix} a^{\ddagger} & u \\ 0 & b^{\ddagger} \end{bmatrix}_{e} = \begin{bmatrix} b^{\ddagger} & 0 \\ u & a^{\ddagger} \end{bmatrix}_{1-e}.$$

The proof of (i) is completed.

(ii) We prove $b^{\ddagger} = [(1-e)x(1-e)]^{\ddagger} = (1-e)x^{\ddagger}(1-e)$.

Since $x \in \mathscr{A}^{pD}$, $a \in \mathscr{A}_1^{pD}$, then $x \in \mathscr{A}^{d}$, $a \in \mathscr{A}_1^{d}$ and $x^d = x^{\ddagger}$, $a^d = a^{\ddagger}$. According to Theorem 2.3 (ii) of [2], it follows that

$$\begin{bmatrix} a^d & u \\ 0 & b^d \end{bmatrix}_{e} = x^d = \begin{bmatrix} ex^de & ex^d(1-e) \\ (1-e)x^de & (1-e)x^d(1-e) \end{bmatrix}_{e},$$

where u is defined as (8). Thus, $(1-e)x^de = 0$, i.e. $(1-e)x^{\ddagger}e = 0$, which implies that $(1-e)x^{\ddagger}(1-e) = (1-e)x^{\ddagger}$. Noting that $(1-e)xe = 0$, we can get $(1-e)x(1-e) = (1-e)x$. Therefore, we only prove $[(1-e)x]^{\ddagger} = (1-e)x^{\ddagger}$.

Let $v = (1-e)x^{\ddagger}$.

(a) $[(1-e)x]v = (1-e)x(1-e)x^{\ddagger} = (1-e)xx^{\ddagger} = (1-e)x^{\ddagger}x = [(1-e)x^{\ddagger}](1-e)x = v[(1-e)x]$.

(b) $v[(1-e)x]v = (1-e)x^{\ddagger}(1-e)x(1-e)x^{\ddagger} = (1-e)x^{\ddagger}(1-e)xx^{\ddagger} = (1-e)x^{\ddagger}xx^{\ddagger} = (1-e)x^{\ddagger} = v$.

(c) First, we prove $[(1-e)(x-x^2x^{\ddagger})]^n = (1-e)(x-x^2x^{\ddagger})^n$ for any $n \geq 1$ by induction.

It is obvious for $n = 1$.

Assume $[(1-e)(x-x^2x^{\ddagger})]^n = (1-e)(x-x^2x^{\ddagger})^n$.

For the $n+1$ case, we have

$$\begin{aligned}
& [(1-e)(x-x^2x^{\ddagger})]^{n+1} \\
= \;& (1-e)(x-x^2x^{\ddagger})[(1-e)(x-x^2x^{\ddagger})]^n \\
= \;& [(1-e)(x-x^2x^{\ddagger})(1-e)](x-x^2x^{\ddagger})^n \\
= \;& [(1-e)x(1-e)-(1-e)x^2x^{\ddagger}(1-e)](x-x^2x^{\ddagger})^n \\
= \;& [(1-e)x-(1-e)x(1-e)x(1-e)x^{\ddagger}(1-e)](x-x^2x^{\ddagger})^n \\
= \;& [(1-e)x-(1-e)x(1-e)x(1-e)x^{\ddagger}](x-x^2x^{\ddagger})^n \\
= \;& [(1-e)x-(1-e)x(1-e)xx^{\ddagger}](x-x^2x^{\ddagger})^n \\
= \;& [(1-e)x-(1-e)x^2x^{\ddagger}](x-x^2x^{\ddagger})^n \\
= \;& (1-e)(x-x^2x^{\ddagger})^{n+1}.
\end{aligned}$$

Since there exists $k \geq 0$ such that $(x-x^2x^{\ddagger})^k \in J(\mathscr{A})$,

$$\begin{aligned}
& \{(1-e)x-[(1-e)x]^2v\}^k \\
= \;& \{(1-e)x-[(1-e)x]^2(1-e)x^{\ddagger}\}^k \\
= \;& [(1-e)x-(1-e)x^2x^{\ddagger}]^k \\
= \;& [(1-e)(x-x^2x^{\ddagger})]^k \\
= \;& (1-e)(x-x^2x^{\ddagger})^k \in J(\mathscr{A}) \bigcap \mathscr{A}_2 = J(\mathscr{A}_2).
\end{aligned}$$

Hence $b^{\ddagger} = (1-e)x^{\ddagger}$. Using (i), we see $x^{\ddagger}$ is given by (7) and (8).

