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Positivity of a Differential Operator with Nonlocal Conditions

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Abstract. In the present paper, the positivity of the differential operator with nonlocal boundary conditions is established. The structure of fractional spaces is investigated. In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for parabolic equations.

1. Introduction

Various local and nonlocal boundary value problems for partial differential equations can be considered as an abstract boundary value problem for ordinary differential equations in a Banach space with a densely defined unbounded space operator. The theory of differential and difference operators and their related applications has been investigated by many researchers (see, for example,[1–17]).

In the present paper, we consider the differential operator A^x defined by the formula

$$A^x u = -\frac{d^2u}{dx^2} + \delta u, \tag{1}$$

with the domain

$$D(A^x) = \left\{u \in C^{(2)}[0,1] : u(0) = 0, u(1) = u(\mu), 0 \leq \mu < 1\right\},$$

where $\delta > 0$.

We will introduce the Banach space $C^\beta[0,1]$ $(0 < \beta < 1)$ of all continuous functions φ defined on $[0,1]$ and satisfying a Hölder condition for which the following norm is finite:

$$\left\|\varphi\right\|_{C^\beta[0,1]} = \sup_{x\in[0,1]} \left|\varphi(x)\right| + \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{\left|\varphi(y) - \varphi(x)\right|}{\left|y-x\right|^\beta}.$$

Let $\overset{\circ}{C}{}^{\beta}[0,1] = \left\{u \in C^\beta[0,1] : u(0) = 0, u(1) = u(\mu), 0 \leq \mu < 1\right\}$.

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Moreover, for a positive operator A in the Banach space E, let us introduce the fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \alpha < 1)$ consisting of those $v \in E$ for which norms

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \|A(\lambda + A)^{-1} v\|_E$$

are finite.

We will investigate the resolvent of the operator $-A^x$, that is, we consider the equation

$$A^x u + \lambda u = f \tag{2}$$

or

$$\begin{aligned} &-\frac{d^2u(x)}{dx^2} + \delta u(x) + \lambda u(x) = f(x), \\ &u(0) = 0, u(1) = u(\mu), 0 \le \mu < 1. \end{aligned} \tag{3}$$

We establish the positivity of the differential operator A^x in $C[0,1]$. The structure of fractional spaces $E_\alpha(C[0,1], A^x)$ will be investigated. It is established that for any $0 < \alpha < 1/2$ the norms in the spaces $E_\alpha(C[0,1], A^x)$ and $\overset{\circ}{C}{}^{2\alpha}[0,1]$ are equivalent. This result permits us to prove the positivity of A^x in $\overset{\circ}{C}{}^{2\alpha}[0,1]$. In applications, we will obtain new coercive inequalities for the solution of a nonlocal boundary value problem for parabolic equation.

2. Green's Function and Positivity of A^x in $C[0,1]$

In this section, we will explain the proof of the positivity in $C[0,1]$ of the operator A^x defined by formula (1).

Lemma 2.1. *Let $\lambda > 0$. Then, the following equation*

$$A^x u + \lambda u = f \tag{4}$$

is uniquely solvable, and the formula holds:

$$u(x) = (A^x + \lambda)^{-1} f(x) = \int_0^1 G(x, s, \mu; \lambda + \delta) f(s) ds, \tag{5}$$

where

$$\begin{aligned} G(x, s, \mu; \lambda + \delta) = &-(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)}) \frac{T}{2\sqrt{\delta+\lambda}} (e^{-\sqrt{\delta+\lambda}(1-s)} - e^{-\sqrt{\delta+\lambda}(1+s)}) \\ &\times [(1 - e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1} (1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1} (e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)}) + 1] \\ &+ (e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)}) (1 - e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1} (1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1} \frac{1}{2\sqrt{\delta+\lambda}} \\ &\times (e^{-\sqrt{\delta+\lambda}|\mu-s|} - e^{-\sqrt{\delta+\lambda}(\mu+s)}) + \frac{1}{2\sqrt{\delta+\lambda}} (e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}). \end{aligned} \tag{6}$$

Here

$$T = (1 - e^{-2\sqrt{\delta+\lambda}})^{-1}.$$

Proof. We see that problem (3) can be obviously rewritten as the equivalent nonlocal boundary value problem for the second order linear differential equation

$$-\frac{d^2u}{dx^2}+(\delta+\lambda)u=f(x), 0<x<1, u\left(0\right)=0, u(1)=u\left(\mu\right), 0\leq\mu<1.$$

We have the following formula:

$$u(x)=T\left\{(e^{-\sqrt{\delta+\lambda}x}-e^{-\sqrt{\delta+\lambda}(2-x)})\varphi+(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})\psi\right. \tag{7}$$

$$\left.-(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})\frac{1}{2\sqrt{\delta+\lambda}}\int_0^1(e^{-\sqrt{\delta+\lambda}(1-s)}-e^{-\sqrt{\delta+\lambda}(1+s)})f(s)ds\right\}$$

$$+\frac{1}{2\sqrt{\delta+\lambda}}\int_0^1(e^{-\sqrt{\delta+\lambda}|x-s|}-e^{-\sqrt{\delta+\lambda}(x+s)})f(s)ds$$

for the solution of the boundary value problem

$$-\frac{d^2u}{dx^2}+(\delta+\lambda)u=f(x), 0<x<1, u(0)=\varphi, u(1)=\psi$$

for second-order linear differential equations. Applying formula (7) and nonlocal boundary conditions $u\left(0\right)=\varphi=0, u(1)=u\left(\mu\right)=\psi$, we get

$$\psi=T\left\{(e^{-\sqrt{\delta+\lambda}(1-\mu)}-e^{-\sqrt{\delta+\lambda}(1+\mu)})\psi-(e^{-\sqrt{\delta+\lambda}(1-\mu)}-e^{-\sqrt{\delta+\lambda}(1+\mu)})\right.$$

$$\left.\times\frac{1}{2\sqrt{\delta+\lambda}}\int_0^1(e^{-\sqrt{\delta+\lambda}(1-s)}-e^{-\sqrt{\delta+\lambda}(1+s)})f(s)ds\right\}$$

$$+\frac{1}{2\sqrt{\delta+\lambda}}\int_0^1(e^{-\sqrt{\delta+\lambda}|\mu-s|}-e^{-\sqrt{\delta+\lambda}(\mu+s)})f(s)ds.$$

