Filomat 30:3 (2016), 733–740
DOI 10.2298/FIL1603733K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

How Effects Efficiency on the Word Problem for Monoids?

Esma Kangal[a]

[a] *Department of Mathematics, Faculty of Science, Selcuk University, Campus, 42075, Konya, Turkey*

Abstract. In this paper, we partially find an answer to the question: Is there a relationship between the algebraic properties efficiency (or inefficiency) and solvability of the word problem?. In fact, by considering the semi-direct product on special monoids, we show that efficiency and inefficiency are not completely independent properties to prove the solvability of the word problem over monoids.

1. Introduction and Preliminaries

In combinatorial group theory, the fundamental decision problems are introduced by Max Dehn in 1911 (see [7]), and so many studies published on them since then (for example, one may see the papers [2, 10, 12, 13] and the citations in them). One of these decision problems is the word problem which is about the existence of an algorithm to decide whether or not any two words on generators of monoids (or groups) represent the same element in these monoids (or groups). That is, if there exists such an algorithm, the word problem is solvable, if otherwise it is not. In fact deciding which monoids (or groups) have solvable problem is quite important specially in computational algebra and so some engineering sciences. In here, we will use the complete rewriting system (cf. [3]) on presentations as a method to find an answer for the word problem of some monoids. It is well known that this method provides us to find normal forms of elements which implies the solvability of word problem. Let us recall this method as in the following two paragraphs:

Let A be a finite alphabet and suppose that A^* consists of all words obtained by the elements of A. A (string) rewriting system on A^* is a subset $R \subseteq A^* \times A^*$. Each element (u, v) of R, also written as $u \to v$, is called a (rewrite) rule of R. The idea for a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. In general, for a given rewriting system R, we write $x \to y$ for $x, y \in A^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. Also we write $x \to^* y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of reductions. Furthermore an element $x \in A^*$ is called irreducible with respect to R if there no possible rewriting $x \to y$; otherwise x is called reducible. The rewriting system R is called:

- Noetherian if there is no infinite chain of rewriting $x \to x_1 \to x_2 \to \cdots$ for any words $x \in A^*$.
- Confluent if whenever $x \to^* y_1$ and $x \to^* y_2$, there is a $z \in A^*$ such that $y_1 \to^* z$ and $y_2 \to^* z$.
- Complete if R is both noetherian and confluent.

Additionally, when R is a (string) rewriting system on A^*, the reduction relation $\to$ is noetherian if and only if there exists an admissible well-founded partial ordering $>$ on A^* (see [3]) such that $u > v$ holds for

2010 *Mathematics Subject Classification*. Primary 20F10; Secondary 16S15, 20F05, 20E22, 20M05
Keywords. Semi-direct product, (complete) rewriting system, efficiency, inefficiency
Received: 10 August 2015; Revised: 03 December 2015; Accepted: 18 December 2015
Communicated by Ljubiša D.R. Kočinac
Email address: esmakangal@selcuk.edu.tr (Esma Kangal)

each rule $(u, v) \in R$. Therefore, to construct a complete rewriting system, we must use an admissible well-founded partial ordering. To explain well founded partial ordering, there are some information needed. Let $>$ be a binary relation on A^*. Firstly, if it is irreflexive, antisymmetric and transitive, then this relation $>$ is called strict partial ordering. Secondly, for all $u, v, x, y \in A^*$ if $u > v$ implies $xuy > xvy$, then it is admissible. Besides them, a strict partial ordering $>$ on A^* is called well founded if there is no infinite chain of the form $x_0 > x_1 > x_2 > \cdots$. Now let us remind some special orderings.

• The length ordering $x > y$ on A^*: if $|x| > \left|y\right|$, then $x > y$.

• The lexicographical ordering $x >_{lex} y$ on A^*: if there is a non-empty string z such that $x = yz$, or $x = ua_iv$ and $y = ua_jz$ for some $u, v, z \in A^*, a_i, a_j \in A$ and $i, j \in \{1, \cdots, n\}$ satisfying $i > j$, then $x >_{lex} y$.

• The length-lexicographical ordering $x >_{ll} y$ on A^*: it is a combination of the length ordering and the lexicographical ordering, that is, if $|x| > \left|y\right|$ or $|x| = \left|y\right|$ and $x >_{lex} y$, then $x >_{ll} y$.

