Filomat 28:8 (2014), 1661–1674
DOI 10.2298/FIL1408661G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nondifferentiable Higher Order Symmetric Duality under Invexity/Generalized Invexity

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Abstract. In this paper, we introduce a pair of nondifferentiable higher-order symmetric dual models. Weak, strong and converse duality theorems for this pair are established under the assumption of higher-order invexity/generalized invexity. Self duality has been discussed assuming the function involved to be skew-symmetric. Several known results are obtained as special cases.

1. Introduction

Duality in linear programming is symmetric, i.e, the dual of the dual is the primal problem. However, the majority of dual formulations in nonlinear programming do not possess this property. Symmetric dual programs have applications in the theory of games. Dorn [10] introduced the concept of symmetric duality in quadratic programming. In nonlinear programming this concept was significantly developed by Dantzig et al. [9]. Pardalos et al. [23] have discussed optimality conditions and duality for nondifferentiable fractional programming with generalized convexity for multiobjective problems.

Chinchuluun and Pardalos [7] have surveyed of recent developments in multiobjective optimization, while Pardalos et al. [22], and Zopounidis and Pardalos [27] have studied recent advances in multicriteria analysis which includes multiobjective optimization.

Mangasarian [19] introduced the concept of second and higher-order duality for nonlinear programming problems, which motivates several authors [1, 2, 11, 12, 15, 16, 25] in this field. The study of second and higher-order duality is significant due to the computational advantage over the first-order duality as it provides tighter bounds for the value of the objective function when approximations are used.

In the past, several generalization have been given for convexity. One of them is invex function, which was introduced by Hanson [13] and Craven [8]. After this work, several authors [4, 16, 17] have also presented extension of invexity and used them to derive duality results. Mond and Zhang [20] obtained duality results for various higher-order dual problems under invexity assumptions. Chen [6] discussed duality

2010 *Mathematics Subject Classification*. Primary 90C46; 49N15; Secondary 90C30
Keywords. Nondifferentiable higher-order dual models; symmetric duality; Duality theorems; Higher-order invexity/generalized invexity; Self duality
Received: 08 August 2013; Accepted: 11 October 2013
Communicated by Predrag Stanimirović
The second author is thankful to the MHRD, Government of India for providing financial support.
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theorems under higher-order generalized $F-$convexity for a pair of nondifferentiable programs. Yang et al. [26] have discussed multiobjective higher-order symmetric duality under invexity assumptions. Ahmad et al. [3] discussed higher-order duality in nondifferentiable multiobjective programming.

Recently, Ahmad [2] has unified higher-order duality in nondifferentiable multiobjective programming. In this paper, we introduce a pair of nondifferentiable higher-order symmetric dual models. Weak, strong and converse duality theorems for this pair are established under the assumption of higher-order invexity/generalized invexity. Self duality has been discussed assuming the function involved to be skew-symmetric. Several known results are obtained as special cases.

2. Preliminaries

Definition 2.1. *The support function $s(x|C)$ of C is defined by*

$$s(x|C) = max\{x^T y : y \in C\}.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For any convex set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0 \ \textit{for all} \ z \in S\}.$$

It is readily verified that for a compact convex set E, y is in $N_E(x)$ if and only if

$$s(y|E) = x^T y.$$

Definition 2.2. *A function $\phi : R^n \mapsto R$ is said to be higher-order invex at $u \in R^n$ with respect to $\eta : R^n \times R^n \mapsto R^n$ and $h : R^n \times R^n \mapsto R$, if for all $(x, p) \in R^n \times R^n$,*

$$\phi(x) - \phi(u) - h(u, p) + p^T \nabla_p h(u, p) \geqq \eta^T(x, u)\{\nabla_x \phi(u) + \nabla_p h(u, p)\}.$$

Definition 2.3. *A function $\phi : R^n \mapsto R$ is said to be higher-order pseudoinvex at $u \in R^n$ with respect to $\eta : R^n \times R^n \mapsto R^n$ and $h : R^n \times R^n \mapsto R$, if for all $(x, p) \in R^n \times R^n$,*

$$\eta^T(x, u)[\nabla_x \phi(u) + \nabla_p h(u, p)] \geqq 0 \Rightarrow \phi(x) - \phi(u) - h(u, p) + p^T \nabla_p h(u, p) \geqq 0.$$

3. Wolfe Type Higher Order Symmetric Duality

We consider the following pair of Wolfe type higher order symmetric duals and establish weak, strong and converse duality theorems.