Solving the equation, we obtain

$$\psi=-(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)}-e^{-\sqrt{\delta+\lambda}(1+\mu)}) \tag{8}$$

$$\times\frac{1}{2\sqrt{\delta+\lambda}}\int_0^1(e^{-\sqrt{\delta+\lambda}(1-s)}-e^{-\sqrt{\delta+\lambda}(1+s)})f(s)ds$$

$$+(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{T^{-1}}{2\sqrt{\delta+\lambda}}\int_0^1(e^{-\sqrt{\delta+\lambda}|\mu-s|}-e^{-\sqrt{\delta+\lambda}(\mu+s)})f(s)ds.$$

Finally, applying formulas (7)-(8), we obtain formula (5). This finishes the proof of Lemma 2.1. □

The function $G(x,s;\lambda+\delta)$ is called the Green function of the resolvent equation (4).

Lemma 2.2. *For all* $0 \leq x \leq 1$, *the following formula holds*

$$\int_0^1 G(x,s,\mu;\lambda+\delta)ds = \frac{1}{\delta+\lambda} - \frac{1}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}x} \tag{9}$$

$$-\frac{e^{-\sqrt{\delta+\lambda}\mu}}{\delta+\lambda}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}.$$

Proof. Applying formula (6) and taking the integral, we get

$$\int_0^1 G(x,s,\mu;\lambda+\delta)ds = -\int_0^1 (e^{-\sqrt{\delta+\lambda}(1-s)} - e^{-\sqrt{\delta+\lambda}(1+s)})ds$$

$$\times(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})\frac{T}{2\sqrt{\delta+\lambda}}$$

$$\times[(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)}) + 1]$$

$$+(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{1}{2\sqrt{\delta+\lambda}}$$

$$\times\int_0^1 (e^{-\sqrt{\delta+\lambda}|\mu-s|} - e^{-\sqrt{\delta+\lambda}(\mu+s)})ds + \frac{1}{2\sqrt{\delta+\lambda}}\int_0^1 (e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)})ds$$

$$= -(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})\frac{T}{2(\delta+\lambda)}(1-2e^{-\sqrt{\delta+\lambda}} + e^{-2\sqrt{\delta+\lambda}})$$

$$\times[(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)}) + 1]$$

$$+(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}$$

$$\times\frac{1}{2(\delta+\lambda)}\left(2-2e^{-\sqrt{\delta+\lambda}\mu} - e^{-\sqrt{\delta+\lambda}(1-\mu)} + e^{-\sqrt{\delta+\lambda}(1+\mu)}\right)$$

$$+\frac{1}{2(\delta+\lambda)}\left(2-2e^{-\sqrt{\delta+\lambda}x} - e^{-\sqrt{\delta+\lambda}(1-x)} + e^{-\sqrt{\delta+\lambda}(1+x)}\right)$$

$$= \frac{1}{\delta+\lambda} - \frac{1}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}x} + \frac{1}{2(\delta+\lambda)}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})$$

$$\times(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Delta,$$

where

$$\Delta = -(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})(1+e^{-\sqrt{\delta+\lambda}(1+\mu)}) - T(1-e^{-\sqrt{\delta+\lambda}x})^2$$

$$\times\left[e^{-\sqrt{\delta+\lambda}(1-\mu)} + e^{-\sqrt{\delta+\lambda}(1+\mu)} + (1-e^{-\sqrt{\delta+\lambda}(1-\mu)})(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})\right]$$

$$+2-2e^{-\sqrt{\delta+\lambda}\mu} - e^{-\sqrt{\delta+\lambda}(1-\mu)} + e^{-\sqrt{\delta+\lambda}(1+\mu)}$$

$$= -2e^{-\sqrt{\delta+\lambda}\mu}(1-e^{-\sqrt{\delta+\lambda}(1-\mu)}).$$

This finishes the proof of Lemma 2.2. □

Lemma 2.3. *For all $\lambda \in R_\varphi = \{\lambda : |\arg\lambda| \le \varphi, \varphi < \pi/2\}$ expressions $1+e^{-\sqrt{\delta+\lambda}(1+\mu)}$, $1-e^{-\sqrt{\delta+\lambda}(1-\mu)}$ and $1-e^{-2\sqrt{\delta+\lambda}}$ are not equal to zero.*

Proof. Let $\lambda = \rho e^{i\varphi} = \rho\cos\varphi + i\sin\varphi$. Then

$$\delta+\lambda = \delta+\rho\cos\varphi + i\rho\sin\varphi = |\delta+\lambda|\, e^{i\psi}$$

and

$$|\delta+\lambda| = \sqrt{\delta^2+2\rho\cos\varphi+\rho^2} \ge \sqrt{\delta^2+|\lambda|^2}.$$

Therefore $(\delta+\lambda)^{1/2} = \left|(\delta+\lambda)^{1/2}\right| e^{i\psi/2}, \left|(\delta+\lambda)^{1/2}\right| = |\delta+\lambda|^{1/2}\frac{\sqrt{2}}{2}$. Here $\tan\psi = \frac{\rho\sin\varphi}{\delta+\rho\cos\varphi} \le \tan\varphi$. From that it follows $|\delta+\lambda|^{1/2} \ge \left(\delta^2+|\lambda|^2\right)^{1/4}$ and we have that $\left|(\delta+\lambda)^{1/2}\right| \ge \left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}$. Therefore, using the triangle inequality, we get

$$\left|1-e^{-2\sqrt{\delta+\lambda}}\right| \ge 1-\left|e^{-2\sqrt{\delta+\lambda}}\right| \ge 1-e^{-2\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}} > 0.$$