With respect to the above information, the length ordering is an admissible well founded (strict) partial ordering. But the lexicographical ordering is not well founded when $|A| > 1$ since we have the infinite descending chain $a_2 >_{lex} a_1a_2 >_{lex} a_1a_1a_2 >_{lex} \cdots >_{lex} a_1^ia_2 >_{lex} a_1^{i+1}a_2 >_{lex} \cdots$ for $A = \{a_1, a_2, \cdots, a_n\}$. However one can still use the length-lexicographical ordering. The fundamental reason is that since the length-lexicographical ordering actually contains the properties of the length ordering, then it is admissible well founded partial ordering. Besides we note that, in some cases, Gröbner (-Shirshov) bases can be used instead of complete rewriting systems to prove the solvability of the word problem (cf. [2, 12, 13, 16]). Up to now, we state (complete) rewriting system in general, but we should also note that in this method there is an important application called Knuth Bendix algorithm (see [3, 8, 11, 19]) which is needed in some of calculations for complete rewriting systems. This algorithm has developed an algorithm for creating a complete rewriting system R' which is equivalent to R, so that any word over A has an (unique) irreducible form with respect to R' and chosen ordering. By considering overlaps of left-hand sides of rules whose left-hand side is greater than right-hand side with respect to chosen ordering, this algorithm basically proceeds forming new rules when two reductions of an overlapping word result in two distinct reduced forms. If all rules with new rules occured at the end of the this process are noetherian and confluent, then this new rewriting system is complete. Since the property of having unique normal forms is guaranteed for any system that is confluent and noetherian (cf. [3]), we can find a unique normal form for this new rewriting system. As a result of this above material, the word problem is decidable for this new (finite reduction) system with the property that every element has a unique normal form as long as there exist algorithm that allow one to compute for a given element that unique normal form and allow one to compare two words to determine whether they are identical. The following lemma plays an important role in the proofs of our main results.

Lemma 1.1. ([3]) *The method of complete rewriting system gives a new algorithm for obtaining normal forms of elements of monoids (or groups). Therefore, there exists a new algorithm for solving the word problem in these monoids (or groups), namely the word problem is solvable.*

Since another tool of this paper is the semi-direct product of monoids, let us give it briefly: For given monoids A and K with presentations $\mathcal{P}_A = [X\,;\,R]$ and $\mathcal{P}_K = [Y\,;\,S]$, respectively, the semi-direct product $K \rtimes_\theta A$ is defined with a presentation $\mathcal{P}_{K\rtimes_\theta A} = [X,\ Y\,;\ R,\ S,\ T]$ (see [9]). In this presentation, T denotes the set of relators of the form $T_{yx} : yx = x(y\theta_x)$, where $x \in X$ and $y \in Y$. It is known that the semi-direct product on algebraic structures defines an extension. There are published real good works in the literature since last decades (see, for instance, [14, 20, 21]). Therefore, generally, extensions are worth to study some properties on them.

On the other hand, the properties efficiency or inefficiency are important as the meaning of classification on given monoids. There are some key studies on them. We may refer [4] and citations in it for the definitions and fundamental properties of them. At this point, we should remark that these properties need to be defined on the minimal number of generators to ensure a health classification. This will imply that no one can find a suitable presentation for the related monoid having one of these properties. Let us remind these algebraic properties. For a monoid presentation $\mathcal{P}$, the efficiency is defined as in the following ([4]).

Let M be a monoid with a presentation $\mathcal{P} = [Y, S]$. Then the Euler characteristic (see, [4, 5]) of $\mathcal{P}$ is defined by $\chi(\mathcal{P}) = 1 - |Y| + |S|$, where $|.|$ denotes the minimal number of elements in the set. Also, we have a bound-parameter other than Euler characteristic which is related to the efficiency, namely homological bound $\delta(M)$. In fact, let $\delta(M) = 1 - rk_{\mathbb{Z}}(H_1(M)) + d(H_2(M))$, where $rk_{\mathbb{Z}}(.)$ denotes the $\mathbb{Z}$-rank of the torsion-free part and $d(.)$ means the minimal number of generators. Then we define $\chi(M) = \min\{\chi(\mathcal{P}) : \mathcal{P} \text{ is a finite presentation for } M\}$. A presentation $\mathcal{P}_0$ for M is called minimal if $\chi(\mathcal{P}_0) \leq \chi(\mathcal{P})$, for all presentations $\mathcal{P}$ of M. A finite presentation $\mathcal{P}$ is called efficient if $\chi(\mathcal{P}) = \delta(M)$. M is called efficient if $\chi(M) = \delta(M)$. In addition to all of them, to guarantee the minimal number of generators on presentations, the geometric approximation, namely p-Cockcroft property ([5, 6]) was often chosen. The next lemma expose the relationship between efficiency and p-Cockcroft property, and it is also so important for our results.

Lemma 1.2. ([5, 17]) *Let $\mathcal{P}$ be a presentation. Then $\mathcal{P}$ is efficient if and only if p-Cockcroft for some prime p.*

In mathematics, to connect different type of algebraic properties on the same structure is important. In fact, based on this approximation, so many problems stated in the literature (see [1, 15, 18]). For example in [1], the authors exposed the relationship between the two different algebraic properties separability and efficiency by considering standard wreath products. To do that they used a geometric technique, namely Cayley graphs. Hence, in the light of this thought, in here, we investigate the word problem of special monoids which has efficient (or inefficient) presentation on the minimal number of generators since not all monoids are efficient with the minimal number of generators. To do that, we consider the semi-direct product of special one relator monoids by infinite monogenic monoids.