Following an analogous strategy as in the proof for y of (i), we have (ii) for y. □

Remark 3.3. Theorem 3.2 (i) is more general than Lemma 2.9. Indeed, let $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} a & d \\ 0 & b \end{bmatrix} = \begin{bmatrix} A & D \\ 0 & B \end{bmatrix}_e$, where $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, $D = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$. Since $a \in \mathscr{A}^{pD}$, $b \in \mathscr{A}^{pD}$, we have $A \in [eM_2(\mathscr{A})e]^{pD}$, $B \in [(1-e)M_2(\mathscr{A})(1-e)]^{pD}$. Thus, using Theorem 3.2 (i), we get $\begin{bmatrix} a & d \\ 0 & b \end{bmatrix} \in [M_2(\mathscr{A})]^{pD}$.

Before proving our main result, we need to prove the following result.

Theorem 3.4. *Let $a \in \mathscr{A}^{pD}$, $b \in \sqrt{J(\mathscr{A})}$. If $aba = 0$, $ab^2 = 0$, then $a + b \in \mathscr{A}^{pD}$ and*

$$(a+b)^{\ddagger} = (a^{\ddagger} + bua)(1 + a^{\ddagger}b), \tag{9}$$

where $u = \sum_{n=0}^{\infty} b^{2n}(a+b)(a^{\ddagger})^{2n+4}$.

Proof. Write $X_1 = \begin{bmatrix} a \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 & b \end{bmatrix}$, then $a + b = X_2X_1$. Let $M = X_1X_2 = \begin{bmatrix} a & ab \\ 1 & b \end{bmatrix}$, then $M^2 = \begin{bmatrix} a^2+ab & a^2b \\ a+b & ab+b^2 \end{bmatrix} = \begin{bmatrix} ab & a^2b \\ 0 & ab \end{bmatrix} + \begin{bmatrix} a^2 & 0 \\ a+b & b^2 \end{bmatrix} := F + G$. The conditions $aba = 0$ and $ab^2 = 0$ imply $FG = 0$, $F^2 = 0$.

Since $a \in \mathscr{A}^{pD}$, then $a^2 \in \mathscr{A}^{pD}$ and $(a^2)^{\ddagger} = (a^{\ddagger})^2$. According to the condition $b \in \sqrt{J(\mathscr{A})}$, then $b^k \in J(\mathscr{A})$, for some $k \geq 1$, which implies $b^{\ddagger} = 0$ by (3). Using Lemma 2.9(ii) , we can get $G \in [M_2(\mathscr{A})]^{pD}$ and $G^{\ddagger} = \begin{bmatrix} (a^{\ddagger})^2 & 0 \\ u & 0 \end{bmatrix}$, where $u = \sum_{n=0}^{\infty} b^{2n}(a+b)(a^{\ddagger})^{2n+4}$.

Because $F^2 = 0$, then $F^{\ddagger} = 0$. Using Lemma 2.10, we deduce that $M^2 \in [M_2(\mathscr{A})]^{pD}$, and $(M^2)^{\ddagger} = G^{\ddagger} + (G^{\ddagger})^2F = \begin{bmatrix} (a^{\ddagger})^2 + (a^{\ddagger})^3b & (a^{\ddagger})^2b \\ u + ua^{\ddagger}b & ua^{\ddagger}ab \end{bmatrix}$. Applying Lemma 2.8, $M \in [M_2(\mathscr{A})]^{pD}$.

Finally, according to Lemma 2.11(ii), we have that $a + b \in \mathscr{A}^{pD}$ and $(a+b)^{\ddagger} = X_2(M^2)^{\ddagger}X_1$. Observe that $a^{\ddagger}ba = 0$ and by a straightforward computation, we obtain (9). □

Next we present our main theorem, which is a generalization of [17, Theorem 5.4].

