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On Strongly Almost Lacunary Statistical A-Convergence Defined by a Musielak-Orlicz Function

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Abstract. We study some new strongly almost lacunary statistical A-convergent sequence space of order α defined by a Musielak-Orlicz function. We also give some inclusion relations between the newly introduced class of sequences with the spaces of strongly almost lacunary A-convergent sequence of order α. Moreover we have examined some results on the Musielak-Orlicz function with respect to these spaces.

1. Introduction

The concept of statistical convergence was initially introduced by Fast [2], which is closely related to the concept of natural density or asymptotic density of subsets of the set of natural numbers N. Later on, it was studied as a summability method by Fridy [4], Fridy and Orhan [6], Freedman and Sember [3], Schoenberg [18], Malafosse and Rakočević [10] and many other mathematicians. Moreover, in recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Also, statistical convergence is closely related to the concept of convergence in probabilty.

By the concept of almost convergence, we have a sequence $x = (x_k) \in \ell_\infty$ if all of its Banach limits coincide. The set $\hat{c}$ denotes set of all almost convergent sequences. Lorentz [8] proved that

$$\hat{c} = \{x \in \ell_\infty : \lim_m t_{mn}(x) \text{ exist uniformly in } n\},$$

where

$$t_{mn}(x) = \frac{x_n + x_{n+1} + ... + x_{n+m}}{m+1}.$$

Similarly, the space of strongly almost convergent sequence was defined as, $[\hat{c}] = \{x \in \ell_\infty : \lim_m t_{m,n}(|x - Le|)$ exists uniformly in n for some $L\}$, where, $e = (1, 1, ...)$ (see Maddox [9]).

A lacunary sequence is defined as an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

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Note: Throughout this paper, the intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by ϕ_r.

2. Preliminary Concepts

Let $0 < \alpha \leq 1$ be given. The sequence (x_k) is said to be statistically convergent of order α if there is a real number L such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. In this case, we write $S^\alpha - \lim x_k = L$. The set of all statistically convergent sequences of order α will be denoted by S^α.

For any lacunary sequence $\theta = (k_r)$, the space N_θ is defined as (Freedman et al. [3])

$$N_\theta = \left\{ (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in J_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The space N_θ is a BK space with the norm

$$||(x_k)||_\theta = \sup_r h_r^{-1} \sum_{k\in J_r} |x_k|.$$

Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given The sequence $x = (x_k) \in w$ is said to be S_θ^α-statistically convergent (or a lacunary statistically convergent sequence of order α) if there is a real number L such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0,$$

where $I_r = (k_{r-1}, k_r]$ and h_r^α denotes the α-th power $(h_r)^\alpha$ of h_r, that is, $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, ...h_r^\alpha, ...)$. We write $S_\theta^\alpha - \lim x_k = L$. The set of all S_θ^α-statistically convergent sequences will be denoted by S_θ^α.

By an Orlicz function we mean a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

The idea of Orlicz function is used to construct the sequence space (see Lindenstrauss and Tzafriri [7])

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

This space ℓ_M with the norm,

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space.

Musielak [12] defined the concept of Musielak-Orlicz function as $\mathcal{M} = (M_k)$. A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, k = 1, 2, ..$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. The Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$

$$h_{\mathscr{M}} = \{x \in w : I_{\mathscr{M}}(cx) < \infty, \forall c > 0\},$$

where $I_{\mathscr{M}}$ is a convex modular defined by

$$I_{\mathscr{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathscr{M}}.$$

It is considered $t_{\mathscr{M}}$ equipped with the Luxemberg norm

$$\|x\| = \inf\left\{k > 0 : I_{\mathscr{M}}\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}(1 + I_{\mathscr{M}}(kx)) : k > 0\right\}.$$

A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in N$ and $u \in R_+$, whenever $M_k(u) \leq a$.

If $A = (a_{nk})_{n,k=1}^{\infty}$ is an infinite matrix, then Ax is the sequence whose nth term is given by $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$.