Throughout this paper the notation $i \cap j$ will denote the intersection of relations i and j as well as K_l $(1 \leq l \leq 2)$ and A will denote the one-relator and infinite monogenic (cyclic) monoids, respectively. Additionally, the lexicographical ordering $>_{lex}$ is denoted by $>$ and all used orderings are the length-lexicographical ordering. We finally note that the set $\mathbb{N}$ will be assumed as start with the natural number 1 in whole text.

2. The Word Problem for $K_l \rtimes_\theta A$

This section will be divided into two subsections to catch up our aim. Each of these subsections will be about the semi-direct product $K_l \rtimes_\theta A$, where the monoid A has a presentation $\mathcal{P}_A = [x\,;\ ]$.

2.1. Case 1

Assume that the one relator monoid K_1 is given by the presentation $\mathcal{P}_{K_1} = [y_1, y_2\,;\ y_2y_1^k = y_1y_2y_1]$ $(k \in \mathbb{N})$, and let ψ_x be the endomorphism defined by $[y_1] \to [y_1^n]$ and $[y_2] \to [y_2]$, where $n \in \mathbb{N}$. So, by [5], we have a presentation

$$\mathcal{P}_D = [y_1, y_2, x\,;\ y_2y_1^k = y_1y_2y_1, xy_1^n = y_1x, xy_2 = y_2x] \tag{1}$$

for the monoid $D = K_1 \rtimes_\theta A$.

By considering Lemma 1.2, in [5], Cevik proved that $\mathcal{P}_D$ in (1) is efficient on the minimal number of generators if and only if $k \equiv 2 \pmod p$ and $n \equiv 1 \pmod p$. However, the same author in different paper ([6]), exposed that $\mathcal{P}_D$ in (1) is minimal but inefficient if $k \neq 2(2^{i-1} - 1)$ and $n = 2$, where $i \in \mathbb{N}$. In our results, we will consider the special case of $\mathcal{P}_D$ in (1) by taking $k = 1$. Let us denote it by $\mathcal{P}_{D^*}$.

Among generators of $\mathcal{P}_{D^*}$, let us consider the ordering $x > y_1 > y_2$ as the lexicographical ordering and use the length-lexicographical ordering. Therefore the first main result of this paper is the following:

Theorem 2.1. *The monoid with a presentation $\mathcal{P}_{D^*}$ has a complete rewriting system with the rules*

$$\begin{array}{ll} 1)\, y_1y_2y_1 = y_2y_1\,, & 2)\, xy_1^n = y_1x\,, \\ 3)\, xy_2 = y_2x\,, & 4)\, y_1y_2^ax^by_1 = y_2^ax^by_1, \end{array}$$

where n is a fixed positive integer, $a \in \mathbb{N}$, $b \in \mathbb{N} \cup \{0\}$.

Before giving the proof, we should note that although the rule 4) in Theorem 2.1 coincides with the rule 1) if we take $a = 1$ and $b = 0$, we prefer to use rule 1) separately since it is being in the presentation $\mathcal{P}_{D^*}$ by its this form which was obtained from $\mathcal{P}_D$ in (1).

Proof. Since we use the length-lexicographical ordering, that is an admissible well founded ordering, and we have finite reduction steps for all overlapping words w as follows, the rewriting system for $\mathcal{P}_{D^*}$ is noetherian. Now, let us show the confluent property. To do that we have the following overlapping words and corresponding critical pairs (shortly *cp*), respectively.

$$\begin{array}{l}
1\cap 1 : w = y_1y_2y_1y_2y_1, \text{ and the cp is } (y_2y_1y_2y_1, y_1y_2^2y_1),\\
1\cap 4 : w = y_1y_2y_1y_2^ax^by_1, \text{ and the cp is } (y_2y_1y_2^ax^by_1,\ y_1y_2^{a+1}x^by_1),\\
2\cap 1 : w = xy_1^ny_2y_1, \text{ and the cp is } (y_1xy_2y_1, xy_1^{n-1}y_2y_1),\\
2\cap 4 : w = xy_1^ny_2^ax^by_1, \text{ and the cp is } (y_1xy_2^ax^by_1,\ xy_1^{n-1}y_2^ax^by_1),\\
4\cap 1 : w = y_1y_2^ax^by_1y_2y_1, \text{ and the cp is } (y_2^ax^by_1y_2y_1,\ y_1y_2^ax^by_2y_1),\\
4\cap 2 : w = y_1y_2^ax^by_1^n, \text{ and the cp is } (y_2^ax^by_1^n,\ y_1y_2^ax^{b-1}y_1x),\\
4\cap 4 : w = y_1y_2^{a_1}x^{b_1}y_1y_2^{a2}x^{b_2}y_1, \text{and the cp is } (y_2^{a_1}x^{b_1}y_1y_2^{a2}x^{b_2}y_1, y_1y_2^{a_1}x^{b_1}y_2^{a2}x^{b_2}y_1).
\end{array}$$