Theorem 3.5. *Let $a, b \in \mathscr{A}^{pD}$ be such that $s = (1-b^{\pi})a(1-b^{\pi}) \in \mathscr{A}^{pD}$. If $b^{\pi}aba = 0$, $b^{\pi}ab^2 = 0$, then $a + b \in \mathscr{A}^{pD}$ if and only if $t = (1-b^{\pi})(a+b)(1-b^{\pi}) \in \mathscr{A}^{pD}$. In which case,*

$$(a+b)^{\ddagger} = t^{\ddagger} + (1 - t^{\ddagger}a)x + \sum_{n=0}^{\infty}(t^{\ddagger})^{n+2}ab^{\pi}(a+b)^n[1-(a+b)x] + \sum_{n=0}^{\infty} t^{\pi}t^n(1-b^{\pi})ax^{n+2}, \tag{10}$$

where $x = \sum\limits_{n=0}^{\infty} b^{\pi}b^n(a^{\ddagger})^{n+1}b^{\pi}(1+a^{\ddagger}b)$.

Proof. According to Theorem 3.1, we consider the matrix representation of a and b relative to the idempotent $e = bb^{\ddagger}$:

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_e, \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_e,$$

where $b_1 \in \mathscr{A}_1^{-1}$, $b_2 \in \sqrt{J(\mathscr{A}_2)}$. The condition $b^{\pi}ab^2 = 0$ expressed in matrix form yields

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_e = b^{\pi}ab^2 = \begin{bmatrix} 0 & 0 \\ a_{21}b_1^2 & a_{22}b_2^2 \end{bmatrix}_e.$$

This gives $a_{21} = 0$, $a_{22}b_2^2 = 0$. Denote $a_1 = a_{11}$, $a_2 = a_{22}$, $a_3 = a_{12}$. Thus,

$$a = \begin{bmatrix} a_1 & a_3 \\ 0 & a_2 \end{bmatrix}_e, \quad a + b = \begin{bmatrix} t & a_3 \\ 0 & a_2 + b_2 \end{bmatrix}_e.$$

Since $a_1 = s \in \mathscr{A}^{pD}$, by Lemma 2.7, we have $a_1 \in \mathscr{A}_1^{pD}$. Also, $a \in \mathscr{A}^{pD}$. Using Theorem 3.2 (ii), we deduce that $a_2 \in \mathscr{A}_2^{pD}$ and

$$a^{\ddagger} = \begin{bmatrix} a_1^{\ddagger} & u_1 \\ 0 & a_2^{\ddagger} \end{bmatrix}_e.$$

From the condition $b^\pi aba = 0$, we can get that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_e = b^\pi aba = \begin{bmatrix} 0 & 0 \\ 0 & a_2b_2a_2 \end{bmatrix}_e,$$

which implies $a_2b_2a_2 = 0$.

Hence, applying Theorem 3.4 to a_2, b_2, we conclude that $a_2 + b_2 \in \mathscr{A}_2^{pD}$ and

$$(a_2 + b_2)^\ddagger = [a_2^\ddagger + \sum_{n=0}^{\infty} b_2^{2n+1}(a_2 + b_2)(a_2^\ddagger)^{2n+3}](1 - e + a_2^\ddagger b_2).$$

In order to give the expression of $(a_2 + b_2)^\ddagger$ in terms of $a, a^\ddagger, b, b^\ddagger$, we calculate $b^\pi a^\ddagger, b^\pi b^{2n+1}(a + b)(a^\ddagger)^{2n+3}, b^\pi a^\ddagger b$ separately in matrix form as follows:

$$b^\pi a^\ddagger = \begin{bmatrix} 0 & 0 \\ 0 & a_2^\ddagger \end{bmatrix}_e, \quad b^\pi a^\ddagger b = \begin{bmatrix} 0 & 0 \\ 0 & a_2^\ddagger b_2 \end{bmatrix}_e,$$

$$b^\pi b^{2n+1}(a + b)(a^\ddagger)^{2n+3} = \begin{bmatrix} 0 & 0 \\ 0 & b_2^{2n+1}(a_2 + b_2)(a_2^\ddagger)^{2n+3} \end{bmatrix}_e.$$

Thus, $b^\pi a^\ddagger = a_2^\ddagger$, $b^\pi b^{2n+1}(a + b)(a^\ddagger)^{2n+3} = b_2^{2n+1}(a_2 + b_2)(a_2^\ddagger)^{2n+3}$ and $b^\pi a^\ddagger b = a_2^\ddagger b_2$. Write $x = (a_2 + b_2)^\ddagger$. Note that $a(a^\ddagger)^{2n+3} = (a^\ddagger)^{2n+2}$ for $n \geq 0$, then we have

$$\begin{aligned} x &= b^\pi[a^\ddagger + \sum_{n=0}^{\infty} b^{2n+1}(a + b)(a^\ddagger)^{2n+3}]b^\pi(1 + a^\ddagger b) \\ &= b^\pi[a^\ddagger + \sum_{n=0}^{\infty} b^{2n+1}(a^\ddagger)^{2n+2} + \sum_{n=0}^{\infty} b^{2n+2}(a^\ddagger)^{2n+3}]b^\pi(1 + a^\ddagger b) \\ &= b^\pi \sum_{n=0}^{\infty} b^n(a^\ddagger)^{n+1}b^\pi(1 + a^\ddagger b). \end{aligned}$$