We consider a sequence $x = (x_k)$ which is said to be strongly almost lacunary statistical A-convergent of order α (or $S_\theta^\alpha(A, \mathscr{M}, (s))$-statistically convergent) if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \sum_{k \in I_r} \left(M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right)^{(s_k)} \geq \varepsilon \right\} \right| = 0, \text{ uniformly in } m,$$

where $I_r = (k_{r-1}, k_r]$ and h_r^α denotes the α-th power (h_r^α) of h_r, that is, $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, ...h_r^\alpha, ...)$ and $\mathscr{M} = (M_k)$ is a Musielak-Orlicz function.

Some particular cases :

If we take $\alpha = 1$, then the strongly almost lacunary statistical A-convergence of order α reduces to the the strongly almost lacunary statistical A-convergence as follows:

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \sum_{k \in I_r} \left(M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right)^{(s_k)} \geq \varepsilon \right\} \right| = 0, \text{ uniformly in } m,$$

where $I_r = (k_{r-1}, k_r]$ and $\mathscr{M} = (M_k)$ is a Musielak-Orlicz function.

If $\theta = (2^r), \alpha = 1$ and $(s_k) = 1, \forall k \in \mathbb{N}$, then sequence (x_k) is said to be $S(A, \mathscr{M})$-statistically convergent to L if,

$$\delta\left(\left\{ k \in I_r : \sum_{k \in I_r} \left(M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right) \geq \varepsilon, \text{ uniformly in } m, \right\}\right) = 0,$$

where $I_r = (k_{r-1}, k_r]$ and $\mathscr{M} = (M_k)$ is a Musielak-Orlicz function.

Similarly, if $M_k(x) = x, \theta = (2^r), \alpha = 1$ and $(s_k) = 1, \forall k \in \mathbb{N}$, then the sequence (x_k) is said to be strongly almost statistical A-convergent to L such that

$$\delta\left(\left\{ k \in I_r : \sum_{k \in I_r} |t_{km}(A_k(x) - L)| \geq \varepsilon, \text{ uniformly in } m, \right\}\right) = 0,$$

where $I_r = (k_{r-1}, k_r]$.

Also, for A a unit matrix, $M_k(x) = x, \theta = (2^r), \alpha = 1$ and $(s_k) = 1, \forall k \in \mathbb{N}$, then the sequence (x_k) is simply strongly almost statistical convergent to L.

We have introduced the space of strongly almost lacunary A-convergent sequences with respect to a Musielak-Orlicz function $\mathscr{M} = (M_k)$ as follows:

$$\hat{N}_\theta^\alpha(A, \mathscr{M}, (s)) = \left\{ (x_k) : \lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} \left(M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right)^{(s_k)} = 0, \text{ for some } L \text{ and } \rho^{(k)} > 0 \right\}.$$

If we take $\theta = (2^r), \alpha = 1$ and $(s_k) = 1, \forall k \in \mathbb{N}$, then $\hat{N}_\theta^\alpha(A, \mathscr{M}, (s))$ will be reduced to $\hat{N}(A, \mathscr{M})$ and if $\theta = (2^r), \alpha = 1$ and $(s_k) = 1, \forall k \in \mathbb{N}$ and $M_k(x) = x$, then we can define $\hat{N}(A)$ instead of $\hat{N}_\theta^\alpha(A, \mathscr{M}, (s))$ as follows:

$$\hat{N}(A) = \left\{ (x_k) : \sum_{k\in I_r} |t_{km}(A_k(x) - L)| = 0, \text{ for some } L \right\}.$$

where $I_r = (k_{r-1}, k_r]$.

We give some inclusion relations between the sets of $S_\theta^\alpha(A, \mathscr{M}, (s))$-statistically convergent sequences and strongly almost lacunary A-convergent sequence space $\hat{N}_\theta^\alpha(A, \mathscr{M}, (s))$. Also, some results defined by a Musielak-Orlicz function are studied with respect to these sequence spaces.