In fact those critical pairs are resolved by reduction steps as in the following:

$$1\cap 1 : w = y_1y_2y_1y_2y_1 \to \left\{\begin{array}{l} y_2y_1y_2y_1 \to y_2^2y_1\\ y_1y_2^2y_1 \to y_2^2y_1\end{array}\right. ,$$

$$1\cap 4 :\ w = y_1y_2y_1y_2^ax^by_1 \to \left\{\begin{array}{l} y_2y_1y_2^ax^by_1 \to y_2^{a+1}x^by_1\\ y_1y_2^{a+1}x^by_1 \to y_2^{a+1}x^by_1\end{array}\right. ,$$

$$2\cap 1 :\ w = xy_1^ny_2y_1 \to \left\{\begin{array}{l} y_1xy_2y_1 \to y_1y_2xy_1 \to y_2xy_1\\ xy_1^{n-1}y_2y_1 \to xy_1^{n-2}y_2y_1 \to xy_1^{n-3}y_2y_1 \to \cdots \to xy_1y_2y_1 \to xy_2y_1 \to y_2xy_1\end{array}\right. ,$$

$$2\cap 4 : w = xy_1^ny_2^ax^by_1 \to \left\{\begin{array}{l} y_1xy_2^ax^by_1 \to y_1y_2xy_2^{a-1}x^by_1 \to y_1y_2^2xy_2^{a-2}x^by_1 \to \cdots \to y_1y_2^{a-1}xy_2x^by_1\\ xy_1^{n-1}y_2^ax^by_1 \to xy_1^{n-2}y_2^ax^by_1 \to \cdots \to xy_1y_2^ax^by_1 \to xy_2^ax^by_1\end{array}\right.$$
$$\begin{array}{l} \to y_1y_2^ax^{b+1}y_1 \to y_2^ax^{b+1}y_1\\ \to y_2xy_2^{a-1}x^by_1 \to y_2^2xy_2^{a-2}x^by_1 \to \cdots \to y_2^{a-1}xy_2x^by_1 \to y_2^ax^{b+1}y_1\end{array} ,$$

$$4\cap 1 : w = y_1y_2^ax^by_1y_2y_1 \to \left\{\begin{array}{l} y_2^ax^by_1y_2y_1 \to y_2^ax^by_2y_1 \to y_2^ax^{b-1}y_2xy_1 \to y_2^ax^{b-2}y_2x^2y_1 \to\\ y_1y_2^ax^by_2y_1 \to y_1y_2^ax^{b-1}y_2xy_1 \to y_1y_2^ax^{b-2}y_2x^2y_1 \to\end{array}\right.$$
$$\begin{array}{l} \cdots \to y_2^axy_2x^{b-1}y_1 \to y_2^{a+1}x^by_1\\ \cdots \to y_1y_2^{a+1}x^by_1 \to y_2^{a+1}x^by_1\end{array} ,$$

$$4\cap 2 :\ w = y_1y_2^ax^by_1^n \to \left\{\begin{array}{l} y_2^ax^by_1^n \to y_2^ax^{b-1}y_1x\\ y_1y_2^ax^{b-1}y_1x \to y_2^ax^{b-1}y_1x\end{array}\right. ,$$

and

$$4\cap 4 : w = y_1y_2^{a_1}x^{b_1}y_1y_2^{a_2}x^{b_2}y_1 \to \left\{\begin{array}{l} y_2^{a_1}x^{b_1}y_1y_2^{a_2}x^{b_2}y_1 \to y_2^{a_1}x^{b_1}y_2^{a_2}x^{b_2}y_1 \to y_2^{a_1+1}x^{b_1}y_2^{a_2-1}x^{b_2}y_1 \to \cdots\\ y_1y_2^{a_1}x^{b_1}y_2^{a_2}x^{b_2}y_1 \to y_1y_2^{a_1+1}x^{b_1}y_2^{a_2-1}x^{b_2}y_1 \to \cdots \to y_1y_2^{a_1+a_2-1}x^{b_1}y_2x^{b_2}y_1\end{array}\right.$$
$$\begin{array}{l} \to y_2^{a_1+a_2-1}x^{b_1}y_2x^{b_2}y_1 \to y_2^{a_1+a_2}x^{b_1+b_2}y_1\\ \to y_1y_2^{a_1+a_2}x^{b_1+b_2}y_1 \to y_2^{a_1+a_2}x^{b_1+b_2}y_1\end{array} .$$

Since the rewriting system is noetherian and confluent, it is complete. Hence the result. □

As a consequence of Theorem 2.1, we have the following result for normal forms of elements with respect to the presentation P_{D^*}.