Now, by Theorem 3.2, we have that $a + b \in \mathscr{A}^{pD}$ if and only if $t \in \mathscr{A}^{pD}$. Moreover,

$$(a + b)^\ddagger = \begin{bmatrix} t^\ddagger & u \\ 0 & x \end{bmatrix}_e,$$

where

$$u = \sum_{n=0}^{\infty}(t^\ddagger)^{n+2}a_3(a_2 + b_2)^n(a_2 + b_2)^\pi + \sum_{n=0}^{\infty} t^\pi t^n a_3 x^{n+2} - t^\ddagger a_3 x. \tag{11}$$

Because $b^\pi ab^2 = 0$, we have $b^\pi ab^\ddagger = 0$. Thus, $a_2 + b_2 = b^\pi(a + b)b^\pi = b^\pi ab^\pi + b^\pi b = b^\pi a(1 - bb^\ddagger) + b^\pi b = b^\pi(a + b)$, which ensures $(a_2 + b_2)^n = b^\pi(a + b)^n b^\pi$ for any $n \geq 1$. Also, we can easily obtain that $b^\pi(a + b)^n b^\pi = b^\pi(a + b)^n$ for any $n \geq 1$ by induction. Note $a_3 = (1 - b^\pi)ab^\pi$. Thus, (11) reduces to

$$u = \sum_{n=0}^{\infty}(t^\ddagger)^{n+2}ab^\pi(a + b)^n[1 - (a + b)x] + \sum_{n=0}^{\infty} t^\pi t^n(1 - b^\pi)ax^{n+2} - t^\ddagger ax. \tag{12}$$

From $(a + b)^\ddagger = t^\ddagger + u + x$, we get that (10) holds. □

Next, we present one special case of the preceding theorem.

Corollary 3.6. *Let $a, b \in \mathscr{A}^{pD}$. If $aba = 0$, $ab^2 = 0$, then $a + b \in \mathscr{A}^{pD}$ and*

$$(a+b)^\ddagger = b^\ddagger a^\pi + (b^\ddagger)^2 aa^\pi + \sum_{n=1}^{\infty}(b^\ddagger)^{n+2}(a^{n+1}a^\pi - a^{n+1}a^\ddagger b + a^n b) + \sum_{n=0}^{\infty} b^\pi b^n(a^\ddagger)^{n+1}(1 + a^\ddagger b) - b^\ddagger a^\ddagger b - (b^\ddagger)^2 aa^\ddagger b. \tag{13}$$

Proof. From $ab^2 = 0$, it follows that $ab^\ddagger = 0$. Thus, we can have that $s = (1 - b^\pi)a(1 - b^\pi) = 0 \in \mathscr{A}^{pD}$, $t = (1 - b^\pi)(a + b)(1 - b^\pi) = b(bb^\ddagger)$. Since $(bb^\ddagger)^\ddagger = bb^\ddagger$, using Proposition 5.2 of [17], we deduce that $t \in \mathscr{A}^{pD}$ and $t^\ddagger = b^\ddagger$. Thus, Theorem 3.5 is applicable.

Furthermore, note that $a^\ddagger b^\ddagger = 0$, $aba^\ddagger = 0$. Let $x = \sum\limits_{n=0}^{\infty} b^\pi b^n (a^\ddagger)^{n+1} b^\pi (1 + a^\ddagger b)$. We have

$$\begin{aligned}
ax &= a \sum_{n=0}^{\infty} b^\pi b^n (a^\ddagger)^{n+1} b^\pi (1 + a^\ddagger b) \\
&= a(1 - bb^\ddagger)a^\ddagger b^\pi (1 + a^\ddagger b) + a(1 - bb^\ddagger)b(a^\ddagger)^2 b^\pi (1 + a^\ddagger b) \\
&= aa^\ddagger b^\pi (1 + a^\ddagger b) + ab(a^\ddagger)^2 b^\pi (1 + a^\ddagger b) \\
&= aa^\ddagger (1 - bb^\ddagger)(1 + a^\ddagger b) \\
&= aa^\ddagger + a^\ddagger b. \\
abx &= ab[\sum_{n=0}^{\infty} b^\pi b^n (a^\ddagger)^{n+1} b^\pi (1 + a^\ddagger b)] \\
&= ab(1 - bb^\ddagger)a^\ddagger b^\pi (1 + a^\ddagger b) \\
&= 0.
\end{aligned}$$

