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# Path-Set Induced Closure Operators on Graphs

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**Abstract.** Given a simple graph, we associate with every set of paths of the same positive length a closure operator on the (vertex set of the) graph. These closure operators are then studied. In particular, it is shown that the connectedness with respect to them is a certain kind of path connectedness. Closure operators associated with sets of paths in some graphs with the vertex set $\mathbb{Z}^2$ are discussed which include the well known Marcus-Wyse and Khalimsky topologies used in digital topology. This demonstrates possible applications of the closure operators investigated in digital image analysis.

## 1. Introduction

Topological spaces in their usual meaning, i.e. those developed by Bourbaki [1], play a basic role in mathematical analysis (where even special topologies are employed, namely the Hausdorff ones). In many other branches of mathematics and in computer science, various generalizations of topological spaces are often used. In particular, closure operators obtained from the Kuratowski ones by omitting some axioms occur in numerous applications. For example, the so-called closure operators on categories studied in categorical topology are supposed to satisfy just the axioms of extensiveness and monotony (and a certain categorical axiom - the axiom of functionality) - see [4]. The closure operators usually employed in algebra are just the extensive, monotone and idempotent ones. If also the condition is satisfied that the closure of any set equals the union of the closures of all its finite subsets, we get the well-known algebraic closure operators (cf. [6]). The convex hull operators used in geometry (cf. [16]) are just the grounded, extensive, monotone and idempotent closure operators. Such closure operators occur also in logic - they are obtained by assigning, to a given set $A$ of formulas, the set of all formulas provable from $A$ - cf.[10]. The closure operators which are, in general, only grounded, extensive and monotone were studied by E.Čech in [2] and those which are moreover additive, i.e. the so-called pretopologies, were studied by the same author in [3]. Pretopologies were employed for solving problems related to digital image processing in [12-14].

In this note we will study closure operators on graphs which are only grounded, extensive and monotone in general (i.e. which are closure operators in the sense of [2]). These closure operators are induced by sets of paths of a given length. We will study properties of the induced closure operators, in particular the connectedness with respect to them. We will show that the connectedness is a path connectedness for certain special paths. To demonstrate possible applications of our results in digital image analysis, we will

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discuss some closure operators associated with sets of paths in certain graphs with the vertex set $\mathbb{Z}^2$ and we will show that these closure operators include some known closure operators on $\mathbb{Z}^2$ used in digital topology, in particular the Marcus-Wyse and Khalimsky topologies.

## 2. Preliminaries

By a *closure operator* $u$ on a set $X$ we mean a map $u$: $\exp X \to \exp X$ (where $\exp X$ denotes the power set of $X$) which is
(i) grounded (i.e. $u\emptyset = \emptyset$),
(ii) extensive (i.e. $A \subseteq X \Rightarrow A \subseteq uA$), and
(iii) monotone (i.e. $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).
The pair $(X, u)$ is then called a *closure space* and, for every subset $A \subseteq X$, $uA$ is called the *closure* of $A$.

A closure operator $u$ on $X$ which is
(iv) additive (i.e. $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and
(v) idempotent (i.e. $uuA = uA$ whenever $A \subseteq X$)
is called a *Kuratowski closure operator* or a *topology* and the pair $(X, u)$ is called a *topological space*.

Given a cardinal $m$, a closure operator $u$ on a set $X$ and the closure space $(X, u)$ are called an *$S_m$-closure operator* and an *$S_m$-closure space* (briefly, an *$S_m$-space*), respectively, if the following condition is satisfied:
$A \subseteq X \Rightarrow uA = \bigcup\{uB; B \subseteq A, \text{card } B < m\}$.

The *algebraic closure operators* are obtained from the idempotent $S_{\aleph_0}$-closure operators by omitting the requirement of groundedness. In [3], $S_2$-closure operators and $S_2$-spaces are called *quasi-discrete*. $S_2$-topological spaces are usually called *Alexandroff spaces* - see [6]. Of course, any $S_2$-closure operator is additive and is given by closures of the singleton subsets. It is also evident that any $S_\alpha$-closure operator is an $S_\beta$-closure operator whenever $\alpha < \beta$. Since any closure operator on a set $X$ is obviously an $S_\alpha$-closure operator for each cardinal $\alpha$ with $\alpha >$ card $X$, there exists a least cardinal $\alpha$ such that $u$ is an $S_\alpha$-closure operator. Such a cardinal is then an important invariant of the closure operator $u$. Evidently, if $\alpha \leq \aleph_0$, then any additive $S_\alpha$-closure operator is an $S_2$-closure operator.

Many concepts known for topological spaces (see e.g. [5]) may be naturally extended to closure spaces. Given a closure space $(X, u)$, a subset $A \subseteq X$ is called *closed* if $uA = A$, and it is called *open* if $X - A$ is closed. A closure space $(X, u)$ is said to be a *subspace* of a closure space $(Y, v)$ if $uA = vA \cap X$ for each subset $A \subseteq X$. We will speak briefly about a subspace $X$ of $(Y, v)$. A closure space $(X, u)$ is said to be *connected* if $\emptyset$ and $X$ are the only subsets of $X$ which are both closed and open. A subset $X \subseteq Y$ is connected in a closure space $(Y, v)$ if the subspace $X$ of $(Y, v)$ is connected. A maximal connected subset of a closure space is called a *component* of this space. All the basic properties of connected sets and components in topological spaces are preserved also in closure spaces. A closure space $(X, u)$ is said to be a *$T_0$-space* if, for any points $x, y \in X$, from $x \in u\{y\}$ and $y \in u\{x\}$ it follows that $x = y$, and it is called a *$T_{\frac{1}{2}}$-space* if each singleton subset of $X$ is closed or open. Given closure spaces $(X, u)$ and $(Y, v)$, a map $\varphi : X \to Y$ is said to be a *continuous map* of $(X, u)$ into $(Y, v)$ if $f(uA) \subseteq vf(A)$ for each subset $A \subseteq X$.

## 3. $N$-Path Sets in Graphs and Associated Closure Operators

Given a natural number (i.e. a finite ordinal) $m$, we write briefly $i < m$ and $i \leq m$ instead of $0 \leq i < m$ and $0 \leq i \leq m$, respectively, because $i$ is considered to be a natural number, too (i.e. the linear order $<$ and the corresponding partial order $\leq$ are considered to be defined on just the set of natural numbers).