Corollary 2.2. *Let $N(w)$ be a normal form of words $w \in \mathcal{P}_{D^*}$. Then*

$$N(w) = y_2^cx^{a_1}y_1^{b_1}x^{a_2}y_1^{b_2}x^{a_3}\cdots x^{a_m}y_1^{b_m}y_2^d,$$

where $a_i, b_i, c, d \in \mathbb{N}\cup\{0\}$, $0 \le b_i < n$, and $1 \le i \le m$.

Hence, by Corollary 2.2 and Lemma 1.1, we have the following:

Theorem 2.3. *The word problem is solvable for the presentation* $\mathcal{P}_{D^*}$.

Corollary 2.4. *Any monoid with a presentation* $\mathcal{P}_{D^*}$ *that is inefficient, as in* (1) *with* $k = 1$, *has solvable word problem since it has minimal number of generators.*

2.2. Case 2

In this section, we will follow similar process as in the previous section. Thus, let us consider the one-relator monoid K_2 with a presentation $\mathcal{P}_{K_2} = [y_1, y_2\ ;\ y_1^k y_2 = y_2 y_1^k]$, and let ψ_x be the endomorphism given by $[y_1] \to [y_1^n]$ and $[y_2] \to [y_2^m]$, where $k,\ n,\ m \in \mathbb{N}$. Hence, again by [5], we have a presentation

$$\mathcal{P}_E = [y_1, y_2, x\ ;\ y_1^k y_2 = y_2 y_1^k, xy_1^n = y_1 x,\ xy_2^m = y_2 x] \tag{2}$$

for the monoid $E = K_2 \rtimes_\theta A$. In [5], as an application of Lemma 1.2, it has been proved that $\mathcal{P}_E$ in (2) is efficient on the minimal number of generators if and only if $mn \equiv 1 \pmod p$ by the author. Moreover, in [6], it has been proved that $\mathcal{P}_E$ is minimal (but inefficient) if $(n, m) = (1, 2)$ or $(n, m) = (2, 1)$. In our results, as special cases, we will only consider the sub-cases $k = n$ with a presentation

$$\mathcal{P}_{E^*} = [y_1, y_2, x\ ;\ y_1^k y_2 = y_2 y_1^k, xy_1^k = y_1 x, xy_2^m = y_2 x]\,, \tag{3}$$

and $n = 1$ with a presentation

$$\mathcal{P}_{E^{**}} = [y_1, y_2, x\ ;\ y_1^k y_2 = y_2 y_1^k, xy_1 = y_1 x, xy_2^m = y_2 x]\,. \tag{4}$$

Similarly as in the previous case, let us consider the ordering $x > y_1 > y_2$ (length-lexicographical ordering) among generators of the presentations given in (3) and (4). Thus the other main results of the paper are stated and proved in Theorems 2.5 and 2.8 below.

Theorem 2.5. *The monoid with a presentation* $\mathcal{P}_{E^*}$ *as in* (3) *has a complete rewriting system with rules*

$$\begin{array}{lll} 1)\ y_1^k y_2 = y_2 y_1^k\,, & 2)\ xy_1^k = y_1 x\,, & 3)\ xy_2^m = y_2 x\,, \\ 4)\ xy_2^e y_1^k = y_1 x y_2^e\,, & 5)\ y_1 y_2^a y_1^b x = y_2^a y_1^{b+1} x\,, & \end{array}$$

where $k,\ m$ *are fixed positive integers and* $a, b, e \in \mathbb{N} \cup \{0\}$ $(0 \le e < m)$.

Proof. By the same reason as in the proof of Theorem 2.1, the rewriting system for $\mathcal{P}_{E^*}$ is noetherian. Further, we must show that it satisfies the confluent property by take into account the following overlapping words and corresponding critical pairs (again shortly as *cp*):