Therefore, $a[1 - (a + b)x] = a - a^2 x - abx = aa^\pi - aa^\ddagger b$.

On the other hand, $a(a + b)^n = a^n (a + b)$ for $n \geq 1$. So, we deduce that

$$\begin{aligned}
& ab^\pi (a + b)^n [1 - (a + b)x] \\
= \; & a(1 - bb^\ddagger)(a + b)^n [1 - (a + b)x] \\
= \; & a(a + b)^n [1 - (a + b)x] \\
= \; & a^n (a + b)[1 - (a + b)x] \\
= \; & a^n a[1 - (a + b)x] + a^n b[1 - (a + b)x] \\
= \; & a^{n+1} a^\pi - a^{n+1} a^\ddagger b + a^n b.
\end{aligned}$$

Observe that $t^\pi t^n (1 - b^\pi) a x^{n+2} = b^\pi (b^2 b^\ddagger)^n (1 - b^\pi) a x^{n+2} = 0$ for $n \geq 0$.

Finally, by using these relations and (10), we get (13). □

Now, we give an example to show that the conditions of Theorem 3.5 are weaker than Corollary 3.6.

Example 3.7. Let $\mathscr{A}$ be the algebra of all complex 2×2 matrices, and let $a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, we can check that a, b satisfy $b^\pi aba = 0$, $b^\pi ab^2 = 0$, but $aba \neq 0$, $ab^2 \neq 0$.

With Corollary 3.6, we recover the case $ab = 0$ studied in [17].

In the following results, we give expressions for $(a + b)^\ddagger$ under certain conditions which do not use $b \in \mathscr{A}^{pD}$.

Theorem 3.8. *Let $a, b, e \in \mathscr{A}$ be such that $a \in \mathscr{A}^{pD}$, $e^2 = e$, $ea = ae$, $be = b$ [resp. $eb = b$]. If $r = (a + b)e \in \mathscr{A}^{pD}$, then $a + b \in \mathscr{A}^{pD}$ and*

$$(a+b)^\ddagger = \sum_{n=0}^{\infty} (1-e)a^\pi a^n b (r^\ddagger)^{n+3}(a+b) - a^\ddagger (1-e) b (r^\ddagger)^2 (a+b) + a^\ddagger (1-e) + \sum_{n=0}^{\infty} (a^\ddagger)^{n+2} (1-e) b (a+b)^n [1 - r^\ddagger (a+b)] + e(r^\ddagger)^2 (a+b) \tag{14}$$

$$[\mathit{resp.}\ \ (a + b)^\ddagger = r^\ddagger + \sum_{n=0}^{\infty} (r^\ddagger)^{n+2} b(1 - e)a^n a^\pi + (1 - r^\ddagger b)(1 - e)a^\ddagger + r^\pi \sum_{n=0}^{\infty} r^n b(1 - e)(a^\ddagger)^{n+2}]. \tag{15}$$

Proof. We consider the matrix representation of e, a, b relative to e. We have

$$e = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}_e, \ a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_e, \ b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_e.$$

The condition $ea = ae$ implies $a_{12} = 0, a_{21} = 0$. We denote $a_1 = a_{11}, a_2 = a_{22}$. Thus

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_e.$$

Observe that $(1-e)a = a(1-e)$ and $(1-e)^{\ddagger} = 1-e$, using Proposition 5.2 of [17], we can conclude that $a_2 = (1-e)a \in \mathscr{A}_2^{pD}$ and $a_2^{\ddagger} = (1-e)a^{\ddagger} = a^{\ddagger}(1-e)$.