Similarly $\left|1+e^{-\sqrt{\delta+\lambda}(1+\mu)}\right| > 0$ and $\left|1-e^{-\sqrt{\delta+\lambda}(1-\mu)}\right| > 0$. Lemma 2.3 is proved. □

Lemma 2.4. *For any $\lambda \in R_\varphi = \{\lambda : |\arg\lambda| \le \varphi, \varphi < \pi/2\}$, $\mu \in [0,1)$ and $x \in [0,1]$ the following estimate holds*

$$\int_0^1 \left|G(x,s,\mu;\lambda+\delta)\right| ds \le \frac{M(\delta,\mu)}{1+|\lambda|}. \tag{10}$$

Proof. There are three possible cases: $0 \le x \le \mu, \mu < x \le \frac{1+\mu}{2}, \frac{+\mu}{2} < x \le 1$. In the first and second cases, it is easy to see that

$$\min\{2-x-s, 1-x+|\mu-s|, |x-s|\} = |x-s| \tag{11}$$

for any $x,s \in [0,1]$. By Lemma 2.3 and estimate (11) we have the following estimate for the Green function of resolvent equation (4)

$$\left|G(x,s,\mu;\lambda+\delta)\right| \le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}. \tag{12}$$

Then, applying (12), we get

$$\int_0^1 \left|G(x,s,\mu;\lambda+\delta)\right| ds \le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} ds$$

$$= \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left[\int_0^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)} ds + \int_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)} ds\right]$$

$$\le \frac{M_2(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/2}}. \tag{13}$$

Estimate (10) for this case follows from the last estimate and the following inequality

$$\frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/2}} \leq \frac{M_1(\delta)}{1+|\lambda|}. \tag{14}$$

In the third case $\frac{1+\mu}{2} < x \leq 1$. It is easy to see that

$$\min\{2-x-s, 1-x+\left|\mu-s\right|, |x-s|\}$$
$$= \begin{cases} 1-x+\mu-s,\ 0\leq s\leq \mu, \\ x-s,\ \mu\leq s\leq x, \\ s-x\ ,\ x\leq s\leq 1 \end{cases} \tag{15}$$

for any $x, s \in [0,1]$. By Lemma 2.3 and formula (15), we have the following estimate for the Green function of resolvent equation (4)

$$\left|G(x,s,\mu;\lambda+\delta)\right| \leq \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \begin{cases} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}\left(1-x+\mu-s\right)}, 0\leq s\leq \mu, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)},\ \mu\leq s\leq x, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)},\ x\leq s\leq 1. \end{cases} \tag{16}$$

Then, applying (16), we get

$$\int_0^1 \left|G(x,s,\mu;\lambda+\delta)\right| ds \leq \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left[\int_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}\left(1-x+\mu-s\right)} ds\right.$$
$$\left. + \int_\mu^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)} ds + \int_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)} ds\right]$$
$$\leq \frac{M_3(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/2}}. \tag{17}$$

Lemma 2.4 is proved. □

Lemma 2.5. *For any $\lambda \in R_\varphi = \{\lambda : \left|\arg\lambda\right| \leq \varphi, \varphi < \pi/2\}$, $\mu \in [0,1)$ and $x \in [0,1]$ the following estimate for the derivative of Green's function of resolvent equation (4) with respect to x holds*

$$\left|G_x(x,s,\mu;\lambda+\delta)\right| \leq M_2(\delta,\mu) e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} \tag{18}$$

for $0 \leq x \leq \mu, \mu < x \leq \frac{1+\mu}{2}$,

$$\left|G(x,s,\mu;\lambda+\delta)\right| \leq M_3(\delta,\mu) \times \begin{cases} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}\left(1-x+\mu-s\right)}, 0\leq s\leq \mu, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)},\ \mu\leq s\leq x, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)},\ x\leq s\leq 1. \end{cases} \tag{19}$$

for $\frac{1+\mu}{2} < x \leq 1$.

Proof. By using the equation (6), if $x-s<0$, then we get

$$G_x(x,s,\mu;\lambda+\delta)=-\frac{T}{2}(e^{-\sqrt{\delta+\lambda}(1-x)}+e^{-\sqrt{\delta+\lambda}(1+x)})(e^{-\sqrt{\delta+\lambda}(1-s)}-e^{-\sqrt{\delta+\lambda}(1+s)})$$
$$\times[(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)}-e^{-\sqrt{\delta+\lambda}(1+\mu)})+1]$$

$$+\frac{1}{2}(e^{-\sqrt{\delta+\lambda}(1-x)}+e^{-\sqrt{\delta+\lambda}(1+x)})(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}$$
$$\times(e^{-\sqrt{\delta+\lambda}|\mu-s|}-e^{-\sqrt{\delta+\lambda}(\mu+s)})+\frac{1}{2}(e^{-\sqrt{\delta+\lambda}(s-x)}+e^{-\sqrt{\delta+\lambda}(x+s)}). \tag{20}$$

If $x-s>0$, then we get

$$G_x(x,s,\mu;\lambda+\delta)=-\frac{T}{2}(e^{-\sqrt{\delta+\lambda}(1-x)}+e^{-\sqrt{\delta+\lambda}(1+x)})(e^{-\sqrt{\delta+\lambda}(1-s)}-e^{-\sqrt{\delta+\lambda}(1+s)})$$
$$\times[(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)}-e^{-\sqrt{\delta+\lambda}(1+\mu)})+1]$$
$$+\frac{1}{2}(e^{-\sqrt{\delta+\lambda}(1-x)}+e^{-\sqrt{\delta+\lambda}(1+x)})(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}$$
$$\times(e^{-\sqrt{\delta+\lambda}|\mu-s|}-e^{-\sqrt{\delta+\lambda}(\mu+s)})+\frac{1}{2}(-e^{-\sqrt{\delta+\lambda}(x-s)}+e^{-\sqrt{\delta+\lambda}(x+s)}). \tag{21}$$