Primal Problem (WHP):

Minimize
$L(x, y, p) = f(x, y) + s(x|C) - y^T z + h(x, y, p) - p^T \nabla_p h(x, y, p) - y^T[\nabla_y f(x, y) + \nabla_p h(x, y, p)]$,

subject to

$$\nabla_y f(x,y) - z + \nabla_p h(x,y,p) \leqq 0, \tag{3.1}$$
$$p^T[\nabla_y f(x,y) - z + \nabla_p h(x,y,p)] \geqq 0, \tag{3.2}$$
$$x,\ y \geqq 0,\ \ z \in D, \tag{3.3}$$

Dual Problem (WHD):

Maximize
$M(u,v,r) = f(u,v) - s(v|D) + u^T w + g(u,v,r) - r^T \nabla_r g(u,v,r) - u^T[\nabla_u f(u,v) + \nabla_r g(u,v,r)]$,

subject to

$$\nabla_u f(u,v) + w + \nabla_r g(u,v,r) \geqq 0, \tag{3.4}$$
$$r^T[\nabla_u f(u,v) + w + \nabla_r g(u,v,r)] \leqq 0, \tag{3.5}$$
$$u,\ v \geqq 0,\ \ w \in C, \tag{3.6}$$

where
(i) $f : R^n \times R^m \mapsto R$, $g : R^n \times R^m \times R^n \mapsto R$ and $h : R^n \times R^m \times R^m \mapsto R$ are twice differentiable functions, and
(ii) $C \subset R^n$ and $D \subset R^m$ are compact convex sets.

Theorem 3.1. *(Weak Duality). Let (x,y,z,p) and (u,v,w,r) be feasible solutions for primal and dual problem, respectively. Suppose that*

(i) $f(.,v) + (.)^T w$ is higher-order invex at u with respect to η_1 and $g(u,v,r)$,
(ii) $-[f(x,.) - (.)^T z]$ is higher-order invex at y with respect to η_2 and $-h(x,y,p)$,
(iii) $\eta_1(x,u) + u + r \geqq 0$,
(iv) $\eta_2(v,y) + y + p \geqq 0$.

Then

$$L(x,y,z,p) \geqq M(u,v,w,r). \tag{3.7}$$

Proof. From the hypothesis (iii) and the dual constraints (3.4), we get

$$(\eta_1(x,u)^T + u + r)[\nabla_x f(u,v) + w + \nabla_r g(u,v,r)] \geqq 0,$$

or

$$(\eta_1(x,u)^T + u)[\nabla_x f(u,v) + w + \nabla_r g(u,v,r)] \geqq -r^T[\nabla_x f(u,v) + w + \nabla_r g(u,v,r)],$$

which on using the dual constraint (3.5) implies that

$$(\eta_1(x,u) + u)^T[\nabla_x f(u,v) + w + \nabla_r g(u,v,r)] \geqq 0. \tag{3.8}$$

Now by the higher order invexity of $f(.,v) + (.)^T w$ at v with respect to η_1 and $g(u,v,r)$, we get

$$f(x,v) + x^T w - f(u,v) - u^T w - g(u,v,r) + r^T \nabla_r g(u,v,r) + u^T[\nabla_x f(u,v) + w + \nabla_r g(u,v,r)] \geqq 0. \tag{3.9}$$

Similarly, hypothesis (iv) along with the primal constraints (3.1) and (3.2), yields

$$(\eta_2(v,y) + y)^T[\nabla_y f(x,y) - z + \nabla_p h(x,y,p)] \leqq 0. \tag{3.10}$$

Therefore, by higher-order invexity of $-[f(x,.)-(.)^Tz]$ at y with respect to η_2 and $-h(x,y,p)$, we obtain

$$f(x,y)-y^Tz-f(x,v)+v^Tz+h(x,y,p)-p^T\nabla_p h(x,y,p)-y^T[\nabla_y f(x,y)-z+\nabla_p h(x,y,p)] \geqq 0. \tag{3.11}$$

Adding inequalities (3.9) and (3.11), we get

$$f(x,y)+x^Tw-y^Tz+h(x,y,p)-p^T\nabla_p h(x,y,p)-y^T[\nabla_y f(x,y)-z+\nabla_p h(x,y,p)]$$
$$\geqq f(u,v)-v^Tz+u^Tw+g(u,v,r)-r^T\nabla_r g(u,v,r)-u^T[\nabla_x f(u,v)+w+\nabla_r g(u,v,r)].$$

Since $x^Tw \leqq s(x|C)$ and $v^Tz \leqq s(v|D)$, the above inequality implies

$$f(x,y)+s(x|C)-y^Tz+h(x,y,p)-p^T\nabla_p h(x,y,p)-y^T[\nabla_y f(x,y)-z+\nabla_p h(x,y,p)]$$
$$\geqq f(u,v)-s(v|D)+u^Tw+g(u,v,r)-r^T\nabla_r g(u,v,r)-u^T[\nabla_x f(u,v)+w+\nabla_r g(u,v,r)],$$

or

$$L(x,y,z,p) \geqq M(u,v,w,r).$$

Thus the result holds. □

Theorem 3.2. *(Strong Duality). Let $(\bar{x},\bar{y},\bar{z},\bar{p})$ be a local optimal solution of (WHP). Assume that*

(i) $\nabla_{pp}h(\bar{x},\bar{y},\bar{p})$ is negative definite,
(ii)$\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p}) \neq 0$,
(iii) $\bar{y}^T[\nabla_y h(\bar{x},\bar{y},\bar{p})-\nabla_p h(\bar{x},\bar{y},\bar{p})+\nabla_{yy}f(\bar{x},\bar{y})\bar{p}]=0 \Rightarrow \bar{p}=0$,
(iv) $h(\bar{x},\bar{y},0)=g(\bar{x},\bar{y},0)$, $\nabla_x h(\bar{x},\bar{y},0)=\nabla_r g(\bar{x},\bar{y},0)$, $\nabla_y h(\bar{x},\bar{y},0)=\nabla_p h(\bar{x},\bar{y},0)$.