Let $G = (V, E)$ be an (undirected simple) graph with $V \neq \emptyset$ the vertex set and $E$ the set of edges. We denote by $\mathcal{P}(G)$ the set of all paths in $G$, i.e. the set of all sequences $(x_i|\, i \leq m)$ where $m$ is a natural number, $x_i \in V$ for every $i \leq m$, and $\{x_i, x_{i+1}\} \in E$ whenever $i < m$. If $\mathcal{B} \subseteq \mathcal{P}(G)$, we put $\mathcal{B}^{-1} = \{(x_i|\, i \leq m);\ (x_{m-i}|\, i \leq m) \in \mathcal{B}\}$.

Given a natural number $n > 0$, let $\mathcal{B}_n \subseteq \mathcal{P}(G)$ be a set of paths of length $n$. $\mathcal{B}_n$ will be called an *$n$-path set* in $G$. We put
$\hat{\mathcal{B}}_n = \{(x_i|\, i \leq m) \in \mathcal{P}(G);\ 0 < m \leq n$ and there exists $(y_i|\, i \leq n) \in \mathcal{B}_n$ such that $x_i = y_i$ for every $i \leq m\}$ (so that $\mathcal{B}_n \subseteq \hat{\mathcal{B}}_n$) and

$\mathcal{B}_n^* = \hat{\mathcal{B}}_n \cup \hat{\mathcal{B}}_n^{-1}$.
The elements of $\mathcal{B}_n^*$ will be called *$\mathcal{B}_n$-initial segments* in $G$.
For any subset $X \subseteq V$ we put
$u_{\mathcal{B}_n}X = X \cup \{x \in V;$ there exists $(x_i|\, i \le m) \in \hat{\mathcal{B}}_n$ with $\{x_i;\ i < m\} \subseteq X$ and $x_m = x\}$.
Clearly, $u_{\mathcal{B}_n}$ is an $S_{n+1}$-closure operator on $V$. It will be said to be *induced by $\mathcal{B}_n$*.

Let $G = (V, E)$ and $H = (U, F)$ be graphs. Let $\mathcal{B}_n$ and $C_n$ be $n$-path sets in $G$ and $H$, respectively ($n > 0$ a natural umber). A map $f : V \to U$ is said to be a $(\mathcal{B}_n, C_n)$-homomorphism from $G$ to $H$ if $(x_i|\, i \le n) \in \mathcal{B}_n$ implies $(f(x_i)|\, i \le n) \in C_n$. Note that, if $n = 1$ and $\mathcal{B}_n$ and $C_n$ are the sets of all paths of length 1 in $G$ and $H$, respectively, we get the usual homomorphism of $G$ into $H$. The following statement is evident:

**Proposition 3.1.** *Let $G = (V, E)$ and $H = (U, F)$ be graphs and let $\mathcal{B}_n$ and $C_n$ be $n$-path sets in $G$ and $H$, respectively. If a map $f : V \to U$ is a $(\mathcal{B}_n, C_n)$-homomorphism from $G$ to $H$, then it is a continuous map of $(V, u_{\mathcal{B}_n})$ into $(U, u_{C_n})$.*

Of course, closure operators induced by $n$-path sets are not additive in general. On the other hand, we have:

**Proposition 3.2.** *Let $G = (V, E)$ be a graph and let $\mathcal{B}_n$ be an $n$-path set in $G$. Then the union of a system of closed subsets of $(V, u_{\mathcal{B}_n})$ is a closed subset of $(V, u_{\mathcal{B}_n})$.*

*Proof.* Let $\{A_j;\ j \in J\}$ be a system of closed subsets of $(V, u_{\mathcal{B}_n})$ and let $x \in u_{\mathcal{B}_n} \bigcup_{j\in J} A_j$ be a vertex such that there exists $(x_i|\, i \le m) \in \hat{\mathcal{B}}_n$ with the property that $x = x_m$ and $x_i \in \bigcup_{j\in J} A_j$ for all $i < m$. In particular, we have $x_0 \in \bigcup_{j\in J} A_j$ and so there exists $j_0 \in J$ such that $x_0 \in A_{j_0}$. Suppose that $\{x_i;\ i \le m\}$ is not a subset of $A_{j_0}$. Then there is a smallest natural number $p \le m$ such that $x_p \notin A_{j_0}$. Consequently, $0 < p$ and $x_i \in A_{j_0}$ for all $i < p$. Since $(x_i|\, i \le p) \in \hat{\mathcal{B}}_n$, we have $x_p \in u_{\mathcal{B}_n} A_{j_0} = A_{j_0}$, which is a contradiction. Therefore, $\{x_i;\ i \le m\} \subseteq A_{j_0}$ and, hence, $x \in A_{j_0} \subseteq \bigcup_{j\in J} A_j$. We have shown that $u_{\mathcal{B}_n} \bigcup_{j\in J} A_j \subseteq \bigcup_{j\in J} A_j$. As the converse inclusion is obvious, the proof is complete. $\square$

**Proposition 3.3.** *Let $G = (V, E)$ be a graph and let $\mathcal{B}_n$ be an $n$-path set in $G$. If $u_{\mathcal{B}_n}$ is idempotent, then $(V, u_{\mathcal{B}_n})$ is an Alexandroff topological space.*

*Proof.* Let $X \subseteq V$ be a subset and $x \in u_{\mathcal{B}_n}X$ be a vertex. If $x \in X$, then $x \in \bigcup_{x\in X} u_{\mathcal{B}_n}\{x\}$ because of the extensiveness of $u_{\mathcal{B}_n}$. Let $x \notin X$. Then there exists $(x_i|\, i \le m) \in \hat{\mathcal{B}}_n$ such that $x = x_m$ and $x_i \in X$ for all $i < m$. Consequently, we have $x \in u_{\mathcal{B}_n}\{x_i;\ i < m\}$ and $\{x_i;\ i < j\} \subseteq u_{\mathcal{B}_n}\{x_i;\ i < j - 1\}$ whenever $1 < j \le m$. Hence, we have $x \in u_{\mathcal{B}_n}\{x_i;\ i < m\} \subseteq u_{\mathcal{B}_n}u_{\mathcal{B}_n}\{x_i;\ i < m - 1\} = u_{\mathcal{B}_n}\{x_i;\ i < m - 1\} \subseteq u_{\mathcal{B}_n}u_{\mathcal{B}_n}\{x_i;\ i < m - 2\} = u_{\mathcal{B}_n}\{x_i;\ i < m - 2\} \subseteq \ldots u_{\mathcal{B}_n}\{x_0\}$. Thus, $x \in \bigcup_{x\in X} u_{\mathcal{B}_n}\{x\}$ and the inclusion $u_{\mathcal{B}_n}X \subseteq \bigcup_{x\in X} u_{\mathcal{B}_n}\{x\}$ is proved. As the converse inclusion follows from the monotony of $u_{\mathcal{B}_n}$, the proof is complete. $\square$

**Definition 3.4.** An $n$-path set $\mathcal{B}_n$ in a graph is said to be *terse* provided that the following condition holds:
If $(x_i|\, i \le n), (y_i|\, i \le n) \in \mathcal{B}_n$ are paths with $\{x_0, x_1\} = \{y_0, y_1\}$, then $(x_i|\, i \le n) = (y_i|\, i \le n)$.