There are three possible cases: $0\le x\le\mu$, $\mu<x\le\frac{1+\mu}{2}$, $\frac{1+\mu}{2}<x\le 1$. In the first and second cases, we have estimate (11). By Lemma 2.3 and estimate (11), we have the following estimate from (20) and (21).

$$\left|G_x(x,s,\mu;\lambda+\delta)\right|\le M_1(\delta,\mu)e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}. \tag{22}$$

Then, applying (22), we get

$$\int_0^1\left|G_x(x,s,\mu;\lambda+\delta)\right|ds\le M_1(\delta,\mu)\int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}ds$$

$$=M_1(\delta,\mu)\left[\int_0^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)}ds+\int_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)}ds\right]$$
$$\le\frac{M_3(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}. \tag{23}$$

Estimate (18) for this case follows from the last estimate and inequality (14). In the third case, by Lemma 2.3 and formula (15) we have the following estimate from (20).

$$\left|G(x,s,\mu;\lambda+\delta)\right|\le M_1(\delta,\mu)\begin{cases} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}\left(1-x+\mu-s\right)}, 0\le s\le\mu,\\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)},\ \mu\le s\le x,\\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)},\ x\le s\le 1.\end{cases} \tag{24}$$

Then, applying (24), we get

$$\int_0^1 \left|G_x(x,s,\mu;\lambda+\delta)\right| ds \le M_1(\delta,\mu)\left[\int_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}ds\right.$$

$$\left.+\int_\mu^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)}ds+\int_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)}ds\right]$$

$$\le \frac{M_3(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}. \tag{25}$$

Estimate (10) for this case follows from the last estimate and inequality (12). Lemma 2.5 is proved. □

Applying formula (7) we can establish the positivity of A^x in C[0,1] as follows.

Theorem 2.6. *For all $\lambda \in R_\varphi = \{\lambda : \left|\arg\lambda\right| \le \varphi, \varphi < \pi/2\}$, the resolvent $(\lambda I + A^x)^{-1}$ defined by formula (7) is subject to the bound*

$$\left\|(\lambda I + A^x)^{-1}\right\|_{C[0,1]\to C[0,1]} = \frac{M(\varphi,\delta)}{(1+|\lambda|)}.$$

Proof. Using the formula (7) and the triangle inequality, we get

$$|u(x)| \le \int_0^1 \left|G(x,s,\mu;\lambda+\delta)\right| ds \max_{0\le s\le 1}\left|f(s)\right|$$

for any $x \in [0,1]$. Then,

$$\max_{x\in[0,1]}|u(x)| \le \max_{x\in[0,1]}\int_0^1 \left|G(x,s,\mu;\lambda+\delta)\right| ds \left\|f(s)\right\|_{C[0,1]},$$

and

$$\left\|(A^x+\lambda)^{-1}\right\|_{C[0,1]} \le \frac{M(\varphi,\delta)}{(1+|\lambda|)}\left\|f(s)\right\|_{C[0,1]}.$$

From that it follows

$$\left\|(A^x+\lambda)^{-1}\right\|_{C[0,1]\to C[0,1]} \le \frac{M(\varphi,\delta)}{(1+|\lambda|)}.$$

□

Second, the positivity of A^x in $C\,[0,1]$ is investigated.

3. The Structure of Fractional Spaces $E_\alpha(A^x, C[0,1])$, Positivity of A^x in $\overset{\circ}{C}{}^{2\alpha}[0,1]$

Clearly, the operators A^x and its resolvent $(\lambda + A^x)^{-1}$ commute. Thus, from the definition of the norm in the space $E_\alpha(C[0,1], A^x)$ it follows that

$$\left\|(\lambda + A^x)^{-1}\right\|_{E_\alpha(C[0,1],A^x)\to E_\alpha(C[0,1],A^x)} \le \left\|(\lambda + A^x)^{-1}\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]\to \overset{\circ}{C}{}^{2\alpha}[0,1]}.$$

Hence, by using Theorem 2.6, we obtain the positivity of the operator A^x in the fractional spaces $E_\alpha(C[0,1], A^x)$. Moreover, the following theorem holds.

Theorem 3.1. *Let $\alpha \in (0, 1/2)$. Then, the norms of the spaces $E_\alpha(C[0,1], A^x)$ and $\overset{\circ}{C}{}^{2\alpha}[0,1]$ are equivalent.*

Proof. For any $\lambda \ge 0$ we have the following equality

$$A^x(\lambda + A^x)^{-1} f(x) = f(x) - \lambda(\lambda + A^x)^{-1} f(x).$$

By formula (5), we can write

$$A^x(\lambda + A^x)^{-1} f(x) = f(x) - \lambda \int_0^1 G(x, s, \mu; \lambda + \delta) f(s) ds$$

$$= \frac{\delta}{\delta+\lambda} f(x) + \frac{\lambda}{\delta+\lambda} f(x) - \lambda \int_0^1 G(x, s, \mu; \lambda + \delta) f(s) ds. \tag{26}$$

From equation (9), we have the following formula

$$\frac{1}{\delta+\lambda} = \int_0^1 G(x, s, \mu; \lambda + \delta) ds + \frac{1}{\delta+\lambda} e^{-\sqrt{\delta+\lambda}x}$$

$$+\frac{e^{-\sqrt{\delta+\lambda}\mu}}{\delta+\lambda}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}. \tag{27}$$