3. Main Results

Theorem 3.1. *Let $\alpha, \beta \in (0, 1]$ be real numbers such that $\alpha \leq \beta$, $\mathscr{M}$ be a Musielak-Orlicz function and $\theta = (k_r)$ be a lacunary sequence. Then $\hat{N}_\theta^\alpha(A, \mathscr{M}, (s)) \subset \hat{S}_\theta^\beta$.*

Proof. Let $x \in \hat{N}_\theta^\alpha(A, \mathscr{M}, (s))$.

For $\varepsilon > 0$ given, let us denote $\sum_1$ as the sum over $k \in I_r$, $|t_{km}(A_k(x) - L)| \geq \varepsilon$. and $\sum_2$ denote the sum over $k \in I_r$, $|t_{km}(A_k(x) - L)| < \varepsilon$ respectively.

As $h_r^\alpha \leq h_r^\beta$ for each r, we may write

$\frac{1}{h_r^\alpha} \sum_{k\in I_r} \left[M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right]^{s_k}$

$= \frac{1}{h_r^\alpha} \left[\sum_1 \left[M_k \left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}} \right) \right]^{s_k} + \sum_2 \left[M_k \left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}} \right) \right]^{s_k} \right]$

$\geq \frac{1}{h_r^\beta} \left[\sum_1 \left[M_k \left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}} \right) \right]^{s_k} + \sum_2 \left[M_k \left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}} \right) \right]^{s_k} \right]$

$\geq \frac{1}{h_r^\beta} \sum_1 \left[M_k \left(\frac{\varepsilon}{\rho^{(k)}} \right) \right]^{s_k}$

$\geq \frac{1}{h_r^\beta} \sum_1 \min([M_k(\varepsilon_1)]^h, [M_k(\varepsilon_1)]^H), \varepsilon_1 = \frac{\varepsilon}{\rho^{(k)}}$

$\geq \frac{1}{h_r^\beta} |\{k \in I_r : |t_{km}(A(x)) - L| \geq \varepsilon\}| \min([M_k(\varepsilon_1)]^h, [M_k(\varepsilon_1)]^H$.

As $x \in \hat{N}_\theta^\alpha(A, \mathscr{M}, (s))$, the left hand side of the above inequality tends to zero as $r \to \infty$.

Therefore, the right hand side of the above inequality tends to zero as $r \to \infty$, hence $x \in \hat{S}_\theta^\beta$. □

Corollary 3.2. *Let $0 < \alpha \leq 1$, $\mathscr{M}$ be a Musielak-Orlicz function and $\theta = (k_r)$ be a lacunary sequence. Then*

$$\hat{N}_\theta^\alpha(A, \mathscr{M}, (s)) \subset \hat{S}_\theta^\alpha.$$

Theorem 3.3. *Let $\mathscr{M}$ be a Musielak-Orlicz function, $x = (x_k)$ be a bounded sequence and $\theta = (k_r)$ be a lacunary sequence. If $\lim_{r\to\infty} \frac{h_r}{h_r^\alpha} = 1$, then $x \in \hat{S}_\theta^\alpha \Rightarrow x \in \hat{N}_\theta^\alpha(A, \mathscr{M}, (s))$.*

Proof. Suppose that $x = (x_k)$ is a bounded sequence that is $x \in \ell_\infty$ and $\hat{S}_\theta^\alpha - \lim x_k = L$. As $x \in \ell_\infty$, then there is a constant $T > 0$ such that $|x_k| \leq T$. Given $\varepsilon > 0$, we have

$\frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right]^{s_k}$

$\frac{1}{h_r^\alpha} \sum_1 \left[M_k \left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}} \right) \right]^{s_k} + \frac{1}{h_r^\alpha} \sum_2 \left[M_k \left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}} \right) \right]^{s_k}$

$\leq \frac{1}{h_r^\alpha} \sum_1 \max \left\{ \left[M_k \left(\frac{T}{\rho^{(k)}} \right) \right]^h, \left[M_k \left(\frac{T}{\rho^{(k)}} \right) \right]^H \right\} + \frac{1}{h_r^\alpha} \sum_2 \left[M_k \left(\frac{\varepsilon}{\rho^{(k)}} \right) \right]^{s_k}$