From $be = b$, it follows that $b_{12} = 0,\ b_{22} = 0$. Denote $b_1 = b_{11}, b_3 = b_{21}$. Hence ,

$$a+b = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_e + \begin{bmatrix} b_1 & 0 \\ b_3 & 0 \end{bmatrix}_e = \begin{bmatrix} a_1+b_1 & 0 \\ b_3 & a_2 \end{bmatrix}_e.$$

Since $be = b$, then $a_1 + b_1 = e(a+b)e = e(a+b)$ which implies $e(a+b)^n e = e(a+b)^n$, $[e(a+b)]^n = e(a+b)^n$ for any $n \geq 1$ by induction. From the condition $r = (a+b)e \in \mathscr{A}^{pD}$ and Lemma 2.11(i), we deduce that $a_1 + b_1 \in \mathscr{A}_1^{pD}$ and $(a_1+b_1)^{\ddagger} = e(r^{\ddagger})^2(a+b)$. According to Theorem 3.2 (i), we obtain that $a+b \in \mathscr{A}^{pD}$ and

$$(a+b)^{\ddagger} = \begin{bmatrix} (a_1+b_1)^{\ddagger} & 0 \\ u & a_2^{\ddagger} \end{bmatrix}_e,$$

where

$$u = \sum_{n=0}^{\infty}(a_2^{\ddagger})^{n+2}b_3(a_1+b_1)^n(a_1+b_1)^{\pi} + \sum_{n=0}^{\infty}a_2^{\pi}a_2^n b_3[(a_1+b_1)^{\ddagger}]^{n+2} - a_2^{\ddagger}b_3(a_1+b_1)^{\ddagger}. \tag{16}$$

Note that

$$\begin{aligned}
& (a_2^{\ddagger})^{n+2}b_3(a_1+b_1)^n(a_1+b_1)^{\pi} \\
= {} & [a^{\ddagger}(1-e)]^{n+2}(1-e)be[e(a+b)]^n[e-e(a+b)e(r^{\ddagger})^2(a+b)] \\
= {} & (a^{\ddagger})^{n+2}(1-e)be(a+b)^n e[1-(a+b)e(r^{\ddagger})^2(a+b)] \\
= {} & (a^{\ddagger})^{n+2}(1-e)b(a+b)^n[1-r^{\ddagger}(a+b)], \\
& a_2^{\pi}a_2^n b_3[(a_1+b_1)^{\ddagger}]^{n+2} \\
= {} & [(1-e)-(1-e)aa^{\ddagger}(1-e)][(1-e)a]^n(1-e)be[e(r^{\ddagger})^2(a+b)]^{n+2} \\
= {} & (1-e)(1-aa^{\ddagger})(1-e)a^n(1-e)be[e(r^{\ddagger})^{n+3}(a+b)] \\
= {} & (1-e)a^{\pi}a^n b(r^{\ddagger})^{n+3}(a+b), \\
& a_2^{\ddagger}b_3(a_1+b_1)^{\ddagger} \\
= {} & a^{\ddagger}(1-e)(1-e)bee(r^{\ddagger})^2(a+b) \\
= {} & a^{\ddagger}(1-e)b(r^{\ddagger})^2(a+b).
\end{aligned}$$

Therefore we have (14).

The proof for the case of $eb = b$ is analogous. □

In [4], expressions of the Drazin inverse of $a+b$ in the additive category are given under the following conditions:

(1) a is Drazin invertible, $r = (a+b)a^{\pi}$ is Drazin invertible, $a^D b = 0$;

(2) a is Drazin invertible, $r = (a+b)aa^D$ is Drazin invertible, $aa^D b = b$. Here, we consider expressions of $(a+b)^{\ddagger}$ under the similar conditions in a Banach algebra.

Corollary 3.9. *Let $a \in \mathscr{A}^{pD}$, $b \in \mathscr{A}$ such that $ba^{\ddagger} = 0$ [resp. $a^{\ddagger}b = 0$], $r = (a+b)a^{\pi} \in \mathscr{A}^{pD}$. Then $a+b \in \mathscr{A}^{pD}$ and*

$$(a+b)^{\ddagger} = \sum_{n=0}^{\infty}(a^{\ddagger})^{n+2}b(a+b)^n[1-r^{\ddagger}(a+b)] + a^{\ddagger} + [1-a^{\ddagger}(a+b)](r^{\ddagger})^2(a+b) \tag{17}$$

$$[\textit{resp. } (a+b)^{\ddagger} = a^{\ddagger} + r^{\ddagger} + r^{\pi}\sum_{n=0}^{\infty}r^n b(a^{\ddagger})^{n+2} - r^{\ddagger}ba^{\ddagger}]. \tag{18}$$