By using (26) and (27), we get

$$A^x(\lambda + A^x)^{-1} f(x) = \frac{\delta}{\delta+\lambda} f(x) + \frac{\lambda}{\delta+\lambda}\Big[e^{-\sqrt{\delta+\lambda}x}$$

$$+e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Big] f(x)$$

$$+\lambda \int_0^1 G(x, s, \mu; \lambda + \delta)\,(f(x) - f(s)) ds.$$

Then

$$\lambda^\alpha A^x(\lambda + A^x)^{-1} f(x) = \frac{\delta\lambda^\alpha}{\delta+\lambda} f(x) + \frac{\lambda^{\alpha+1}}{\delta+\lambda}\Big[e^{-\sqrt{\delta+\lambda}x}$$

$$+e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Big]f(x)$$

$$+\lambda^{\alpha+1}\int_0^1 G(x,s,\mu;\lambda+\delta)\,(f(x)-f(s))ds$$

$$=P_1(x)+P_2(x)+P_3(x)$$

Using the definition of norm space $\overset{\circ}{C}{}^{2\alpha}[0,1]$ and $\frac{\lambda^\alpha\delta^{1-\alpha}}{\delta+\lambda}\leq 1$, we can write

$$|P_1(x)|\leq\frac{\delta^\alpha\lambda^\alpha\delta^{1-\alpha}}{\delta+\lambda}\left|f(x)\right|\leq\delta^\alpha\max_{x\in[0,1]}\left|f(x)\right|\leq\delta^\alpha\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}$$

for any $x\in[0,1]$.

Then,

$$\max_{x\in[0,1]}|P_1(x)|\leq\delta^\alpha\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}$$

or

$$\|P_1\|_{C[0,1]}\leq\delta^\alpha\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}. \tag{28}$$

We have that

$$P_2(x)=\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}x}\left[f(x)-f(0)\right]+\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu-x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\left[f(x)-f(1)\right]$$

$$+\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\left[-f(x)+f(1)\right]$$

$$+\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu-x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\left[f(\mu)-f(0)\right]$$

$$-\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}f(1).$$

Then, using the triangle inequality and the estimate $p^\alpha e^{-pt}\leq\frac{M}{t^\alpha}$, $p>0$, we can write

$$|P_2(x)|\leq\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}x}\frac{\left|f(x)-f(0)\right|}{x^{2\alpha}}x^{2\alpha}$$

$$+\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}(1+\mu-x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{\left|f(x)-f(1)\right|}{(1-x)^{2\alpha}}(1-x)^{2\alpha}$$

$$+\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{\left|f(x)-f(1)\right|}{(1-x)^{2\alpha}}(1-x)^{2\alpha}$$

$$+\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{\left|f(\mu)-f(0)\right|}{\mu^{2\alpha}}\mu^{2\alpha}$$

$$+\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}f(1)$$

$$\leq M\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}+M\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}+M\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}+M\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}$$

$$+\max_{x\in[0,1]}\left|f(x)\right|\frac{1}{(1+\mu+x)^{2\alpha}}\leq M_1\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}$$

for any $x\in[0,1]$. Then,

$$\|P_2\|_{C[0,1]}\leq M\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}. \tag{29}$$

Now, we will estimate $P_3(x)$.

From (12) for $0\leq x\leq\mu$ and $\mu<x\leq\frac{1+\mu}{2}$, we get

$$|P_3(x)|\leq\lambda^{\alpha+1}\int_0^1\left|G(x,s,\mu;\lambda+\delta)\right|\left|f(x)-f(s)\right|ds$$

$$\leq\frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}\left|f(x)-f(s)\right|ds$$

$$\leq\frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}\left|f(x)-f(s)\right|\frac{|x-s|^{2\alpha}}{|x-s|^{2\alpha}}ds$$

$$\leq M\left\|f\right\|_{C^{2\alpha}[0,1]}\frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}|x-s|^{2\alpha}\,ds.$$

Using the substitution $y=\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|$ we have the following inequality

$$|P_3(x)|\leq M_2\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}\frac{\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}}\int_0^\infty e^{-y}y^{2\alpha}ds$$

$$\leq M_2\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}\frac{\lambda^{\alpha+1}\Gamma(2\alpha+1)}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}}\leq M_2\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}\Gamma(2\alpha+1).$$

And using estimate (16) for $\frac{1+\mu}{2}<x\leq 1$, we get

$$P_3(x)\leq\lambda^{\alpha+1}\int_0^1 G(x,s,\mu;\lambda+\delta)\left(f(x)-f(s)\right)ds=P_{31}(x)+P_{32}(x)+P_{32}(x)$$

where

$$P_{31}(x)=\frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}\left(f(x)-f(s)\right)ds,$$

$$P_{32}(x)=\frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int_\mu^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)}\left(f(x)-f(s)\right)ds,$$

$$P_{33}(x)=\frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)}\left(f(x)-f(s)\right)ds.$$

Let us estimate $|P_{31}(x)|$.

$$(P_{31}(x)) = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}\left(\left|f(x)-f(1)+f(\mu)-f(s)\right|\right)ds.$$

By the triangle inequality, we have

$$|P_{31}(x)| \le \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}\left[\left|f(x)-f(1)\right|+\left|f(\mu)-f(s)\right|\right]ds.$$

Using the definition of norm, the following inequality holds.

$$\begin{aligned}
|P_{31}(x)| &\le \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}\left[\int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}(1-x)^{2\alpha}ds\right.\\
&\left.+\int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}(\mu-s)^{2\alpha}ds\right]\\
&\le \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\left\|f\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)}(1-x)^{2\alpha}\left[\int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(\mu-s)}ds\right.\\
&\left.+e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)}\int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(\mu-s)}(\mu-s)^{2\alpha}ds\right].
\end{aligned}$$

Let us put

$$I_1 = e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)}(1-x)^{2\alpha}$$

$$I_2 = \int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(\mu-s)}ds$$

$$I_3 = \int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}(\mu-s)^{2\alpha}ds.$$

Then, by the inequality $e^{-at} \le \frac{M}{(at)^\theta}; 0<\theta<1$,

$$I_1 \le \frac{(1-x)^{2\alpha}}{\left(\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)\right)^{2\alpha}} = \frac{M}{\left(\delta^2+|\lambda|^2\right)^{2\alpha/4}}.$$