Then

(I) $(\bar{x},\bar{y},\bar{z},\bar{r}=0)$ is feasible for (WHD) and
(II) $L(\bar{x},\bar{y},\bar{z},\bar{p})=M(\bar{x},\bar{y},\bar{w},\bar{r})$.

Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (WHP) and (WHD), then $(\bar{x},\bar{y},\bar{z},\bar{p}=0)$ and $(\bar{x},\bar{y},\bar{w},\bar{r}=0)$ are global optimal solutions of (WHP) and (WHD), respectively.

Proof. Since $(\bar{x},\bar{y},\bar{p})$ is a local optimal solution of (WHP), there exist $\alpha,\ \delta \in R,\ \beta \in R^m$ and $\mu,\xi \in R^n$ such that the following Fritz-John conditions [18, 24] are satisfied at $(\bar{x},\bar{y},\bar{z},\bar{p})$:

$$\alpha[\nabla_x f(\bar{x},\bar{y})+\xi+\nabla_x h(\bar{x},\bar{y},\bar{p})-\nabla_{px}h(\bar{x},\bar{y},\bar{p})\bar{p}]+[\nabla_{yx}f(\bar{x},\bar{y})+\nabla_{px}h(\bar{x},\bar{y},\bar{p})](\beta-\alpha\bar{y}-\delta\bar{p})-\mu=0, \tag{3.12}$$

$$\alpha[\nabla_y h(\bar{x},\bar{y},\bar{p})-\nabla_{py}h(\bar{x},\bar{y},\bar{p})\bar{p}-\nabla_p h(\bar{x},\bar{y},\bar{p})]+[\nabla_{yy}f(\bar{x},\bar{y})+\nabla_{py}h(\bar{x},\bar{y},\bar{p})](\beta-\alpha\bar{y}-\delta\bar{p})=0, \tag{3.13}$$

$$\nabla_{pp}h(\bar{x},\bar{y},\bar{p})(\beta-\alpha\bar{p}-\alpha\bar{y}-\delta\bar{p})-\delta[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0, \tag{3.14}$$

$$\beta^T[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0, \tag{3.15}$$

$$\delta\bar{p}^T[\nabla_y f(\bar{x},\bar{y}) - \bar{z} + \nabla_p h(\bar{x},\bar{y},\bar{p})] = 0, \tag{3.16}$$

$$-\beta + \mu\bar{p} \in N_D(\bar{z}), \tag{3.17}$$

$$\xi^T\bar{x} = s(\bar{x}|C), \quad \xi \in C, \tag{3.18}$$

$$\bar{x}^T\mu = 0, \tag{3.19}$$

$$(\alpha,\beta,\gamma,\delta,\mu) \neq 0, \tag{3.20}$$

$$(\alpha,\beta,\gamma,\delta,\mu) \geqq 0. \tag{3.21}$$

Premultiplying $(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})$ in equation (3.14), we get

$$(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})^T\nabla_{pp}h(\bar{x},\bar{y},\bar{p})(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})$$
$$-\delta(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})^T[\nabla_y f(\bar{x},\bar{y}) - \bar{z} + \nabla_p h(\bar{x},\bar{y},\bar{p})] = 0,$$

which along with equations (3.15), (3.16) yields

$$(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})^T\nabla_{pp}h(\bar{x},\bar{y},\bar{p})(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})$$
$$+\alpha\delta\bar{y}^T[\nabla_y f(\bar{x},\bar{y}) - \bar{z} + \nabla_p h(\bar{x},\bar{y},\bar{p})] = 0,$$

Now from equations (3.1), (3.3) and (3.21), we obtain

$$(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})^T\nabla_{pp}h(\bar{x},\bar{y},\bar{p})(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p}) \geqq 0.$$

Using hypothesis (i), we have

$$\beta = \alpha\bar{p} + \alpha\bar{y} + \delta\bar{p}. \tag{3.22}$$

This together with hypothesis (ii) and equation (3.14), yields

$$\delta = 0. \tag{3.23}$$

Now, we claim that $\alpha \neq 0$. Indeed if $\alpha = 0$, then equation (3.22) and (3.23), gives

$$\beta = 0.$$

Which together with equations (3.12) and (3.23), yields $\mu = 0$, a contradiction to equation (3.21). Hence

$$\alpha > 0. \tag{3.24}$$

Using equations (3.13) and (3.22), we have

$$\alpha[\nabla_y h(\bar{x},\bar{y},\bar{p}) - \nabla_p h(\bar{x},\bar{y},\bar{p}) + \nabla_{yy} f(\bar{x},\bar{y})\bar{p}] = 0, \tag{3.25}$$

which by hypothesis (iii) and equation (3.24), yields

$$\bar{p} = 0. \tag{3.26}$$

Moreover, equation (3.12) along with (3.22), (3.26) and hypothesis (iv) yields

$$\alpha[\nabla_x f(\bar{x},\bar{y}) - \xi + \nabla_r g(\bar{x},\bar{y},\bar{p})] = \mu,$$

as $\alpha > 0$,

$$\nabla_x f(\bar{x}, \bar{y}) - \xi + \nabla_r g(\bar{x}, \bar{y}, \bar{p}) = \frac{\mu}{\alpha} \geqq 0, \tag{3.27}$$