$\leq \max\{[M_k(K)]^h, [M_k(K)]^H\} \frac{1}{h_r^\alpha} |\{k \in I_r : |t_{km}(A_k(x) - L)| \geq \varepsilon\}| + \frac{h_r}{h_r^\alpha} \max\{[M_k(\varepsilon_1)]^h, [M_k(\varepsilon_1)]^H\}, \ \frac{T}{\rho^{(k)}} = K, \frac{\varepsilon}{\rho^{(k)}} = \varepsilon_1.$

Hence, $x \in \hat{N}_\theta^\alpha(A, \mathscr{M}, (s))$. □

Theorem 3.4. *If* $\lim s_k > 0$ *and* $x = (x_k)$ *is strongly* $\hat{N}_\theta^\alpha(A, \mathscr{M}, (s))$*-summable to L with respect to the Musielak-Orlicz function* $\mathscr{M}$*, then* $\hat{N}_\theta^\alpha(A, \mathscr{M}, (s)) - lim x_k$ *is unique.*

Proof. Let $\lim s_k = s > 0$. Suppose that $\hat{N}_\theta^\alpha(A, \mathscr{M}, (s)) - \lim x_k = L$, and $\hat{N}_\theta^\alpha(A, \mathscr{M}, (s)) - \lim x_k = L_1$. Then

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho_1^{(k)}} \right) \right]^{s_k} = 0, \text{ for some } \rho_1^{(k)} > 0$$

and

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho_2^{(k)}} \right) \right]^{s_k} = 0, \text{ for some } \rho_2^{(k)} > 0.$$

Define $\rho^{(k)} = \max(2\rho_1^{(k)}, 2\rho_2^{(k)})$. As $\mathscr{M}$ is nondecreasing and convex, we have

$\frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_k \left(\frac{|L - L_1|}{\rho^{(k)}} \right) \right]^{s_k}$

$\leq \frac{D}{h_r^\alpha} \sum_{k \in I_r} \frac{1}{2^{s_k}} \left(\left[M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho_1^{(k)}} \right) \right]^{s_k} + \left[M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho_2^{(k)}} \right) \right]^{s_k} \right)$

$\leq \frac{D}{h_r^\alpha} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho_1^{(k)}} \right) \right]^{s_k} + \frac{D}{h_r^\alpha} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km}(A_k(x) - L)|}{\rho_2^{(k)}} \right) \right]^{s_k} \to 0, (r \to \infty),$

where $\sup_k s_k = H$ and $D = \max(1, 2^{H-1})$. Hence,

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_k \left(\frac{|L - L_1|}{\rho^{(k)}} \right) \right]^{s_k} = 0.$$

As $\lim_{k \to \infty} s_k = s$, we have

$$\lim_{k \to \infty} \left[M_k \left(\frac{|L - L_1|}{\rho^{(k)}} \right) \right]^{s_k} = \left[M_k \left(\frac{|L - L_1|}{\rho^{(k)}} \right) \right]^{s}$$

and so $L = L_1$. Thus the limit is unique. □

Theorem 3.5. *Let* $A = (a_{mk})$ *be an infinite matrix of complex numbers and let* $\mathscr{M} = (M_k)$ *be a Musielak-Orlicz function satisfying* Δ_2*-condition . If x is a strongly almost lacunary A-convergent sequence with respect to* $\mathscr{M}$*, then* $\hat{N}_\theta^\alpha(A) \subset \hat{N}_\theta^\alpha(A, \mathscr{M})$*.*

Proof. Let $x \in \hat{N}_\theta^\alpha(A)$. Then $\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} |t_{km}(A(x) - L)| = 0$, uniformly in m.