Thus, for example, a 1-path set in a graph $G = (V, E)$ is terse if, for every edge $\{x, y\} \in E$, it contains at most one of the paths $(x, y)$ and $(y, x)$.

**Theorem 3.5.** *Let $\mathcal{B}_n$ be a terse $n$-path set in a graph $G = (V, E)$. Then $(v, u_{\mathcal{B}_n})$ is a $T_0$-space.*

*Proof.* Let $x, y \in V$, $x \in u_R\{y\}$, $y \in u_R\{x\}$. Then there is a path $(x_i|\, i < \alpha) \in \mathcal{B}_n$ with $x_0 = y$ and $x_1 = x$, and there is a path $(y_i|\, i < \alpha) \in \mathcal{B}_n$ with $y_0 = x$ and $y_1 = y$. As $\mathcal{B}_n$ is terse, $(x_i|\, i < \alpha) = (y_i|\, i < \alpha)$. Therefore, $x = y$, so that $(X, u_{\mathcal{B}_n})$ is a $T_0$-space. $\square$

**Theorem 3.6.** *Let $\mathcal{B}_n$ be a terse $n$-path set in a graph $G = (V, E)$. Then $\mathcal{B}_n$ is a minimal element (with respect to the set inclusion) of the set of all $n$-path sets $\mathcal{D}_n$ in $G$ satisfying $u_{\mathcal{D}_n} = u_{\mathcal{B}_n}$.*

*Proof.* Let $\mathcal{D}_n$ be an $n$-path set in $G$ with $u_{\mathcal{D}_n} = u_{\mathcal{B}_n}$ and let $\mathcal{D}_n \subseteq \mathcal{B}_n$. Let $(x_i|\ i \le n) \in \mathcal{B}_n$ be an arbitrary element. Then $(x_0, x_1) \in \mathcal{B}_n$, hence $x_1 \in u_{\mathcal{B}_n}\{x_0\}$. Consequently, $x_1 \in u_{\mathcal{D}_n}\{x_0\}$, so that $(x_0, x_1) \in \hat{\mathcal{B}}_n$. Therefore, there is $(y_i|\ i \le n) \in \mathcal{D}_n$ such that $y_0 = x_0$ and $y_1 = x_1$. Since $(y_i|\ i \le n) \in \mathcal{B}_n$ and $\mathcal{B}_n$ is terse, we have $(x_i|\ i \le n) = (y_i|\ i \le n)$. Consequently, $\mathcal{D}_n = \mathcal{B}_n$. □

**Definition 3.7.** An $n$-path set $\mathcal{B}_n$ in a graph is said to be *strongly terse* provided that the following condition holds:

If $(x_i|\ i \le n), (y_i|\ i \le n) \in \mathcal{B}_n$ are paths with $\{x_0, x_1\} = \{y_{i_0}, y_{i_1}\}$ for some $i_0, i_1 \le n$, then $(x_i|\ i \le n) = (y_i|\ i \le n)$.

**Theorem 3.8.** *If $\mathcal{B}_n$ is a strongly terse $n$-path set in a graph $G$, then the correspondence $\mathcal{B}_n \mapsto u_{\mathcal{B}_n}$ is one-to-one.*

*Proof.* Given a natural number $n > 0$, for any closure operator $u$ on $G$ we define an $n$-path set in $G$ as follows: $B_n(u) = \{(x_i| i \le n) \in \mathcal{P}(G);\ (x_i| i \le n)$ has the property that, for each $j$, $0 < j \le n$, and each proper subset $A_j \subset \{x_i;\ i < j\}$, we have $x_j \in u\{x_i; i < j\} - uA_j\}$.
Let $\mathcal{B}_n$ be a strongly terse $n$-path set in $G$ and let $(x_i|\ i \le n) \in \mathcal{B}_n$. If $i_0$, $0 < i_0 \le n$, is an arbitrary natural number, then $x_{i_0} \in u_{\mathcal{B}_n}\{x_i;\ i < i_0\}$. Let $A \subseteq \{x_i;\ i < i_0\}$ be an arbitrary subset and let $x_{i_0} \in u_{\mathcal{B}_n}A$. Then there exist a path $(y_j|\ j \le n) \in \mathcal{B}_n$ and a natural number $j_0$, $0 < j_0 \le n$, such that $x_{i_0} = y_{j_0}$ and $\{y_j;\ j < j_0\} \subseteq A$, hence $\{y_j;\ j \le j_0\} \subseteq \{x_i;\ i \le i_0\}$. Thus, $\{y_0, y_1\} \subseteq \{x_i;\ i \le n\}$ and, consequently, $(x_i|\ i \le n) = (y_j|\ j \le n)$. Hence, $i_0 = j_0$ and, therefore, $A = \{x_i;\ i < i_0\}$. We have shown that $x_{i_0} \notin u_{\mathcal{B}_n}A$ whenever $A \subseteq \{x_i;\ i < i_0\}$ is a proper subset. It follows that $(x_i|\ i \le n) \in \mathcal{B}_n(u_{\mathcal{B}_n})$ which results in $\mathcal{B}_n \subseteq \mathcal{B}_n(u_{\mathcal{B}_n})$.