Using equation (3.24)

$$\bar{x}^T[\nabla_x f(\bar{x}, \bar{y}) - \xi + \nabla_r g(\bar{x}, \bar{y}, \bar{p})] = \frac{\bar{x}^T\mu}{\alpha} = 0. \tag{3.28}$$

Now taking $\xi = \bar{w} \in C$, we find $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ satisfies the constraints (3.4)-(3.6), that is, it is a feasible solution for the dual problem (WHD).

Also, since $\beta = \gamma\bar{y}$ and $\alpha > 0$, equation (3.17) gives $\bar{y} \in N_D(\bar{z})$ and therefore

$$\bar{y}^T\bar{z} = s(\bar{y}|D). \tag{3.29}$$

Hence using equations (3.18), (3.26), (3.29) and hypothesis (iv), we get

$$f(\bar{x}, \bar{y}) + s(\bar{x}|C) + h(\bar{x}, \bar{y}, \bar{p}) - \bar{p}^T\nabla_p h(\bar{x}, \bar{y}, \bar{p}) - \bar{y}^T[\nabla_y f(\bar{x}, \bar{y}) + \nabla_p h(\bar{x}, \bar{y}, \bar{p})$$

$$= f(\bar{x}, \bar{y}) - s(\bar{y}|D) + g(\bar{x}, \bar{y}, \bar{p}) - \bar{r}^T\nabla_r g(\bar{x}, \bar{y}, \bar{p}) - \bar{x}^T[\nabla_y f(\bar{x}, \bar{y}) + \nabla_r g(\bar{x}, \bar{y}, \bar{p})],$$

i.e.,

$$L(\bar{x}, \bar{y}, \bar{z}, \bar{p}) = M(\bar{x}, \bar{y}, \bar{w}, \bar{r}).$$

Also, by Theorem 3.1, $(\bar{x}, \bar{y}, \bar{w}, \bar{p} = 0)$ and $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ are global optimal solutions of the respective problems.

Theorem 3.3. *(Converse Duality). Let $(\bar{u}, \bar{v}, \bar{w}, \bar{r})$ be a local optimal solution of (WHP). Assume that*

(i) $\nabla_{rr} g(\bar{u}, \bar{v}, \bar{r})$ is positive definite,
(ii)$\nabla_u f(\bar{u}, \bar{v}) - \bar{w} + \nabla_r g(\bar{u}, \bar{v}, \bar{r}) \neq 0$,
(iii)$\bar{y}^T[\nabla_u h(\bar{u}, \bar{v}, \bar{r}) - \nabla_r h(\bar{u}, \bar{v}, \bar{r}) + \nabla_{uu} f(\bar{u}, \bar{v})\bar{r}] = 0 \Rightarrow \bar{r} = 0$,
(iv) $g(\bar{u}, \bar{v}, 0) = g(\bar{u}, \bar{v}, 0)$, $\nabla_u g(\bar{u}, \bar{v}, 0) = \nabla_r g(\bar{u}, \bar{v}, 0)$, $\nabla_v g(\bar{u}, \bar{v}, 0) = \nabla_p h(\bar{u}, \bar{v}, 0)$.

Then
(I) $(\bar{u}, \bar{v}, \bar{z}, \bar{p} = 0)$ is feasible for (WHP) and
(II) $L(\bar{u}, \bar{v}, \bar{z}, \bar{p}) = M(\bar{u}, \bar{v}, \bar{w}, \bar{r})$.

Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (WHP) and (WHD), then $(\bar{u}, \bar{v}, \bar{w}, \bar{r} = 0)$ and $(\bar{u}, \bar{v}, \bar{z}, \bar{p} = 0)$ are global optimal solutions of (WHD) and (WHP), respectively.

Proof. Follows on the line of Theorem 3.2.

3.1. Self Duality

A mathematical programming problem is said to be self-dual if primal problem having equivalent dual formulation, that is, if the dual can be rewritten as in the form of the primal. In general, (WHP) and (WHD) are not self-duals without some added restrictions on f, g and h.