Proof. Let $e = a^\pi$ in Theorem 3.8. □

Corollary 3.10. *Let $a \in \mathscr{A}^{pD}$, $b \in \mathscr{A}$ with $baa^\ddagger = b$ [resp. $aa^\ddagger b = b$], $r = (a+b)aa^\ddagger \in \mathscr{A}^{pD}$. Then $a+b \in \mathscr{A}^{pD}$ and*

$$(a+b)^\ddagger = (1-a^\pi)(r^\ddagger)^2(a+b) + \sum_{n=0}^{\infty} a^\pi a^n b (r^\ddagger)^{n+3}(a+b) \tag{19}$$

$$[resp.\ (a+b)^\ddagger = r^\ddagger + \sum_{n=0}^{\infty} (r^\ddagger)^{n+2} b a^n a^\pi]. \tag{20}$$

Proof. Let $e = aa^\ddagger$ in Theorem 3.8. □

In [3], Castro-González, Koliha and Wei studied the necessary and sufficient conditions for $(A+B)^D = (I + A^D B)^{-1} A^D$, where A, B are complex matrices and $I + A^D B$ is invertible. Here, we consider the necessary and sufficient conditions for $(a+b)^\ddagger = (1 + a^\ddagger b)^{-1} a^\ddagger$ in a Banach algebra.

Theorem 3.11. *Let $a \in \mathscr{A}^{pD}$, $b \in \mathscr{A}$ and let $1 + a^\ddagger b \in \mathscr{A}^{-1}$, $a^\pi b = ba^\pi$, $aa^\pi b = baa^\pi$. Then the following conditions are equivalent:*

(i) *$a + b \in \mathscr{A}^{pD}$ and $(a+b)^\ddagger = (1 + a^\ddagger b)^{-1} a^\ddagger$;*

(ii) *$a^\pi b \in \sqrt{J(\mathscr{A})}$.*

Proof. We consider the matrix representation of a and b relative to $e = aa^\ddagger$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_e, \quad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_e,$$

where $a_1 \in \mathscr{A}_1^{-1}$, $a_2 \in \sqrt{J(\mathscr{A}_2)}$. From the matrix form of $a^\pi b = ba^\pi$, it follows that $b_{12} = 0$, $b_{21} = 0$. Denote $b_1 = b_{11}$, $b_2 = b_{22}$. Thus,

$$a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix}_e.$$

Since

$$1 + a^\ddagger b = \begin{bmatrix} e + a_1^{-1} b_1 & 0 \\ 0 & 1 - e \end{bmatrix}_e \in \mathscr{A}^{-1},$$

we have $e + a_1^{-1} b_1 \in \mathscr{A}_1^{-1}$. Thus, $a_1 + b_1 \in \mathscr{A}_1^{-1}$ and $(a_1 + b_1)^{-1} = (e + a_1^{-1} b_1)^{-1} a_1^{-1}$. Calculations show that $(e + a_1^{-1} b_1)^{-1} a_1^{-1} = (1 + a^\ddagger b)^{-1} a^\ddagger$. The condition $aa^\pi b = baa^\pi$ implies $a_2 b_2 = b_2 a_2$.

(ii) ⇒ (i) Since $a^\pi b = b_2 \in \sqrt{J(\mathscr{A})}$, using Lemma 2.5 (ii) and Lemma 2.4, we obtain $a_2 + b_2 \in \sqrt{J(\mathscr{A}_2)}$, which implies $(a_2 + b_2)^\ddagger = 0$. Thus, $(a+b) \in \mathscr{A}^{pD}$ and

$$(a+b)^\ddagger = \begin{bmatrix} (a_1 + b_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}_e.$$

Hence $(a+b)^\ddagger = (1 + a^\ddagger b)^{-1} a^\ddagger$.

(i) ⇒ (ii) From $(a+b)^\ddagger = (a_1 + b_1)^{-1} + (a_2 + b_2)^\ddagger$ and the condition (i), we obtain $(a_2 + b_2)^\ddagger = 0$. Thus, $a_2 + b_2 \in \sqrt{J(\mathscr{A}_2)}$. By Lemma 2.5 (ii) and Lemma 2.4 again, we have that $a^\pi b = b_2 \in \sqrt{J(\mathscr{A})}$. □

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