$$\begin{array}{l}
1 \cap 5:\ w = y_1^k\ y_2^a y_1^b x, \text{ and the cp is } (y_2 y_1^k y_2^{a-1} y_1^b x,\ y_1^{k-1}\ y_2^a y_1^{b+1} x)\,, \\
2 \cap 1:\ w = xy_1^k y_2, \text{ and the cp is } (y_1 x y_2,\ xy_2 y_1^k)\,, \\
2 \cap 5:\ w = xy_1^k y_2^a y_1^b x, \text{ and the cp is } (y_1 x y_2^a y_1^b x,\ xy_1^{k-1} y_2^a y_1^{b+1} x)\,, \\
4 \cap 1:\ w = xy_2^e y_1^k y_2, \text{ and the cp is } (y_1 x y_2^{e+1},\ xy_2^{e+1} y_1^k)\,, \\
4 \cap 5:\ w = xy_2^e y_1^k y_2^a y_1^b x, \text{ and the cp is } (y_1 x y_2^{e+a} y_1^b x,\ xy_2^e y_1^{k-1} y_2^a y_1^{b+1} x)\,, \\
5 \cap 2:\ w = y_1 y_2^a y_1^b x y_1^k, \text{ and the cp is } (y_2^a y_1^{b+1} x y_1^k,\ y_1 y_2^a y_1^{b+1} x)\,, \\
5 \cap 3:\ w = y_1 y_2^a y_1^b x y_2^m, \text{ and the cp is } (y_2^a y_1^{b+1} x y_2^m,\ y_1 y_2^a y_1^b y_2 x)\,, \\
5 \cap 4:\ w = y_1 y_2^a y_1^b x y_2^e y_1^k, \text{ and the cp is } (y_2^a y_1^{b+1} x y_2^e y_1^k, y_1 y_2^a y_1^{b+1} x y_2^e).
\end{array}$$

All critical pairs are resolved by reduction steps. We show them as follows:

$$1 \cap 5:\ w = y_1^k\ y_2^a y_1^b x \to \left\{ \begin{array}{l} y_2 y_1^k y_2^{a-1} y_1^b x \to y_2^2 y_1^k y_2^{a-2} y_1^b x \to \cdots \to y_2^{a-1} y_1^k y_2 y_1^b x \to y_2^a y_1^{k+b} x \\ y_1^{k-1}\ y_2^a y_1^{b+1} x \to y_1^{k-2}\ y_2^a y_1^{b+2} x \to \cdots \to y_1\ y_2^a y_1^{b+k-1} x \to y_2^a y_1^{b+k} x \end{array} \right. ,$$

$$2 \cap 1: \ w = xy_1^k y_2 \to \begin{cases} y_1 x y_2 \\ x y_2 y_1^k \to y_1 x y_2 \end{cases},$$

$$2 \cap 5: \ w = xy_1^k y_2^a y_1^b x \to \begin{cases} y_1 x y_2^a y_1^b x \\ x y_1^{k-1} y_2^a y_1^{b+1} x \to x y_1^{k-2} y_2^a y_1^{b+2} x \to \cdots \to x y_1 y_2^a y_1^{b+k-1} x \to x y_2^a y_1^{b+k} x \to y_1 x y_2^a y_1^b x \end{cases},$$

$$4 \cap 1: \ w = xy_2^e y_1^k y_2 \to \begin{cases} y_1 x y_2^{e+1} \\ x y_2^{e+1} y_1^k \to y_1 x y_2^{e+1} \end{cases},$$

$$4 \cap 5: \ w = xy_2^e y_1^k y_2^a y_1^b x \to \begin{cases} y_1 x y_2^{e+a} y_1^b x \\ x y_2^e y_1^{k-1} y_2^a y_1^{b+1} x \to \cdots \to x y_2^e y_1 y_2^a y_1^{b+k-1} x \to x y_2^{e+a} y_1^{k+b} x \to y_1 x y_2^{e+a} y_1^b x \end{cases},$$

$$5 \cap 2: \ w = y_1 y_2^a y_1^b x y_1^k \to \begin{cases} y_2^a y_1^{b+1} x y_1^k \to y_2^a y_1^{b+2} x \\ y_1 y_2^a y_1^{b+1} x \to y_2^a y_1^{b+2} x \end{cases},$$

$$5 \cap 3: \ w = y_1 y_2^a y_1^b x y_2^m \to \begin{cases} y_2^a y_1^{b+1} x y_2^m \to y_2^a y_1^{b+1} y_2 x \\ y_1 y_2^a y_1^b y_2 x \to y_2^a y_1^{b+1} y_2 x \end{cases},$$

and

$$5 \cap 4: w = y_1 y_2^a y_1^b x y_2^e y_1^k \to \begin{cases} y_2^a y_1^{b+1} x y_2^e y_1^k \to y_2^a y_1^{b+2} x y_2^e \\ y_1 y_2^a y_1^{b+1} x y_2^e \to y_2^a y_1^{b+2} x y_2^e \end{cases}.$$

The rewriting system is noetherian and confluent, so it is complete. Hence the result. □

By Theorem 2.5, we have the following result for normal forms of elements with respect to the presentation P_{E^*}.