Conversely, let $(x_i|\ i \le n) \in \mathcal{B}_n(u_{\mathcal{B}_n})$. As $x_1 \in u_{\mathcal{B}_n}\{x_0\}$ and $\mathcal{B}_n$ is strongly terse, there is a unique path $(y_i|\ i \le n) \in \mathcal{B}_n$ with $x_0 = y_0$ and $x_1 = y_1$. Let $i_0$, $1 < i_0 \le n$, be a natural number such that $x_i = y_i$ for all $i < i_0$. Since $x_{i_0} \in u_R\{x_i;\ i < i_0\}$, there are a path $(z_j|\ j \le n) \in \mathcal{B}_n$ and a natural number $j_0$, $0 < j_0 \le n$, such that $x_{i_0} = z_{j_0}$ and $\{z_j;\ j < j_0\} \subseteq \{x_i;\ i < i_0\}$. Then $x_{i_0} \in u_{\mathcal{B}_n}\{z_j;\ j < j_0\}$, hence $\{z_j;\ j < j_0\} = \{x_i;\ i < i_0\} = \{y_i;\ i < i_0\}$. We have $i_0 = j_0$, so that $x_{i_0} = z_{i_0} = y_{i_0}$ (because $(y_i|\ i \le n)$ is unique). Consequently, $(x_i|\ i \le i_0) = (y_i|\ i \le i_0)$. Now the principle of mathematical induction implies $(x_i|\ i \le n) = (y_i|\ i \le n)$. We have shown that $(x_i|\ i \le n) \in \mathcal{B}_n$, which results in $\mathcal{B}_n(u_{\mathcal{B}_n}) \subseteq \mathcal{B}_n$. Therefore, $\mathcal{B}_n(u_{\mathcal{B}_n}) = \mathcal{B}_n$ and the proof is complete. □

We will need the following

**Lemma 3.9.** *Let $\mathcal{B}_n$ be a strongly terse $n$-path set in a graph $G$, let $p, q$ be natural numbers with $0 < p < q \le n$, and let $(x_i|\ i \le p), (y_i|\ i \le q)$ be $\mathcal{B}_n$-initial segments in $G$ such that $\{x_i;\ i \le p\} \subseteq \{y_i;\ i \le q\}$ and $x_0 = y_0$. Then there exists a unique path $(z_i|\ i < n) \in \mathcal{B}_n$ such that $x_i = z_i$ for all $i \le p$ and $y_i = z_i$ for all $i \le q$.*

*Proof.* As $(x_i|\ i \le p)$ is a $\mathcal{B}_n$-initial segment, there is a path $(t_i|\ i \le n) \in \mathcal{B}_n$ such that 1) $x_i = t_i$ for all $i \le p$ or 2) $x_i = t_{p-i}$ for all $i \le p$. Similarly, as $(y_i|\ i \le q)$ is a $\mathcal{B}_n$-initial segment, there is a path $(z_i|\ i \le n) \in \mathcal{B}_n$ such that 1′) $y_i = z_i$ for all $i \le q$ or 2′) $y_i = z_{q-i}$ for all $i \le q$. Admit that 1) and 2′) are valid. Then $t_0 = x_0 = y_0 = z_q$ and $t_1 = x_1 \in \{y_i;\ i \le q\} \subseteq \{z_i;\ i \le n\}$, hence $(t_i|\ i \le n) = (z_i|\ i \le n)$. We have $z_0 = t_0 = z_q$, which is a contradiction. Admit that 2) and 1′) are valid. Then $t_0 = x_p \in \{y_i;\ i \le q\} \subseteq \{z_i;\ i \le n\}$ and $t_1 = x_{p-1} \in \{y_i;\ i \le q\} \subseteq \{z_i;\ i \le n\}$, hence $(t_i|\ i \le n) = (z_i|\ i \le n)$. We have $t_0 = z_0 = y_0 = x_0 = t_p$, which is a contradiction. Admit that 2) and 2′) are valid. Then $t_0 = x_p \in \{y_i;\ i \le q\} \subseteq \{z_i;\ i \le n\}$ and $t_1 = x_{p-1} \in \{y_i;\ i \le q\} \subseteq \{z_i;\ i \le n\}$, hence $(t_i|\ i \le n) = (z_i|\ i \le n)$. We have $t_p = x_0 = y_0 = z_q = t_q$, which is a contradiction because $p < q$. Consequently, only 1) and 1′) may be valid simultaneously. Then $t_0 = x_0 = y_0 = z_0$ and $t_1 = x_1 \in \{y_i;\ i \le q\} \subseteq \{z_i;\ i \le n\}$, hence $(t_i|\ i \le n) = (z_i|\ i \le n)$. The existence of $(z_i|\ i < n)$ is proved. As $\mathcal{B}_n$ is (strongly) terse, $(z_i|\ i < n)$ is clearly unique. □

**Definition 3.10.** An $n$-path set $\mathcal{B}_n$ in a graph $G = (V, E)$ is said to be *plain* provided that the following condition is satisfied:

If $(x_i|\ i \le n), (y_i|\ i \le n) \in \mathcal{B}_n$ and there are natural numbers $i_0$, $i_1$, $i'_0$, $i'_1 \le n$, $i_0 \ne i_1$, such that $x_{i_0} = y_{i'_0}$ and $x_{i_1} = y_{i'_1}$, then $(x_i|\ i \le n) = (y_i|\ i \le n)$.

Clearly, every plain $n$-path set is strongly terse and every 1-path set is plain.

**Theorem 3.11.** *Let $\mathcal{B}_n$ be a plain n-path set in a graph $G = (V, E)$, let $x \in V$ and let $A, B \subseteq V$ be minimal (with respect to set inclusion) subsets with the properties $x \in u_{\mathcal{B}_n}A - A$ and $x \in u_{\mathcal{B}_n}B - B$, respectively. Then $A \cap B = \emptyset$ or $A = B$.*

*Proof.* Suppose that $A \cap B \neq \emptyset$. There exist a path $(x_i|\ i \leq n) \in \mathcal{B}_n$ and a natural number $i_0$, $0 < i_0 \leq n$, such that $x = x_{i_0}$ and $x_i \in A$ for each $i < i_0$. Similarly, there exist a path $(y_i|\ i \leq n) \in \mathcal{B}_n$ and a natural number $i_1$, $0 < i_1 \leq n$, such that $x = y_{i_1}$ and $y_i \in B$ for each $i < i_1$. Since $A$ and $B$ are minimal, we have $A = \{x_i;\ i < i_0\}$ and $B = \{y_i;\ i < i_1\}$. Let $y \in A \cap B$ be a point. Then $y \neq x$ and we have $\{x, y\} \subseteq \{x_i;\ i \leq n\} \cap \{y_i;\ i \leq n\}$. Consequently, $(x_i|\ i \leq n) = (y_i|\ i \leq n)$. It follows that $A = B$ because $i_0 = i_1$. □