Let us define two sequences y and z such that

$$(|t_{km}(A_k(y) - L)|) = \begin{cases} (|t_{km}(A_k(x) - L)|) & \text{if } (|t_{km}(A_k(x) - L)|) > 1; \\ \theta & \text{if } (|t_{km}(A_k(x) - L)|) \le 1. \end{cases}$$

$$(|t_{km}(A_k(z) - L)|) = \begin{cases} \theta & \text{if } (|t_{km}(A_k(x) - L)|) > 1; \\ (|t_{km}(A_k(x) - L)|) & \text{if } (|t_{km}(A_k(x) - L)|) \le 1. \end{cases}$$

Hence, $(|t_{km}(A_k(x) - L)|) = (|t_{km}(A_k(y) - L)|) + (|t_{km}(A_k(z) - L)|)$.
Also, $(|t_{km}(A_k(y) - L)|) \le (|t_{km}(A_k(x) - L)|)$ and $(|t_{km}(A_k(z) - L)|) \le (|t_{km}(A_k(x) - L)|)$.
Since, $\hat{N}_\theta^\alpha(A)$ is normal, so we have $y, z \in \hat{N}_\theta^\alpha(A)$.
Let $\sup_k M_k(2) = T$

Then

$$\begin{aligned}
&\tfrac{1}{h_r^\alpha} \sum_{k\in I_r} \left[M_k\left(\frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right] \\
&= \tfrac{1}{h_r^\alpha} \sum_{k\in I_r} \left[M_k\left(\frac{|t_{km}(A_k(y) - L)| + |t_{km}(A_k(z) - L)|}{\rho^{(k)}} \right) \right] \\
&\le \tfrac{1}{h_r^\alpha} \sum_{k\in I_r} \left[\frac{1}{2} M_k\left(\frac{2|t_{km}(A_k(y) - L)|}{\rho^{(k)}} \right) + \frac{1}{2} M_k\left(\frac{2|t_{km}(A_k(z) - L)|}{\rho^{(k)}} \right) \right] \\
&< \tfrac{1}{2} \tfrac{1}{h_r^\alpha} \sum_{k\in I_r} K_1 \left(\frac{|t_{km}(A_k(y) - L)|}{\rho^{(k)}} \right) M_k(2) + \frac{1}{2} \frac{1}{h_r^\alpha} \sum_{k\in I_r} K_2 \left(\frac{|t_{km}(A_k(z) - L)|}{\rho^{(k)}} \right) M_k(2) \\
&\le \tfrac{1}{2} \tfrac{1}{h_r^\alpha} \sum_{k\in I_r} K_1 \left(\frac{|t_{km}(A_k(y) - L)|}{\rho^{(k)}} \right) \sup M_k(2) + \frac{1}{2} \frac{1}{h_r^\alpha} \sum_{k\in I_r} K_2 \left(\frac{|t_{km}(A_k(z) - L)|}{\rho^{(k)}} \right) \sup M_k(2) \\
&\to 0 \text{ as } r \to \infty.
\end{aligned}$$

Hence $x \in \hat{N}_\theta^\alpha(A, \mathscr{M})$. This completes the proof. □

Theorem 3.6. *Let $A = (a_{mk})$ be an infinite matrix of complex numbers and let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function satisfying Δ_2-condition. If*

$$\lim_{\nu\to\infty} \inf_k \frac{M_k\left(\frac{\nu}{\rho^{(k)}}\right)}{\frac{\nu}{\rho^{(k)}}} > 0, \text{ for some } \rho^{(k)} > 0,$$

then $\hat{N}_\theta^\alpha(A) = \hat{N}_\theta^\alpha(A, \mathscr{M})$.