Corollary 2.6. *Let $N(w)$ be a normal form of words $w \in \mathcal{P}_{E^*}$. Then*

$$N(w) = y_2^a y_1^b x^{c_1} y_2^{a_1} y_1^{b_1} x^{c_2} y_2^{a_2} y_1^{b_2} \cdots y_2^{a_q} y_1^{b_q} x^{c_q},$$

where $a, b, c_i \in \mathbb{N} \cup \{0\}$, $0 \leq a_i < m$ and $0 \leq b_i < k$ $(1 \leq i \leq q)$.

Hence, by Corollary 2.6 and Lemma 1.1, we have

Theorem 2.7. *The word problem for $\mathcal{P}_{E^*}$ given in* (3) *is solvable.*

Finally let us consider the presentation $\mathcal{P}_{E^{**}}$ in (4). Then we obtain the following theorem.

Theorem 2.8. *The monoid defined with a presentation $\mathcal{P}_{E^{**}}$ has a complete rewriting system with the rules*

$$\begin{array}{ll} 1)\, y_1^k y_2 = y_2 y_1^k, & 2)\, x y_1 = y_1 x, \\ 3)\, x y_2^m = y_2 x, & 4)\, x y_2^a y_1^k = y_1^k x y_2^a \end{array},$$

where k, m are fixed positive integers and $0 < a < m$.

Proof. As in the previous similar theorems, it is clear that the rewriting system of $\mathcal{P}_{E^{**}}$ is noetherian. For the confluent property, we have

$$\begin{array}{l} 2 \cap 1: \ w = x y_1^k y_2, \text{ and the cp is } (y_1^k x y_2,\ x y_2 y_1^k), \\ 4 \cap 1: \ w = x y_2^a y_1^k y_2, \text{ and the cp is } (y_1^k x y_2^{a+1},\ x y_2^{a+1} y_1^k). \end{array}$$

As before, all these critical pairs are resolved by reduction steps.

$$2 \cap 1: \ w = x y_1^k y_2 \to \begin{cases} y_1 x y_1^{k-1} y_2 \to y_1^2 x y_1^{k-2} y_2 \to \cdots \to y_1^k x y_2 \\ x y_2 y_1^k \to y_1^k x y_2 \end{cases},$$

and

$$4 \cap 1: \ w = x y_2^a y_1^k y_2 \to \begin{cases} y_1^k x y_2^{a+1} \\ x y_2^{a+1} y_1^k \to y_1^k x y_2^{a+1} \end{cases}.$$

Hence the result. □

By Theorem 2.8, we have the following result for normal forms of elements with respect to the presentation $P_{E^{**}}$. In fact we did prefer to write these elements very clearly to make a better understand on them.

Corollary 2.9. *Let $N(w)$ be a normal form of words $w \in \mathcal{P}_{E^{**}}$. Suppose that*

$$\begin{aligned}
A_1 &= y_2^{a_{11}} y_1^{b_{11}} y_2^{a_{21}} y_1^{b_{21}} \cdots y_2^{a_{\alpha 1}} y_1^{b_{\alpha 1}}\,, \\
&\quad \textit{where } a_{11}, a_{21}, \cdots, a_{\alpha 1}, b_{\alpha 1} \in \mathbb{N} \cup \{0\} \textit{ and } 0 \leq b_{11}, b_{21}, \cdots, b_{(\alpha-1)1} < k\,, \\
A_2 &= y_2^{a_{12}} y_1^{b_{12}} y_2^{a_{22}} y_1^{b_{22}} \cdots y_2^{a_{\beta 2}} y_1^{b_{\beta 2}}\,, \\
&\quad \textit{where } a_{22}, a_{32}, \cdots, a_{\beta 2}, b_{\beta 2} \in \mathbb{N} \cup \{0\} \textit{ and } 0 \leq b_{12}, b_{22}, \cdots, b_{(\beta-1)2} < k;\;\; a_{12} < m\,, \\
A_3 &= y_2^{a_{13}} y_1^{b_{13}} y_2^{a_{23}} y_1^{b_{23}} \cdots y_2^{a_{\gamma 3}} y_1^{b_{\gamma 3}}\,, \\
&\quad \textit{where } a_{23}, a_{33}, \cdots, a_{\gamma 3}, b_{\gamma 3} \in \mathbb{N} \cup \{0\} \textit{ and } 0 \leq b_{13}, b_{23}, \cdots, b_{(\gamma-1)3} < k;\;\; a_{13} < m\,, \\
\vdots & \qquad\qquad\quad \vdots \\
A_r &= y_2^{a_{1r}} y_1^{b_{1r}} y_2^{a_{2r}} y_1^{b_{2r}} \cdots y_2^{a_{\delta r}} y_1^{b_{\delta r}}\,, \\
&\quad \textit{where } a_{2r}, a_{3r}, \cdots, a_{\delta r}, b_{\delta r} \in \mathbb{N} \cup \{0\} \textit{ and } 0 \leq b_{1r}, b_{2r}, \cdots, b_{(\delta-1)r} < k;\; a_{1r} < m\,.
\end{aligned}$$

Therefore

$$N(w) = A_1 x^{c_1} A_2 x^{c_2} A_3 \cdots A_r x^{c_r}\,,$$

where $c_i \in \mathbb{N} \cup \{0\}$; $1 \leq i \leq r$; $\alpha, \beta, \gamma, ..., \delta \in \mathbb{N}$.