## 4. Connectedness

**Definition 4.1.** Let $\mathcal{B}_n$ be an $n$-path set in a graph $G = (V, E)$. A sequence $C = (x_i|\ i \leq m)$, $m > 0$, of vertices of $V$ is called a $\mathcal{B}_n$*-walk* in $G$ if there is an increasing sequence $(j_k|\ k \leq p)$ of natural numbers with $j_0 = 0$ and $j_p = m$ such that $j_k - j_{k-1} \leq n$ and $(x_j|\ j_{k-1} \leq j \leq j_k) \in \mathcal{B}_n^*$ for every $k$ with $k \leq p$. The sequence $(j_k|\ k \leq p)$ is said to be a *binding sequence* of $C$.

If the members of $C$ are pairwise different, then $C$ is called a $\mathcal{B}_n$*-path* in $G$.

A $\mathcal{B}_n$-walk $C$ is said to be *closed* if $x_0 = x_m$, and it is said to be a $\mathcal{B}_n$*-cycle* if, for every pair $i_0, i_1$ of different natural numbers with $i_0, i_1 \leq m$, $x_{i_0} = x_{i_1}$ is equivalent to $\{i_0, i_1\} = \{0, m\}$.

Of course, every $\mathcal{B}_n$-walk ($\mathcal{B}_n$-path) in a graph $G = (V, E)$ is a walk (path) in $G$ and both concepts coincide if $\mathcal{B}_n = \mathcal{B}_1 = E'$ where $E'$ is an arbitrary set of paths of length 1 such that, (1) for every $\{x, y\} \in E$, $(x, y) \in E'$ and $(y, x) \in E'$ and (2) for every $(x, y) \in E'$, $\{x, y\} \in E$.

Observe that, if $(x_0, x_1, ..., x_m)$ is a $\mathcal{B}_n$-walk in $G$, then $(x_m, x_{m-1}, ..., x_0)$ is a $\mathcal{B}_n$-walk in $G$, too. Further, if $(x_i|\ i \leq m)$ and $(y_i|\ i \leq p)$ are $\mathcal{B}_n$-walks in $G$ with $x_m = y_0$, then, putting $z_i = x_i$ for all $i \leq m$ and $z_i = y_{i-m}$ for all $i$ with $m \leq i \leq m + p$, we get a $\mathcal{B}_n$-walk $(z_i|\ i \leq m + p)$ in $G$.

**Definition 4.2.** Let $\mathcal{B}_n$ be an $n$-path set in a graph $G = (V, E)$. A subset of $V$ is said to be $\mathcal{B}_n$*-connected* if it is connected in $(V, u_{\mathcal{B}_n})$. A maximal $\mathcal{B}_n$-connected subset of $V$ is called a $\mathcal{B}_n$*-component* of $G$.

Of course, every initial segment of $G$ is $\mathcal{B}_n$-connected.

**Theorem 4.3.** *Let $\mathcal{B}_n$ be an n-path set in a graph $G = (V, E)$. A subset $A \subseteq V$ is $\mathcal{B}_n$-connected if and only if any two different vertices from $A$ may be joined by a $\mathcal{B}_n$-walk in $G$ contained in $A$.*

*Proof.* If $A = \emptyset$, then the statement is trivial. Let $A \neq \emptyset$. If any two vertices from $A$ can be joined by a $\mathcal{B}_n$*-walk* in $G$ contained in $A$, then $A$ is clearly $\mathcal{B}_n$-connected. Conversely, let $A$ be $\mathcal{B}_n$-connected and suppose that there are vertices $x, y \in A$ which can not be joined by a $\mathcal{B}_n$-walk in $G$ contained in $A$. Let $B$ be the set of all vertices from $A$ which can be joined with $x$ by a $\mathcal{B}_n$-it walk in $G$ contained in $A$. Let $z \in u_{\mathcal{B}_n}B \cap A$ be a vertex and assume that $z \notin B$. Then there are a path $(x_i|\ i \leq n) \in \mathcal{B}_n$ and a natural number $i_0$, $0 < i_0 \leq n$, such that $z = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq B$. Thus, $x$ and $x_0$ can be joined by a $\mathcal{B}_n$-walk in $G$ contained in $A$, and also $x_0$ and $z$ can be joined by a $\mathcal{B}_n$-walk in $G$ contained in $A$ - namely by the $\mathcal{B}_n$-initial segment $(x_i|\ i \leq i_0) \in \mathcal{B}_n^*$. It follows that $x$ and $z$ can be joined by a $\mathcal{B}_n$-walk in $G$ contained in $A$, which is a contradiction. Therefore, $z \in B$, i.e. $u_{\mathcal{B}_n}B \cap A = B$. Consequently, $B$ is closed in the subspace $A$ of $(V, u_{\mathcal{B}_n})$.

Further, let $z \in u_{\mathcal{B}_n}(A - B) \cap A$ be a vertex and assume that $z \in B$. Then $z \notin A - B$, thus there are a path $(x_i|\ i < n) \in \mathcal{B}_n$ and a natural number $i_0$, $0 < i_0 \leq n$, such that $z = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq A - B$. Since $x$ can be joined with $z$ by a $\mathcal{B}_n$-walk in $G$ contained in $A$ (because we have assumed that $z \in B$) and $z$ can be joined with $x_0$ by a $\mathcal{B}_n$-walk in $G$ contained in $A$ - namely by the initial segment $(x_{i_0-i}|\ i \leq i_0) \in \mathcal{B}_n^*$, also $x$ and $x_0$ can be joined by a $\mathcal{B}_n$-walk in $G$ contained in $A$. This is a contradiction with $x_0 \notin B$. Thus, $z \notin B$, i.e. $u_{\mathcal{B}_n}(A - B) \cap A = A - B$. Consequently, $A - B$ is closed in the subspace $A$ of $(V, u_{\mathcal{B}_n})$. Hence, $A$ is the union of the nonempty disjoint sets $B$ and $A - B$ closed in the subspace $A$ of $(V, u_{\mathcal{B}_n})$. But this is a contradiction because $A$ is $\mathcal{B}_n$-connected. Therefore, any two points of $A$ can be joined by a $\mathcal{B}_n$-walk in $G$ contained in $A$. □

Note that, if two vertices of a graph $G$ can be joined by a $\mathcal{B}_n$-*walk* in $G$, they need not be joined by a $\mathcal{B}_n$-*path* in $G$.