Proof. If $\hat{N}_\theta^\alpha(A) = \hat{N}_\theta^\alpha(A, \mathscr{M})$ for some $\rho^{(k)} > 0$, then there exists a number $\gamma > 0$ such that

$$M_k\left(\frac{\nu}{\rho^{(k)}}\right) \ge \gamma \left(\frac{\nu}{\rho^{(k)}}\right), \forall \nu > 0, \text{ and some } \rho^{(k)} > 0.$$

Let $x \in \hat{N}_\theta^\alpha(A, \mathscr{M})$. Then

$$\begin{aligned}\frac{1}{h_r^{\alpha}}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}}\right)\right] &\geq \frac{1}{h_r^{\alpha}}\sum_{k\in I_r}\nu\left[\gamma\left(\frac{t_{km}(A_k(x)-L)|}{\rho^{(k)}}\right)\right]\\ &= \gamma\frac{1}{h_r^{\alpha}}\sum_{k\in I_r}\left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}}\right)\end{aligned}$$

Hence, $x \in \hat{N}_{\theta}^{\alpha}(A)$. This completes the proof. □

Theorem 3.7. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, where (M_k) is pointwise convergent. Then $\hat{N}_{\theta}^{\alpha}(A, \mathcal{M}, (s)) \subset \hat{S}_{\theta}^{\alpha}(A, \mathcal{M}, (s))$ if and only if $\lim_k M_k\left(\frac{\nu}{\rho^{(k)}}\right) > 0$ for some $\nu > 0, \rho^{(k)} > 0$.*

Proof. Let $\varepsilon > 0$ and $x \in \hat{N}_{\theta}^{\alpha}(A, \mathcal{M}, (s))$.

Also, if $\lim_k M_k\left(\frac{\nu}{\rho^{(k)}}\right) > 0$, then there exists a number $c > 0$ such that

$$M_k\left(\frac{\nu}{\rho^{(k)}}\right) \geq c, \text{ for } \nu > \varepsilon.$$

Let us consider, $I_r^1 = \left\{i \in I_r : \left[M_k\left(\frac{|t_{km}(A(x)-L)|}{\rho^{(k)}}\right)\right] \geq \varepsilon\right\}$.
Then

$$\begin{aligned}\frac{1}{h_r^{\alpha}}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(A(x)-L)|}{\rho^{(k)}}\right)\right]^{s_k} &\geq \frac{1}{h_r^{\alpha}}\sum_{k\in I_r^1}\left[M_k\left(\frac{|t_{km}(A(x)-L)|}{\rho^{(k)}}\right)\right]^{s_k}\\ &\geq c\frac{1}{h_r^{\alpha}}|t_{km}(A_0(\varepsilon)|\end{aligned}$$

Hence, it follows that $x \in \hat{S}_{\theta}^{\alpha}(A, \mathcal{M}, (s))$.

Conversely, let us assume that the condition does not hold good. For a number $\nu > 0$, let $\lim_k M_k\left(\frac{\nu}{\rho^{(k)}}\right) = 0$ for some $\rho > 0$. Now, we select a lacunary sequence $\theta = (n_r)$ such that $M_k\left(\frac{\nu}{\rho^{(k)}}\right) < 2^{-r}$ for any $k > n_r$.

Let $A = I$ and define a sequence x by putting

$$A_k(x) = \begin{cases} \nu & \text{if } n_{r-1} < k \leq \frac{n_r+n_{r-1}}{2}; \\ \theta & \text{if } \frac{n_r+n_{r-1}}{2} < k \leq n_r. \end{cases}$$

Therefore,

$$\begin{aligned}\frac{1}{h_r^{\alpha}}\sum_{k\in I_r}\left[M_k\left(\frac{|A_k(x)|}{\rho^{(k)}}\right)\right]^{s_k} &= \frac{1}{h_r^{\alpha}}\sum_{n_{r-1}<k\leq\frac{(n_r+n_{r-1})}{2}} M_k\left(\frac{\nu}{\rho^{(k)}}\right)\\ &< \frac{1}{h_r^{\alpha}}\frac{1}{2^{r-1}}\left[\frac{n_r+n_{r-1}}{2} - n_{r-1}\right]\\ &= \frac{1}{2^r} \to 0 \text{ as } r \to \infty.\end{aligned}$$

Thus we have $x \in \hat{N}_{\theta}^{\alpha 0}(A, \mathcal{M}, (s))$.