(i) If we assume $f : R^n \times R^m \to R$ and $g : R^n \times R^m \times R^n \to R$ to be skew symmetric, i.e.,

$$f(u,v) = -f(v,u), \quad g(u,v,r) = -g(v,u,r)$$

(ii) $C = D$

then we shall show that (WHP) and (WHD) are self-duals. By recasting the dual problem (WHD) as a minimization problem, we have

Minimize

$$M(u,v,r) = -\{f(u,v) - s(v|D) + u^T w + g(u,v,r) - r^T \nabla_r g(u,v,r) - u^T [\nabla_u f(u,v) + w + \nabla_r g(u,v,r)]$$

subject to

$$\begin{aligned}
&\nabla_u f(u,v) + w + \nabla_r g(u,v,r) \geqq 0,\\
&r^T[\nabla_u f(u,v) + w + \nabla_r g(u,v,r)] \leqq 0,\\
&u,\ v \geqq 0, \quad w \in C.
\end{aligned}$$

Now as f and g are skew symmetric, i.e.,

$$\nabla_u f(u,v) = -\nabla_u f(v,u)$$

$$\nabla_r g(u,v,r) = -\nabla_r g(v,u,r),$$

the above problem recasting as :

Minimize

$$M(u,v,r) = f(v,u) + s(v|C) - u^T w + g(v,u,r) - r^T \nabla_r g(v,u,r) - u^T [\nabla_u f(v,u) + w + \nabla_r g(v,u,r)]$$

subject to

$$\begin{aligned}
&\nabla_u f(v,u) + w + \nabla_r g(v,u,r) \leqq 0,\\
&r^T[\nabla_u f(v,u) + w + \nabla_r g(v,u,r)] \geqq 0,\\
&u,\ v \geqq 0, \quad w \in D,
\end{aligned}$$

Which shows that it is identical to (WHP), i.e., the objective and the constraint functions are identical. Thus, the problem (WHP) is self-dual.

It is obvious that (x, y, w, p) is feasible for (WHP), then (y, x, w, p) is feasible for (WHD) and vice versa.

4. Mond-Weir type Higher Order Symmetric Duality

We consider the following pair of higher order symmetric duals and establish weak, strong and converse duality theorems.

Primal Problem (MHP):

Minimize $L(x, y, p) = f(x, y) + s(x|C) - y^T z + h(x, y, p) - p^T \nabla_p h(x, y, p)$

subject to

$$\nabla_y f(x, y) - z + \nabla_p h(x, y, p) \leqq 0, \tag{4.1}$$

$$y^T[\nabla_y f(x, y) - z + \nabla_p h(x, y, p)] \geqq 0, \tag{4.2}$$

$$p^T[\nabla_y f(x, y) - z + \nabla_p h(x, y, p)] \geqq 0, \tag{4.3}$$

$$x \geqq 0, \quad z \in D, \tag{4.4}$$

Dual Problem (MHD):

Maximize $M(u, v, r) = f(u, v) - s(v|D) + u^T w + g(u, v, r) - r^T \nabla_r g(u, v, r)$

subject to

$$\nabla_u f(u, v) + w + \nabla_r g(u, v, r) \geqq 0, \tag{4.5}$$

$$u^T[\nabla_u f(u, v) + w + \nabla_r g(u, v, r)] \leqq 0, \tag{4.6}$$

$$r^T[\nabla_u f(u, v) + w + \nabla_r g(u, v, r)] \leqq 0, \tag{4.7}$$

$$v \geqq 0, \quad w \in C, \tag{4.8}$$

where
(i) $f : R^n \times R^m \mapsto R$, $g : R^n \times R^m \times R^n \mapsto R$ and $h : R^n \times R^m \times R^m \mapsto R$ are twice differentiable functions, and
(ii) $C \subset R^n$ and $D \subset R^m$ are compact convex sets.

Theorem 4.1. *(Weak Duality). Let (x, y, z, p) and (u, v, w, r) be feasible solutions for primal and dual problem, respectively. Suppose that*

(i) $f(., v) + (.)^T w$ is higher-order pseudo-invex at u with respect to η_1 and $g(u, v, r)$,
(ii) $-[f(x, .) - (.)^T z]$ is higher-order pseudo-invex at y with respect to η_2 and $-h(x, y, p)$,
(iii) $\eta_1(x, u) + u + r \geqq 0$,
(iv) $\eta_2(v, y) + y + p \geqq 0$.

Then

$$L(x, y, z, p) \geqq M(u, v, w, r). \tag{4.9}$$

Proof. From the hypothesis (iii) and dual constraints (4.5), we get

$$(\eta_1(x, u)^T + u + r)[\nabla_x f(u, v) + w + \nabla_r g(u, v, r)] \geqq 0,$$

or

$$\eta_1(x, u)^T[\nabla_x f(u, v) + w + \nabla_r g(u, v, r)] \geqq -(u + r)^T[\nabla_x f(u, v) + w + \nabla_r g(u, v, r)],$$

which on using dual constraints (4.6) and (4.7) implies that

$$\eta_1(x, u)^T[\nabla_x f(u, v) + w + \nabla_r g(u, v, r)] \geqq 0. \tag{4.10}$$

Now by the higher order pseudo-invexity of $f(.,v) + (.)^T w$ at v with respect to η_1 and $g(u, v, r)$, we get

$$f(x, v) + x^T w - f(u, v) - u^T w - g(u, v, r) + r^T \nabla_r g(u, v, r) \geqq 0. \tag{4.11}$$