Therefore, by Corollary 2.9 and Lemma 1.1, the final result of this paper can be obtained as follows:

Theorem 2.10. *The word problem for $\mathcal{P}_{E^{**}}$ given in* (4) *is solvable.*

Corollary 2.11. *The monoids with presentations as in* (3) *and* (4) *have always solvable word problem either they are efficient or not since they have minimal number of generators.*

A final note can be stated as follows for exposing the importance of our results:

Remark 2.12. The processes used in this paper did mainly show that the minimal number of generators in given presentations play an important role to prove the solvability of the word problem. In other words, since we did use the lexicographical ordering in solvability cases, these minimum numbers of generators did play an important role to obtain complete rewriting systems. In fact to use minimal number of generators, we considered the presentations as efficient or inefficient since the efficiency case implies directly the minimal number of generators while for the inefficiency case we ensure that the presentation must be defined by the minimal number of generators. As a result of both these cases, no one can find any other presentation having less generators than those.

Acknowledgement

I would like to express my deepest thankfulness to the referees for their careful reading and kind helpful suggestions and comments. I also would like to thank A.S. Cevik and E.G. Karpuz who discussed this problem with me.

References

[1] F. Ates, A.S. Cevik, Separability and efficiency under standard wreath products in terms of Cayley graphs, Rocky Mountain J. Math. 38 (2008) 779–800.
[2] F. Ates, E.G. Karpuz, C. Kocapinar, A.S. Cevik, Gröbner-Shirshov bases of some monoids, Discrete Math. 311 (2011) 1064–1071.
[3] R.V. Book, F. Otto, String rewriting systems, Springer-Verlag, New York, 1993.
[4] A.S. Cevik, The efficiency of standard wreath product, Proc. Edinburgh Math. Soc. 43 (2000) 415–423.
[5] A.S. Cevik, The p-Cockcroft property of semi-direct products of monoids, Intern. J. Algebra Comput. 13 (2003) 1–16.
[6] A.S. Cevik, Minimal but inefficient presentations of the semi-direct product of some monoids, Semigroup Forum 66 (2003) 1–17.
[7] M. Dehn, Über unendliche diskontinuerliche gruppen, Math. Ann. 69 (1911) 116–144.
[8] G. Huet, Confluent reductions: Abstract properties and applications to term rewriting systems, J. Assoc. Comput. Mach. 27 (1980) 797–821.
[9] D.L. Johnson, Presentation of Groups, LMSST Series 15, Cambridge University Press, 1990.
[10] E.G. Karpuz, A.S. Cevik, The word and generalized word problem for semigroups under wreath products, Bull. Math. Soc. Sci. Math. Roumanie 52 (2009) 151–160.
[11] D.E. Knuth, P. Bendix, Simple word problems in universal algebras, In: J. Leech (ed.), Computational Problems in Abstract Algebra (1970) 263–297.
[12] C. Kocapinar, E.G. Karpuz, F. Ates, A.S. Cevik, Gröbner-Shirshov bases of the generalized Bruck-Reilly $*$-extension, Algebra Coll. 19 (2012) 813–820.
[13] E.G. Karpuz, A.S. Cevik, Gröbner-Shirshov bases for extended modular, extended Hecke and Picard groups, Math. Notes 92 (2012) 636–642.
[14] D. Kuziak, I. Peterin, I.G. Yero, Computing the (k-)monopoly number of direct product of graphs, Filomat 29 (2015) 1163–1171.
[15] V.D. Mazurov, E.I. Khukro, Kourovka Notebook, No. 18, Russian Academy of Sciences, 2014.
[16] B. Malešević, M. Obradović, An application of Gröbner bases to planarity of intersection of surfaces, Filomat 23 (2009) 43–55.
[17] S.J. Pride, Identities among relations of group presentations, Group Theory from a Geometrical Viewpoint (1990) 687-717.
[18] M. Sapir, Some group theory problems, Intern. J. Algebra Comput. 17 (2007) 1189–1214
[19] C.C. Sims, Computation for Finitely Presented Groups, Cambridge University Press, 1994.
[20] M. Stojanović, Fundamental simplices with outer vertices for hyperbolic groups and their group extensions for truncations, Filomat 24 (2010) 1–19.
[21] M. Stojanović, Coxeter groups as automorphism groups of solid transitive 3-simplex tilings, Filomat 28 (2014) 557–577.