Taking the integral, we have

$$I_2 \le \frac{\left[1-e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}\mu}\right]}{\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}} \le \frac{M}{\left(\delta^2+|\lambda|^2\right)^{1/4}}$$

and for $y = (\delta^2 + |\lambda|^{21/4} \frac{\sqrt{2}}{2}(\mu - s)$

$$I_3 \leq \frac{M}{\left(\delta^2 + |\lambda|^2\right)^{\frac{2\alpha+1}{4}}} \int_0^\infty e^{-y} y^{2\alpha} ds \leq \frac{M\Gamma(2\alpha+1)}{\left(\delta^2 + |\lambda|^2\right)^{\frac{2\alpha+1}{4}}} \leq \frac{M}{\left(\delta^2 + |\lambda|^2\right)^{\frac{2\alpha+1}{4}}}.$$

Consequently, if we say $e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(1-x)} \leq 1, \Gamma(2\alpha+1) \leq 1$ and using the results of I_1, I_2, I_3 the following inequality holds

$$|P_{31}(x)| \leq \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2 + |\lambda|^2\right)^{1/4}} \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \left[\frac{M}{\left(\delta^2 + |\lambda|^2\right)^{2\alpha/4}} \cdot \frac{M}{\left(\delta^2 + |\lambda|^2\right)^{1/4}} + \frac{M}{\left(\delta^2 + |\lambda|^2\right)^{\frac{2\alpha+1}{4}}}\right]$$

$$|P_{31}(x)| \leq M_2(\delta,\mu) \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}}{\left(\delta^2 + |\lambda|^2\right)^{\frac{1+\alpha}{2}}} \leq M_2(\delta,\mu) \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}.$$

Let us estimate $|P_{32}(x)|$.

$$P_{32}(x) = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2 + |\lambda|^2\right)^{1/4}} \int_\mu^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(x-s)} (f(x) - f(s))\, ds,$$

$$|P_{32}(x)| \leq M \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2 + |\lambda|^2\right)^{1/4}} \int_\mu^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(x-s)} (x-s)^{2\alpha}\, ds.$$

Using the substitution $y = \left(\delta^2 + |\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(x-s)$ we have the following inequality

$$|P_{32}(x)| \leq M_2 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}}{\left(\delta^2 + |\lambda|^2\right)^{\frac{1+\alpha}{2}}} \left(-\int_\infty^0 e^{-y} y^{2\alpha} ds\right)$$

$$\leq M_2 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}}{\left(\delta^2 + |\lambda|^2\right)^{\frac{1+\alpha}{2}}} \int_0^\infty e^{-y} y^{2\alpha} ds$$

$$\leq M_2 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}\Gamma(2\alpha+1)}{\left(\delta^2 + |\lambda|^2\right)^{\frac{1+\alpha}{2}}} \leq M_3 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \Gamma(2\alpha+1).$$

Let us estimate $|P_{33}(x)|$.

$$P_{33}(x) = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2 + |\lambda|^2\right)^{1/4}} \int_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(s-x)} (f(x) - f(s))\, ds.$$

$$|P_{33}(x)| \leq M \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2 + |\lambda|^2\right)^{1/4}} \int_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(s-x)} |x-s|^{2\alpha}\, ds.$$

Using the substitution $y=\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)$, we have the following inequality

$$|P_{33}(x)|\le M_2\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}\frac{\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}}\int\limits_0^\infty e^{-y}y^{2\alpha}ds$$

$$\le M_2\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}\frac{\lambda^{\alpha+1}\Gamma(2\alpha+1)}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}}\le M_3\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}\Gamma(2\alpha+1).$$

Then, we can write the following

$$\max_{x\in[0,1]}|P_3(x)|\le\max_{x\in[0,1]}|P_{31}(x)|+\max_{x\in[0,1]}|P_{32}(x)|+\max_{x\in[0,1]}|P_{33}(x)|$$

or

$$\|P_3\|_{C[0,1]}\le M(\alpha)\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}. \tag{30}$$

Using estimates (28),(29) and (30), we get

$$\max_{x\in[0,1]}\left|\lambda^\alpha A^x(\lambda+A^x)^{-1}f(x)\right|\le M(\delta,\mu)\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}+M(\alpha)\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]} \tag{31}$$

for any $\lambda\ge 0$. Hence,

$$\left\|f\right\|_{E_\alpha(C[0,1],A^x)}\le M\left\|f\right\|_{\overset{\circ}{C}^{2\alpha}[0,1]}.$$

Now, let us prove the opposite inequality. For any positive operator A^x in the Banach space, we can write

$$I=\int\limits_0^\infty A^x(\lambda+A^x)^{-2}d\lambda, \tag{32}$$

where I is the identity operator.

From formulas (5) and (32) it follows that

$$f(x)=\int\limits_0^\infty(\lambda+A^x)^{-1}A^x(\lambda+A^x)^{-1}f(x)d\lambda$$

$$=\int\limits_0^\infty\int\limits_0^1 G(x,s,\mu;\lambda+\delta)A^x(\lambda+A^x)^{-1}f(s)dsd\lambda$$

Consequently,

$$f(x+\tau)-f(x)=\int\limits_0^\infty\int\limits_0^1\left[G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right]A^x(\lambda+A^x)^{-1}f(s)dsd\lambda$$

$$=\int\limits_0^\infty\lambda^{-\alpha}\int\limits_0^1\left[G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right]\lambda^\alpha A^x(\lambda+A^x)^{-1}f(s)dsd\lambda.$$

Hence,

$$\left|f(x+\tau)-f(x)\right| \le \left(\int_0^\infty \lambda^{-\alpha}\int_0^1 \left|G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right| ds d\lambda\right)\left\|f\right\|_{E_\alpha(C[0,1],A^x)}.$$