Similarly, hypothesis (iv) along with primal constraints (4.1)-(4.3), yields

$$\eta_2(v, y)^T [\nabla_y f(x, y) - z + \nabla_p h(x, y, p)] \leqq 0. \tag{4.12}$$

Therefore, by higher-order pseudo-invexity of $-[f(x,.)-(.)^T z]$ at y with respect to η_2 and $-h(x, y, p)$, we obtain

$$f(x, y) - y^T z - f(x, v) + v^T z + h(x, y, p) - p^T \nabla_p h(x, y, p) \geqq 0. \tag{4.13}$$

Adding inequalities (4.11) and (4.13), we get

$$f(x, y) + x^T w - y^T z + h(x, y, p) - p^T \nabla_p h(x, y, p)$$
$$\geqq f(u, v) - v^T z + u^T w + g(u, v, r) - r^T \nabla_r g(u, v, r).$$

Since $x^T w \leqq s(x|C)$ and $v^T z \leqq s(v|D)$, the above inequality implies

$$f(x, y) + s(x|C) - y^T z + h(x, y, p) - p^T \nabla_p h(x, y, p)$$
$$\geqq f(u, v) - s(v|D) + u^T w + g(u, v, r) - r^T \nabla_r g(u, v, r),$$

or

$$L(x, y, z, p) \geqq M(u, v, w, r).$$

Thus the result holds. □

Theorem 4.2. *(Strong Duality). Let $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ be a local optimal solution of (MHP). Assume that*

(i) $\nabla_{pp} h(\bar{x}, \bar{y}, \bar{p})$ is positive or negative definite,
(ii)$\nabla_y f(\bar{x}, \bar{y}) - \bar{z} + \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \neq 0$,
(iii) $\bar{p}^T [\nabla_y f(\bar{x}, \bar{y}) - \bar{z} + \nabla_p h(\bar{x}, \bar{y}, \bar{p})] = 0 \Rightarrow \bar{p} = 0$,
(iv) $h(\bar{x}, \bar{y}, 0) = g(\bar{x}, \bar{y}, 0)$, $\nabla_x h(\bar{x}, \bar{y}, 0) = \nabla_r g(\bar{x}, \bar{y}, 0)$, $\nabla_y h(\bar{x}, \bar{y}, 0) = \nabla_p h(\bar{x}, \bar{y}, 0)$.

Then

(I) $(\bar{x}, \bar{y}, \bar{z}, \bar{r} = 0)$ is feasible for (MHD) and
(II) $L(\bar{x}, \bar{y}, \bar{z}, \bar{p}) = M(\bar{x}, \bar{y}, \bar{w}, \bar{r})$.

Also, if the hypotheses of Theorem 4.1 are satisfied for all feasible solutions of (MHP) and (MHD), then $(\bar{x}, \bar{y}, \bar{z}, \bar{p} = 0)$ and $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ are global optimal solutions of (MHP) and (MHD), respectively.

Proof. Since $(\bar{x}, \bar{y}, \bar{p})$ is a local optimal solution of (MHP), there exist $\alpha,\ \gamma,\ \delta \in R$, $\beta \in R^m$ and $\mu, \xi \in R^n$ such that the following Fritz-John conditions [18, 24] are satisfied at $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$:

$$\alpha[\nabla_x f(\bar{x}, \bar{y}) + \xi + \nabla_x h(\bar{x}, \bar{y}, \bar{p}) - \nabla_{px} h(\bar{x}, \bar{y}, \bar{p})\bar{p}] + [\nabla_{yx} f(\bar{x}, \bar{y}) + \nabla_{px} h(\bar{x}, \bar{y}, \bar{p})](\beta - \gamma\bar{y} - \delta\bar{p}) - \mu = 0, \tag{4.14}$$

$$\alpha[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_y h(\bar{x},\bar{y},\bar{p})-\nabla_{py}h(\bar{x},\bar{y},\bar{p})\bar{p}]+[\nabla_{yy}f(\bar{x},\bar{y})+\nabla_{py}h(\bar{x},\bar{y},\bar{p})](\beta-\gamma\bar{y}-\delta\bar{p})-\gamma[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0, \tag{4.15}$$

$$\nabla_{pp}h(\bar{x},\bar{y},\bar{p})(\beta-\alpha\bar{p}-\gamma\bar{y}-\delta\bar{p})-\delta[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0, \tag{4.16}$$

$$\beta^T[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0, \tag{4.17}$$

$$\gamma\bar{y}^T[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0, \tag{4.18}$$

$$\delta\bar{p}^T[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0, \tag{4.19}$$

$$\alpha\bar{y}-\beta+\gamma\bar{y}+\mu\bar{p}\in N_D(\bar{z}), \tag{4.20}$$

$$\xi^T\bar{x}=s(\bar{x}|C),\quad \xi\in C, \tag{4.21}$$

$$\bar{x}^T\mu=0, \tag{4.22}$$

$$(\alpha,\beta,\gamma,\delta,\mu)\neq 0,\quad (\alpha,\beta,\gamma,\delta,\mu)\geqq 0. \tag{4.23}$$