**Theorem 4.4.** *If $\mathcal{B}_n$ is a plain $n$-path set in $G$, then every $\mathcal{B}_n$-path in $G$ has exactly one binding sequence.*

*Proof.* Let $C = (x_i|\, i \le m)$ be a $\mathcal{B}_n$-path in $G$ and let $(j_k|\, k \le p)$ and $(l_k|\, k \le q)$ be binding sequences of $C$. Clearly, $j_0 = 0 = l_0$. Let $k < p$ be a natural number such that $k \le q$ and $j_k = l_k$. Then $l_k = j_k < j_p = m = l_q$, hence $k < q$. Assume that $j_{k+1} < l_{k+1}$. As both $(x_i|\, j_k \le i \le j_{k+1})$ and $(x_i|\, l_k \le i \le l_{k+1})$ are $\mathcal{B}_n$-initial segments, there exists a path $(y_i|\, i \le n) \in \mathcal{B}_n$ with $y_i = x_{l_k+i}$ for all $i \le l_{k+1} - l_k$ by Lemma 3.9. Since $j_{k+1} < l_{k+1} \le m$, we have $j_{k+1} < m$. Thus, there is a $\mathcal{B}_n$-initial segment $(x_i|\, j_{k+1} \le i \le j_{k+2})$, which means that there is a path $(z_i|\, i \le n) \in \mathcal{B}_n$ such that 1) $x_i = z_{i-j_{k+1}}$ whenever $j_{k+1} \le i \le j_{k+2}$ or 2) $x_i = z_{j_{k+2}-i}$ whenever $j_{k+1} \le i \le j_{k+2}$. Suppose that 1) is valid. Then $z_0 = x_{j_{k+1}} = y_{j_{k+1}-j_k}$ and $z_1 = x_{j_{k+1}+1} = y_{j_{k+1}-j_k+1}$. As $\mathcal{B}_n$ is plain, we have $(y_i|\, i \le n) = (z_i|\, i \le n)$. Suppose that 2) is valid and put $i_0 = j_{k+1} - j_k$, $i_1 = j_{k+2} - j_{k+1} - 1$. Then $y_{i_0} = y_{j_{k+1}-j_k} = x_{j_{k+1}} = z_{j_{k+2}-j_{k+1}} = z_{i_1+1}$ and $z_{i_1} = z_{j_{k+2}-j_{k+1}-1} = x_{j_{k+1}+1} = y_{j_{k+1}-j_k+1} = y_{i_0+1}$. As $\mathcal{B}_n$ is plain, we have $(y_i|\, i \le n) = (z_i|\, i \le n)$. Thus, $y_0 = z_0$ in both cases, which is a contradiction because $y_0 = x_{l_k} = x_{j_k}$ and, on the other hand, $z_0 = x_{j_{k+1}}$ or $z_0 = x_{j_{k+2}}$. Therefore, $j_{k+1} \not< l_{k+1}$. Analogously, by interchanging $j$ and $l$, we can easily show that $l_{k+1} \not< j_{k+1}$. Consequently, $j_{k+1} = l_{k+1}$ and, by the mathematical induction principle, $k \le q$ and $j_k = l_k$ for each $k \le p$. Since $l_p = j_p = m = l_q$ implies $p = q$, we get $(j_k|\, k \le p) = (l_k|\, k \le q)$. The proof is complete. □

The previous theorem enables us to define:

**Definition 4.5.** Let $\mathcal{B}_n$ be a plain $n$-path set in a graph $G$, let $C$ be a $\mathcal{B}_n$-path in $G$ and let $(j_k|\, k \le p)$ be the binding sequence of $C$. Then $C$ is called a $\mathcal{B}_n$-*arc* in $G$ provided that, for every $\mathcal{B}_n$-initial segment $D \in \mathcal{B}_n^*$ with $D \subseteq C$, there is a natural number $k < p$ such that $D$ is a subset of the $\mathcal{B}_n$ initial segment $(x_i|\, j_k \le i \le j_{k+1})$.

**Theorem 4.6.** *Let $\mathcal{B}_n$ be a plain $n$-path set in $G$ and $C$ be a $\mathcal{B}_n$-arc in $G$ joining vertices $x$ and $y$. If $D$ is a $\mathcal{B}_n$-walk in $G$ joining $x$ and $y$ such that $D \subseteq C$, then $C = D$.*