But,

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(A(x))|}{\rho^{(k)}}\right)\right]^{s_k}\geq\varepsilon\right\}\right| &= \lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\left\{k\in\left(n_{r-1},\frac{n_r+n_{r-1}}{2}\right):\sum_{k\in I_r}\left[M_k\left(\frac{\nu}{\rho^{(k)}}\right)\right]^{s_k}\geq\varepsilon\right\}\right|\\ &= \lim_{r\to\infty}\frac{1}{h_r^\alpha}\frac{n_r-n_{r-1}}{2}\\ &= \frac{1}{2}.\end{aligned}$$

So, $x\notin\hat{S}_\theta^\alpha(A,\mathscr{M},(s))$. □

Theorem 3.8. *Let $\mathscr{M}=(M_k)$ be a Musielak-Orlicz function. Then $\hat{S}_\theta^\alpha(A,\mathscr{M},(s))\subset\hat{N}_\theta^\alpha(A,\mathscr{M},(s))$ if and only if $\sup_\nu\sup_k M_k\left(\frac{\nu}{\rho^{(k)}}\right)<\infty$.*

Proof. Let $x\in\hat{S}_\theta^\alpha(A,\mathscr{M},(s))$. Suppose $h(\nu)=\sup_k M_k\left(\frac{\nu}{\rho^{(k)}}\right)$ and $h=\sup_\nu h(\nu)$. Let

$$I_r^2=\left\{k\in I_r:M_k\left(\frac{|t_{km}(A(x)-L)|}{\rho^{(k)}}\right)<\varepsilon\right\}.$$

Now, $M_k(\nu)\leq h$ for all $k,\nu>0$. So,

$$\begin{aligned}\frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(A(x)-L)|}{\rho^{(k)}}\right)\right]^{s_k} &= \frac{1}{h_r^\alpha}\sum_{k\in I_r^1}\left[M_k\left(\frac{|t_{km}(A(x)-L)|}{\rho^{(k)}}\right)\right]^{s_k}\\ &+ \frac{1}{h_r^\alpha}\sum_{k\in I_r^2}\left[M_k\left(\frac{|t_{km}(A(x)-L)|}{\rho^{(k)}}\right)\right]^{s_k}\\ &\leq h\frac{1}{h_r^\alpha}|t_{km}(A_0(\varepsilon)|+h(\varepsilon).\end{aligned}$$

Hence, as $\varepsilon\to0$, it follows that $x\in\hat{N}_\theta^\alpha(A,\mathscr{M},(s))$.
Conversely, suppose that

$$\sup_\nu\sup_k M_k\left(\frac{\nu}{\rho^{(k)}}\right)=\infty.$$

Then we have

$$0<\nu_1<\nu_2<...<\nu_{r-1}<\nu_r<...$$

so that $M_{n_r}\left(\frac{\nu_r}{\rho^{(k)}}\right)\geq h_r^\alpha$ for $r\geq1$. Let $A=I$. We set a sequence $x=(x_k)$ by,

$$A_k(x)=\begin{cases}\nu_r & \text{if } k=n_r \text{ for some } r=1,2,..;\\ \theta & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\left[\sum_{k\in I_r}M_k\left(\frac{|t_{km}(A_k(x))|}{\rho^{(k)}}\right)\right]^{s_k}\geq\varepsilon\right\}\right| &= \lim_{r\to\infty}\frac{1}{h_r^\alpha}\\ &= 0\end{aligned}$$

Hence, $x\in\hat{S}_\theta^\alpha(A,\mathscr{M},(s))$.

But,

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{h_r^{\alpha}}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(A_k(x)-L)|}{\rho^{(k)}}\right)\right] &= \lim_{r\to\infty}\frac{1}{h_r^{\alpha}}\left[M_{n_r}\left(\frac{|v_r-L|}{\rho^{(k)}}\right)\right]\\ &\geq \lim_{r\to\infty}\frac{1}{h_r^{\alpha}}h_r^{\alpha}\\ &= 1\end{aligned}$$

So, $x\in\hat{N}_{\theta}^{\alpha}(A,\mathscr{M},(s))$. □

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