Premultiplying equation (4.16) by $(\beta-\alpha\bar{p}-\gamma\bar{y}-\delta\bar{p})$ and then using equations (4.17)-(4.19), we get

$$(\beta-\alpha\bar{p}-\gamma\bar{y}-\delta\bar{p})^T\nabla_{pp}h(\bar{x},\bar{y},\bar{p})(\beta-\alpha\bar{p}-\gamma\bar{y}-\delta\bar{p})=0,$$

which along with hypothesis (i) yields

$$\beta=\alpha\bar{p}+\gamma\bar{y}+\delta\bar{p}. \tag{4.24}$$

This together with hypothesis (ii) and equation (4.16), yields

$$\delta=0. \tag{4.25}$$

Now, we claim that $\alpha\neq 0$. Indeed if $\alpha=0$, then equation (4.24) and (4.25), gives

$$\beta=\gamma\bar{y}.$$

Which together with equations (4.14) and (4.15), yields

$$\mu=0,$$

and

$$\gamma[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})=0, \text{respectively.}$$

Now by/using hypothesis (ii), $\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})\neq 0$, implies $\gamma=0$ and therefore $\beta=0$, thus $(\alpha,\beta,\gamma,\delta,\mu)=0$, a contradiction to equation (4.23). Hence

$$\alpha>0. \tag{4.26}$$

Using equations (4.17)-(4.19), we have

$$(\beta-\gamma\bar{y}-\delta\bar{p})^T[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0.$$

This with equation (4.24) gives

$$\alpha\bar{p}^T[\nabla_y f(\bar{x},\bar{y})-\bar{z}+\nabla_p h(\bar{x},\bar{y},\bar{p})]=0, \tag{4.27}$$

which by hypothesis (iii) and equation (4.26), yields

$$\bar{p} = 0. \tag{4.28}$$

Therefore equation (4.24) reduces to

$$\beta = \gamma \bar{y}. \tag{4.29}$$

Also, it follows from equations (4.15), (4.24), (4.28) and hypothesis (iv) that

$$(\alpha - \gamma)[\nabla_y f(\bar{x}, \bar{y}) - \bar{z} + \nabla_p h(\bar{x}, \bar{y}, \bar{p})] = 0,$$

which together with/along with hypothesis (iii) gives

$$\gamma = \alpha > 0. \tag{4.30}$$

So equation (4.29) implies

$$\bar{y} = \frac{\beta}{\gamma} \geqq 0. \tag{4.31}$$

Moreover, equation (4.14) along with (4.24), (4.28) and hypothesis (iv) yields

$$\alpha[\nabla_y f(\bar{x}, \bar{y}) - \xi + \nabla_r g(\bar{x}, \bar{y}, \bar{p})] = \mu,$$

as $\alpha > 0$,

$$\nabla_y f(\bar{x}, \bar{y}) - \xi + \nabla_r g(\bar{x}, \bar{y}, \bar{p}) = \frac{\mu}{\alpha} \geqq 0, \tag{4.32}$$

Using equation (4.26)

$$\bar{x}^T[\nabla_y f(\bar{x}, \bar{y}) - \xi + \nabla_r g(\bar{x}, \bar{y}, \bar{p})] = \frac{\bar{x}^T \mu}{\alpha} = 0. \tag{4.33}$$

Now taking $\xi = \bar{w} \in C$, we find $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ satisfies the constraints (4.5)-(4.8), that is, it is a feasible solution for the dual problem (MHD).

Also, since $\beta = \gamma \bar{y}$ and $\alpha > 0$, equation (4.20) gives $\bar{y} \in N_D(\bar{z})$ and therefore

$$\bar{y}^T \bar{z} = s(\bar{y}|D). \tag{4.34}$$

Hence using equations (4.21), (4.28), (4.34) and hypothesis (iv), we get

$$f(\bar{x}, \bar{y}) + s(\bar{x}|C) - \bar{y}^T \bar{z} + h(\bar{x}, \bar{y}, \bar{p}) - \bar{p}^T \nabla_{\bar{p}} h(\bar{x}, \bar{y}, \bar{p})$$
$$= f(\bar{x}, \bar{y}) + s(\bar{y}|D) - \bar{x}^T \bar{w} + g(\bar{x}, \bar{y}, 0) - \bar{r}^T \nabla_{\bar{r}} g(\bar{x}, \bar{y}, 0),$$

i.e.,

$$L(\bar{x}, \bar{y}, \bar{z}, \bar{p}) = M(\bar{x}, \bar{y}, \bar{w}, \bar{r}).$$

Also, by Theorem 4.1, $(\bar{x}, \bar{y}, \bar{w}, \bar{p} = 0)$ and $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ are global optimal solutions of the respective problems.