*Proof.* Let $C = (x_i|\, i \le m)$ and let $D = (y_i|\, i \le s)$ be a $\mathcal{B}_n$-walk in $G$ with $y_0 = x$ and $y_s = y$ such that $\{y_i;\ i \le s\} \subseteq \{x_i;\ i \le m\}$. Let $(j_k|\, k \le p)$ be the binding sequence of $C$ and for each $k < p$ put $C_k = (x_i|\, j_k \le i \le j_{k+1})$. Let $(l_k|\, k \le q)$ be the binding sequence of $(y_i|\, i \le s)$ and for each $k < q$ put $D_k = (y_i|\, l_k \le i \le l_{k+1})$. Clearly, $x_{j_0} = x_0 = x = y_0 = y_{l_0}$. Let $k^* < q$ be a natural number with $k^* \le p$, $j_{k^*} = l_{k^*}$, and $x_i = y_i$ for all $i \le j_{k^*}$ (such $k^*$ exists because we may put $k^* = 0$). As $x_{j_{k^*}} = y_{l_{k^*}} \ne y_{l_{q-1}} = y$, we have $k^* < p$. Since $D_{k^*} \subseteq C$, there is a $\mathcal{B}_n$-initial segment $C_{k_0}$ with $D_{k^*} \subseteq C_{k_0}$. As $y_{l_{k^*}} = x_{j_{k^*}} \in D_{k^*} \cap C_{k^*}$, we have $k_0 = k^*$. (Of course, we also have $y_{l_{k^*}} = x_{j_{k^*}} \in D_{k^*} \cap C_{k^*-1}$, but $D_{k^*} \nsubseteq C_{k^*-1}$ because $y_{l_{k^*+1}} \in D_{k^*} - \{y_i;\ i \le l_{k^*}\} = \{x_i;\ i \le j_{k^*}\}$ and $C_{k^*-1} \subseteq \{x_i;\ i \le j_{k^*}\}$.) Admit that $l_{k^*+1} \ne j_{k^*+1}$. Then $l_{k^*+1} < j_{k^*+1}$ (because $D_{k^*} \subseteq C_{k^*}$) and, by Lemma 3.9, there is a path $(z_i|\, i \le n) \in \mathcal{B}_n$ such that $y_i = z_{i-l_{k^*}}$ for all $i$ with $l_{k^*} \le i \le l_{k^*+1}$ and $x_i = z_{i-l_{k^*}}$ for all $i$ with $j_{k^*} \le i \le j_{k^*+1}$. Thus, $l_{k^*+1} < j_{k^*+1}$ and $x_i = y_i$ for all $i$ with $k_{k^*} \le i \le l_{k^*+1}$. If $k^* + 1 = q$, then $x_{l_{k^*+1}} = y_{l_{k^*+1}} = y = x_{j_{p-1}}$, which is a contradiction because $l_{k^*+1} < j_{k^*+1} \le j_p$. So, we have $k^* + 1 < q$. Since $D_{k^*+1} \subseteq C$, there is an initial segment $C_{k_1}$ with $D_{k^*+1} \subseteq C_{k_1}$. As $y_{l_{k^*+1}} = x_{l_{k^*+1}} \in C_{k^*}$ and $x_{j_{k^*}} \ne x_{l_{k^*+1}} \ne x_{j_{k^*+1}}$, we have $k_1 = k^*$. Hence $D_{k^*+1} \subseteq C_{k^*} \subseteq \{z_i;\ i \le n\}$. Clearly, we have $D_{k^*} \subseteq D_{k^*+1}$ or $D_{k^*+1} \subseteq D_{k^*}$. This is a contradiction because $y_{l_{k^*}} \in D_{k^*} - D_{k^*+1}$ and $y_{l_{k^*+2}} \in D_{k^*+1} - D_{k^*}$. Therefore, $l_{k^*} = j_{k^*}$ and, consequently, $x_i = y_i$ for all $i \le j_{k^*+1}$. Thus, according to the mathematical induction principle, whenever $k \le q$, we have $k \le p$, $j_k = l_k$ and $x_i = y_i$ for all $i \le j_k$. As $x_{j_{q-1}} = y_{l_{q-1}} = y$, we get $j_{q-1} = j_{p-1}$, i.e. $p = q$. Hence $(x_i|\, i \le m) = (y_i|\, i \le s)$. □

The previous statement, if $\mathcal{B}_n$ is plain, then $\mathcal{B}_n$-arcs are minimal (with respect to set inclusion) $\mathcal{B}_n$-walks joining a given pair of vertices.

**Definition 4.7.** Let $\mathcal{B}_n$ be a plain $n$-path set in $G$. A closed $\mathcal{B}_n$-walk $C = (x_i|\, i \le m)$ in $G$ is said to be *simple* if its binding sequence $(j_k|\, k \le p)$ has the property that $p \ge 3$ and $(x_{j_{k+l(\mathrm{mod}\, p)}}|\, l < p)$ is an arc in $G$ whenever $k < p$.

It is evident that every simple closed $\mathcal{B}_n$-walk in $G$ is a $\mathcal{B}_n$-cycle in $G$.

Let $\mathcal{B}_n$ be an $n$-path set in $G = (V, E)$, $U \subseteq V$ be a subset, and $H = (U, F)$ be the induced subgraph of $G$. We denote by $\mathcal{B}_n/U$ the subset of $\mathcal{B}_n$ consisting of the paths belonging to $\mathcal{B}_n$ that are paths in $H$. Thus, $\mathcal{B}_n/U$ is an $n$-path set in $H$. We say that a subset $J \subseteq V$ *separates* $G$ into two components if, putting $U = V - J$, the induced subgraph $(U, F)$ of $G$ has exactly two $\mathcal{B}_n/U$-components.

**Definition 4.8.** A simple closed $\mathcal{B}_n$-path $(x_i|\ i \le m)$ in $G$ is said to be a *Jordan curve* if $\{x_i|\ i \le m\}$ separates $G$ into two components.

The following statement is evident:

**Theorem 4.9.** *A simple closed $\mathcal{B}_n$-path $(x_i|\ i \le m)$ in $G$ is a Jordan curve if and only if there are $\mathcal{B}_n$-connected subsets $X_1, X_2 \subseteq V$ with $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = V - \{x_i|\ i \le m\}$ such that a subset $A \subseteq X_1 \cup X_2$ is not $\mathcal{B}_n$-connected whenever $A \supset X_1$ or $A \supset X_2$.*

## 5. Closure Operators Induced by $N$-Path Sets in Graphs with the Vertex Set $\mathbb{Z}^2$

It is well known [15] that closure operators which are more general than the Kuratowski ones have useful applications in computer science. By Theorem 4.3, $\mathcal{B}_n$-connectedness in graphs is a certain type of path connectedness. This enables us to apply closure operators induced by $n$-path sets in digital topology, which is a theory that has arisen for the study of geometric and topological properties of digital images (cf. [11]). One of the problems of digital topology is to find structures for the digital plane $\mathbb{Z}^2$ convenient for the study of digital images. A basic criterion of such a convenience is the validity of a digital analogue of the Jordan curve theorem (which says that every simple closed curve in the real plane separates this plane into precisely two connected components). The thing is that digital Jordan curves represent borders of objects on digital pictures. Though the classical approach to the problem is based on using graph theory (the 4-adjacency and 8-adjacency relations) for structuring the digital plane, a new approach based on using topological structures is being developed since the beginning of the 90's of the last century - see [8]. There are two well-known topologies on $\mathbb{Z}^2$ which are employed in digital topology, the so-called *Marcus-Wyse topology* [9] and *Khalimsky topology* [7]. Recall that the Marcus-Wyse and Khalimsky topologies are the Alexandroff topologies $s$ and $t$, respectively, on $\mathbb{Z}^2$ with the closures of singleton subsets given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$s\{z\} = \begin{cases} \{z\} \cup A_4(z) \text{ if } x + y \text{ is even,} \\ \{z\} \text{ otherwise} \end{cases}$$