Let

$$T=\tau^{-2\alpha}\left(\int_0^\infty \lambda^{-\alpha}\int_0^1 \left|G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right| ds d\lambda\right)$$

$$=\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_0^x \left|G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right| ds d\lambda$$

$$+\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_x^{x+\tau} \left|G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right| ds d\lambda$$

$$+\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_{x+\tau}^1 \left|G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right| ds d\lambda$$

$$=\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_0^x\int_x^{x+\tau} \left|G_z(z,,s,\mu;\lambda+\delta)\right| dz ds d\lambda$$

$$+\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_x^{x+\tau} \left|G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right| ds d\lambda$$

$$+\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_{x+\tau}^1\int_x^{x+\tau} \left|G_z(z,,s,\mu;\lambda+\delta)\right| dz ds d\lambda$$

$$=T_1+T_2+T_3.$$

Then for any $x,\tau\in R^+$, we have that

$$\frac{\left|f(x+\tau)-f(x)\right|}{|\tau|^{-2\alpha}} \le T\left\|f\right\|_{E_\alpha(C^{2\alpha}[0,1],A^x).}$$

Now, we will prove that

$$T\le \frac{M(\delta)}{2\alpha\left(1-2\alpha\right)}. \tag{33}$$

We will estimate T_1, T_2 and T_3. First, let us estimate T_1.

$$T_1=\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_0^x\int_x^{x+\tau} \left|G_z(z,,s,\mu;\lambda+\delta)\right| dz ds d\lambda$$

$$\leq \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_0^x \int_x^{x+\tau} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}|z-s|} dz ds d\lambda$$

$$\leq \tau^{-2\alpha} \int_0^x \int_x^{x+\tau} \int_0^\infty \lambda^{-\alpha} e^{-|\lambda|^{1/2} \frac{\sqrt{2}}{2}|z-s|} d\lambda dz ds = \tau^{-2\alpha} \int_0^x \int_x^{x+\tau} \Delta(z,s) dz ds,$$

where

$$\Delta(z,s) = \int_0^\infty \lambda^{-\alpha} e^{-|\lambda|^{1/2} \frac{\sqrt{2}}{2}|z-s|} d\lambda. \tag{34}$$

By changing variable

$$p = -|\lambda|^{1/2} \frac{\sqrt{2}}{2} |z-s|,$$

we get

$$\lambda = \left(\frac{p}{\frac{\sqrt{2}}{2}|z-s|} \right)^2 = \frac{p^2}{\frac{1}{2}|z-s|^2}, d\lambda = \frac{2pdp}{\frac{1}{2}|z-s|^2}.$$

Then,

$$\Delta = \int_0^\infty \frac{|z-s|^{2\alpha} \frac{1}{2\alpha}}{p^{2\alpha}} e^{-p} \frac{2pdp}{\frac{1}{2}|z-s|^2} = |z-s|^{2\alpha-2} 2^{2-\alpha} \int_0^\infty e^{-p} p^{1-2\alpha} dp$$

$$= |z-s|^{2\alpha-2} 2^{2-\alpha} \Gamma(2-2\alpha) = M_0 |z-s|^{2\alpha-2}.$$

Then,

$$T_1 \leq M_1 \tau^{-2\alpha} \int_0^x \int_x^{x+\tau} (z-s)^{2\alpha-2} dz ds \leq \frac{M_2}{(1-2\alpha)2\alpha}. \tag{35}$$

Second, let us estimate T_2.

$$T_2 = \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_x^{x+\tau} \left| G(x+\tau, s, \mu; \lambda+\delta) - G(x, s, \mu; \lambda+\delta) \right| ds d\lambda$$

$$= \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_x^{x+\tau} \left[\frac{e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}|z-s|}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} + \frac{e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}|z+\tau-s|}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \right] ds d\lambda$$

$$\leq \frac{M}{\sqrt{\lambda}} \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_x^{x+\tau} \left[e^{-\sqrt{\lambda} \frac{\sqrt{2}}{2}|z-s|} + e^{-\sqrt{\lambda} \frac{\sqrt{2}}{2}|z+\tau-s|} \right] ds d\lambda$$

$$= \frac{M}{\sqrt{\lambda}} \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_x^{x+\tau} \left[e^{-\sqrt{\lambda} \frac{\sqrt{2}}{2}(s-x)} + e^{-\sqrt{\lambda} \frac{\sqrt{2}}{2}(z+\tau-s)} \right] ds d\lambda.$$

$$= T_{21} + T_{22},$$

where

$$T_{21} = \frac{M}{\sqrt{\lambda}}\tau^{-2\alpha}\int_{0}^{\infty}\lambda^{-\alpha}\int_{x}^{x+\tau}e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(s-x)}dsd\lambda,$$

$$T_{22} = \frac{M}{\sqrt{\lambda}}\tau^{-2\alpha}\int_{0}^{\infty}\lambda^{-\alpha}\int_{x}^{x+\tau}e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(z+\tau-s)}dsd\lambda.$$

By changing variable

$$\sqrt{\lambda}\frac{\sqrt{2}}{2}(s-x) = p,$$

we get

$$T_{21} = \frac{M}{\sqrt{\lambda}}\tau^{-2\alpha}\int_{x}^{x+\tau}\int_{0}^{\infty}\lambda^{-\alpha}e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(s-x)}d\lambda ds \leq M_1\tau^{-2\alpha}\int_{x}^{x+\tau}\int_{0}^{\infty}\frac{(s-x)^{2\alpha+1}}{p^{2\alpha+1}}\frac{e^{-p}pdp}{(s-x)^2}ds$$

$$= M_1\tau^{-2\alpha}\int_{x}^{x+\tau}(s-x)^{2\alpha-1}\,ds\int_{0}^{\infty}e^{-p}p^{\,-2\alpha}\,dp = \frac{M_2}{2\alpha}.$$