Theorem 4.3. *(Converse Duality). Let* $(\bar{u}, \bar{v}, \bar{w}, \bar{r})$ *be a local optimal solution of (MHP). Assume that*

(i) $\nabla_{rr} g(\bar{u}, \bar{v}, \bar{r})$ *is positive or negative definite,*
(ii)$\nabla_u f(\bar{u}, \bar{v}) - \bar{w} + \nabla_r g(\bar{u}, \bar{v}, \bar{r}) \neq 0,$
(iii) $\bar{r}^T[\nabla_u f(\bar{u}, \bar{v}) - \bar{w} + \nabla_r g(\bar{u}, \bar{v}, \bar{r})] = 0 \Rightarrow \bar{r} = 0,$
(iv) $g(\bar{u}, \bar{v}, 0) = g(\bar{u}, \bar{v}, 0)$, $\nabla_u g(\bar{u}, \bar{v}, 0) = \nabla_r g(\bar{u}, \bar{v}, 0)$, $\nabla_v g(\bar{x}, \bar{y}, 0) = \nabla_p h(\bar{x}, \bar{y}, 0)$.

Then

(I) $(\bar{u}, \bar{v}, \bar{z}, \bar{p} = 0)$ *is feasible for (MHP) and*
(II) $L(\bar{u}, \bar{v}, \bar{z}, \bar{p}) = M(\bar{u}, \bar{v}, \bar{w}, \bar{r})$.

Also, if the hypotheses of Theorem 4.1 are satisfied for all feasible solutions of (MHP) and (MHD), then $(\bar{u}, \bar{v}, \bar{w}, \bar{r} = 0)$ *and* $(\bar{u}, \bar{v}, \bar{z}, \bar{p} = 0)$ *are global optimal solutions of (MHD) and (MHP), respectively.*

Proof. Follows on the line of Theorem 4.2.

4.1. Self Duality

A mathematical programming problem is said to be self-dual if primal problem having equivalent dual formulation, that is, if the dual can be recast in the form of the primal. In general, (WHP) and (WHD) are not self-duals without some added restrictions on f, g and h.

(i) If we assume $f : R^n \times R^m \to R$ and $g : R^n \times R^m \times R^n \to R$ to be skew symmetric, i.e.,

$f(u, v) = -f(v, u), \;\; g(u, v, r) = -g(v, u, r)$

(ii) $C = D$

then we shall show that (MHP) and (MHD) are self-duals. By recasting the dual problem (MHD) as a minimization problem, we have

Minimize
$M(u, v, r) = -[f(u, v) - s(v|D) + u^T w + g(u, v, r) - r^T \nabla_r g(u, v, r)]$

subject to

$$\begin{aligned}
&\nabla_u f(u, v) + w + \nabla_r g(u, v, r) \geqq 0,\\
&u^T[\nabla_u f(u, v) + w + \nabla_r g(u, v, r)] \leqq 0,\\
&r^T[\nabla_u f(u, v) + w + \nabla_r g(u, v, r)] \leqq 0,\\
&u, \; v \geqq 0, \;\; w \in C.
\end{aligned}$$

Now as f and g are skew symmetric, i.e.,

$$\begin{aligned}
&\nabla_u f(u, v) = -\nabla_u f(v, u)\\
&\nabla_r g(u, v, r) = -\nabla_r g(v, u, r),
\end{aligned}$$

therefore the above problem recasting as :

Minimize
$M(u,v,r) = f(v,u) + s(v|C) - u^T w + g(v,u,r) - r^T \nabla_r g(v,u,r)$

subject to

$$\nabla_u f(u,v) + w + \nabla_r g(u,v,r) \leqq 0,$$
$$u^T[\nabla_u f(v,u) + w + \nabla_r g(v,u,r)] \geqq 0$$
$$r^T[\nabla_u f(v,u) + w + \nabla_r g(v,u,r)] \geqq 0,$$
$$u,\ v \geqq 0,\ \ w \in D,$$

Which shows that it is identical to (MHP), i.e., the objective and the constraint functions are identical. Thus, the problem (MHP) is self-dual.

It is obvious that (x,y,w,p) is feasible for (MHP), then (y,x,w,p) is feasible for (MHD) and vice versa.

5. Special Cases

(1) After removing inequalities (4.3), (4.7), (MHP) and (MHD) become the nondifferentiable programming problems (SP) and (SD) considered by Chen [6]. Also if $h(x,y,p) = (1/2)p^T \nabla_{yy} f(x,y)p$ and $g(u,v,r) = (1/2)r^T \nabla_{xx} f(u,v)r$, then (MHP) and (MHD) become the problems considered by Hou and Yang [14].

(2) If $p = 0$ and $r = 0$, then (MHP) and (MHD) become a pair of nondifferentiable symmetric dual programs considered in Mond and Schechter [21]. In addition if the C and D are null matrices, then (MHP) and (MHD) become a pair of single objective differentiable symmetric dual programs considered in [5] with the omission of nonnegativity constraints from (MP) and (MD).

(3) We can also construct a pair of Wolfe and Mond-Weir type higher-order symmetric dual programs by taking $C = Ay : y^T Ay \leqq 1$ and $D = Bx : x^T Bx \leqq 1$ in our models, where A and B are positive semidefinite matrices. For C and D so defined, $(x^T Ax)^1/2 = S(x|C)$ and $(y^T By)^1/2 = S(y|D)$. Thus, duality results for such a dual pair are obtained.
Also our model can be reduced to a number of other existing models in literature.

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