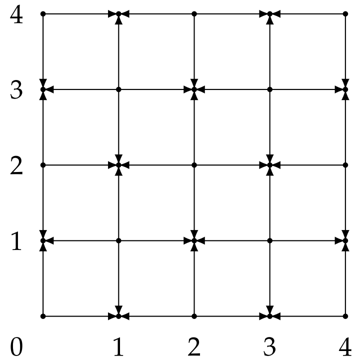
and

$$t\{z\} = \begin{cases} \{z\} \cup A_8(z) \text{ if } x, y \text{ are even,} \\ \{(x+i, y); i \in \{-1, 0, 1\}\} \text{ if } x \text{ is even and } y \text{ is odd,} \\ \{(x, y+j); j \in \{-1, 0, 1\}\} \text{ if } x \text{ is odd and } y \text{ is even,} \\ \{z\} \text{ otherwise.} \end{cases}$$

We will show that both of these topologies and also the closure operators discussed in [12] may be obtained as closure operators induced by $n$-path sets in certain graphs with the vertex set $\mathbb{Z}^2$.

While the concept of $\mathcal{B}_n$-connectedness is useless if $n = 1$ (because then it coincides with the usual graph connectedness), a different situation occurs if $n > 1$.

**Example 5.1.** Put $\mathcal{B}_1 = \{((x_i, y_i)|\ i \le 1);\ (x_i, y_i) \in \mathbb{Z}^2 \text{ for every } i \le 1,\ |x_0 - x_1| + |y_0 - y_1| = 1,\ x_0 + y_0 \text{ even}\}$ and let $G = (\mathbb{Z}^2, E)$ be an arbitrary graph (with the vertex set $\mathbb{Z}^2$) such that $\mathcal{B}_1 \subseteq \mathcal{P}(G)$. Then $\mathcal{B}_1$ is a plain 1-path set in $G$. A portion of the set $\mathcal{B}_1$ is shown in the following figure where the 1-paths from $\mathcal{B}_1$ are represented by arrows oriented from first to last vertex.

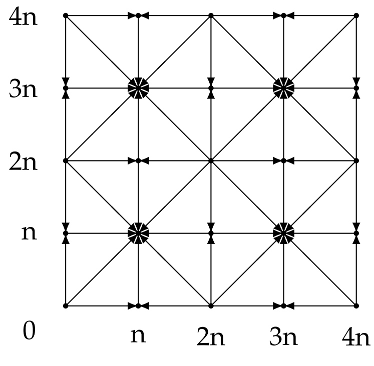



Clearly, $(\mathbb{Z}^2, u_{\mathcal{B}_1})$ is a connected Alexandroff $T_{\frac{1}{2}}$-space in which the points $(x, y) \in \mathbb{Z}^2$ with $x + y$ even are open while those with $x + y$ odd are closed. The closure operator $u_{\mathcal{B}_1}$ coincides with the Marcus-Wyse topology.

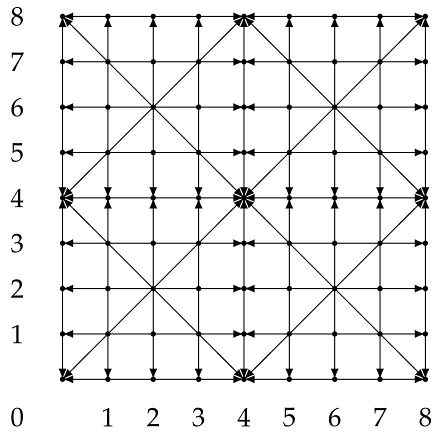
**Example 5.2.** For an arbitrary natural number $n$, let $\mathcal{B}_n$ be the set of all sequences $((x_i, y_i)|\ i \leq n)$ such that $(x_i, y_i) \in \mathbb{Z}^2$ for every $i \leq n$ and one of the following eight conditions is satisfied:

(1) $x_0 = x_1 = ... = x_n$ and there is $k \in \mathbb{Z}$ such that $y_i = 2kn + i$ for all $i \leq n$,
(2) $x_0 = x_1 = ... = x_n$ and there is $k \in \mathbb{Z}$ such that $y_i = 2kn - i$ for all $i \leq n$,
(3) $y_0 = y_1 = ... = y_n$ and there is $k \in \mathbb{Z}$ such that $x_i = 2kn + i$ for all $i \leq n$,
(4) $y_0 = y_1 = ... = y_n$ and there is $k \in \mathbb{Z}$ such that $x_i = 2kn - i$ for all $i \leq n$,
(5) there is $k \in \mathbb{Z}$ such that $x_i = 2kn + i$ for all $i \leq n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2ln + i$ for all $i \leq n$,
(6) there is $k \in \mathbb{Z}$ such that $x_i = 2kn + i$ for all $i \leq n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2ln - i$ for all $i \leq n$,
(7) there is $k \in \mathbb{Z}$ such that $x_i = 2kn - i$ for all $i \leq n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2ln + i$ for all $i \leq n$,
(8) there is $k \in \mathbb{Z}$ such that $x_i = 2kn - i$ for all $i \leq n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2ln - i$ for all $i \leq n$.

Let $G = (\mathbb{Z}^2, E)$ be an arbitrary graph (with the vertex set $\mathbb{Z}^2$) such that $\mathcal{B}_n \subseteq \mathcal{P}(G)$. Then $\mathcal{B}_n$ is a plain $n$-path set in $G$. A portion of the set $\mathcal{B}_n$ is demonstrated in the following figure. The paths belonging to $\mathcal{B}_n$ are represented by arrows oriented from first to last term. Between any pair of neighboring parallel horizontal or vertical arrows (having the same orientation), there are $n - 1$ more parallel arrows with the same orientation that are not displayed.




Clearly, $(\mathbb{Z}^2, u_{\mathcal{B}_n})$ is a connected Alexandroff $T_0$-space. The closure operators $u_{\mathcal{B}_n}$ coincide with the closure operators on $\mathbb{Z}^2$ studied in [12] where a Jordan curve theorem was proved for these operators thus showing their convenience for applications in digital topology. In particular, $u_{\mathcal{B}_1}$ coincides with the Khalimsky topology on $\mathbb{Z}^2$ (for which a Jordan curve theorem was proved in [7]). Note that for $n > 1$, the $\mathcal{B}_n$-connectedness in $G_n$ cannot be obtained as the usual graph connectedness. In particular, for $n = 2$, the above portion of the graph $G_n$ looks, in full detail, as follows:

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