Similarly, by changing variable

$$\sqrt{\lambda}\frac{\sqrt{2}}{2}(x+\tau-s) = p,$$

we get,

$$T_{21} = \frac{M}{\sqrt{\lambda}}\tau^{-2\alpha}\int_{x}^{x+\tau}\int_{0}^{\infty}\lambda^{-\alpha}e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(x+\tau-s)}d\lambda ds \leq M_1\tau^{-2\alpha}\int_{x}^{x+\tau}\int_{0}^{\infty}\frac{(x+\tau-s)^{2\alpha+1}}{p^{2\alpha+1}}\frac{e^{-p}pdp}{(x+\tau-s)^2}ds$$

$$\leq M_1\tau^{-2\alpha}\int_{x}^{x+\tau}(x+\tau-s)^{2\alpha-1}\,ds\int_{0}^{\infty}e^{-p}p^{\,-2\alpha}\,dp = \frac{M_2}{2\alpha}.$$

Then,

$$T_2 \leq \frac{M_2}{2\alpha}.$$

Finally, let us estimate T_3.

$$T_3 = \tau^{-2\alpha}\int_{0}^{\infty}\lambda^{-\alpha}\int_{x+\tau}^{1}\int_{x}^{x+\tau}\left|G_z(z,,s,\mu;\lambda+\delta)\right|dzdsd\lambda$$

$$\leq \tau^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \int\limits_{x+\tau}^{1} \int\limits_{x}^{x+\tau} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}|z-s|} dz ds d\lambda$$

$$\leq \tau^{-2\alpha} \int\limits_{x+\tau}^{1} \int\limits_{x}^{x+\tau} \int\limits_0^\infty \lambda^{-\alpha} e^{-|\lambda|^{1/2} \frac{\sqrt{2}}{2}|z-s|} d\lambda dz ds.$$

Using (34), we get

$$T_3 \leq M_1 \tau^{-2\alpha} \int\limits_{x+\tau}^{1} \int\limits_{x}^{x+\tau} (s-z)^{2\alpha-2} dz ds = \frac{M_1}{(1-2\alpha)\, 2\alpha}. \tag{36}$$

Finally,

$$T \leq \frac{M}{1-2\alpha} + \frac{M}{2\alpha} \leq \frac{M}{2\alpha\,(1-2\alpha)}.$$

Theorem 3.1 is proved. □

4. An Application

Now, we consider an application of Theorem 3.1. First, we consider the boundary value problem

$$-\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \delta u(t,x) = f(t,x), 0 < t < T, x \in [0,1] \tag{37}$$

$$u(0,x) = \varphi(x), x \in [0,1]$$

$$u(t,0) = 0, u(t,1) = u(t,\mu), 0 \leq \mu < 1, 0 < t < T.$$

Here, $\varphi(x)$ and $f(t,x)$ are sufficiently smooth functions in x and they satisfy and compatibility conditions which guarantee problem (37) has a smooth solution $u(t,x)$.

Theorem 4.1. *Let $0 < 2\alpha < 1$. Then, for the solution of the initial value problem (37), we have the following coercive stability inequality:*

$$\|u_t\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,1])} + \|u\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha+2}[0,1])}$$

$$\leq M(\alpha) \left[\left\|\varphi\right\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,1]} + \left\|f\right\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,1])} \right].$$

The proof of Theorem 4.1 is based on Theorem 3.1 on the structure of the fractional spaces $E_\alpha = E_\alpha(C(0,1), A^x)$ and Theorem 2.6 on the positivity of the operator A^x on the following theorems on coercive stability of initial value for the abstract parabolic equation.

Theorem 4.2. *Let A be a strongly positive operator in a Banach space E and $f \in C([0,T], E_\alpha), 0 < \alpha < 1$. Then, for the solution of the nonlocal boundary value problem*

$$u' + Au(t) = f(t), 0 < t < T \tag{38}$$

$$u(0) = \varphi$$

in a Banach space E with positive operator A, we have the coercive inequality

$$\|u'\|_{C([0,T],E_\alpha)} + \|Au\|_{C([0,T],E_\alpha)}$$
$$\leq M\left[\left\|A\varphi\right\|_{E_\alpha} + \frac{M}{\alpha(1-\alpha)}\left\|f\right\|_{C([0,T],E_\alpha)}\right].$$

Second, we consider the nonlocal boundary value problem for the parabolic equation

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = f(t,x), 0 < t < T, x \in [0,1] \tag{39}$$
$$u(0,x) = u(T,x), x \in [0,1]$$
$$u(t,0) = 0, u(t,1) = u(t,\mu), 0 \leq \mu < 1, 0 < t < T.$$

Here, $f(t,x)$ is a sufficiently smooth function in x and it satisfies any compatibility conditions which guarantee problem (39) has a smooth solution $u(t,x)$.

Theorem 4.3. *Let* $0 < 2\alpha < 1$. *Then for the solution of boundary value problem (37), we have the following coercive stability inequality*

$$\|u_t\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,1])} + \|u\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha+2}[0,1])} \leq M(\alpha)\left\|f\right\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,1])}.$$

The proof of Theorem 4.3 is based on Theorem 3.1 on the structure of the fractional spaces $E_\alpha = E_\alpha(C(0,1), A^x)$, and Theorem 2.6 on the positivity of the operator A^x on the following theorem on the coercive stability of the nonlocal boundary value for the abstract parabolic equation.

Theorem 4.4. *Let A be a positive operator in a Banach space E and* $f \in C([0,T], E_\alpha), 0 < \alpha < 1$. *Then for the solution of the nonlocal boundary value problem*

$$u' + Au(t) = f(t), 0 < t < T, \tag{40}$$
$$u(0) = u(T)$$

in a Banach space E with positive operator A the coercive inequality

$$\|u'\|_{C([0,T],E_\alpha)} + \|Au\|_{C([0,T],E_\alpha)} \leq \frac{M}{\alpha(1-\alpha)}\left\|f\right\|_{C([0,T],E_\alpha)}$$

holds.

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