Filomat 30:4 (2016), 1077–1080
DOI 10.2298/FIL1604077O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# On the Residual Algebraic Free Extension of a Valuation on $K$ to $K(x)$

**Figen Öke**[a]

[a]*Department of Mathematics, Faculty of Science University of Trakya TR-22030 Edirne, Turkey*

**Abstract.** In this study the residual algebraic free extension of a valuation on a field $K$ to $K(x)$ is studied. It is assumed that $v$ is a valuation on $K$ with $rankv = 2$ and the residual algebraic free extension $w$ of $v$ to $K(x)$ with $rankw = 3$ is defined for a special case.

## 1. Introduction

Defining all extensions of a valuation $v$ on a field $K$ to $K(x_1, ..., x_n)$ is an old and important problem. Residual transcendental extensions of $v$ to $K(x)$ were described in [1-2]. All extensions of $v$ to $K(x)$ were classified in [3]. The composite of valuations and certain extensions of them were studied in [5-6].

In this paper it is aimed to define a new kind residual algebraic free extension $w$ of $v$ to $K(x)$, where $v$ is a composite valuation $v = v_1 \circ v_2$ with $rankv = 2$.

## 2. Preliminaries

Throughout this paper $K$ is a field, $v$ is a valuation on $K$, $G_v$ is the value group of $v$, $O_v$ is the valuation ring of $v$, $M_v$ is the maximal ideal of $O_v$ and $k_v = O_v/M_v$ is the residue field of $v$, $p_v : O_v \to k_v$ is the canonical homomorphism, $U_v$ is the group of units of $O_v$. If $a \in O_v$ then $a^*$ denotes the natural image of $a$ in $k_v$. $\overline{K}$ is an algebraic closure of $K$ and $\overline{v}$ is a fixed extension of $v$ to $\overline{K}$. $G_{\overline{v}} = \overline{G_v}$ is the divisible closure of $G_v$ and $k_{\overline{v}} = \overline{k_v}$ is an algebraic closure of $k_v$. If $a \in \overline{K}$ then $v_a$ is the restriction of $\overline{v}$ to $K(a)$.

It is said that two valuations $v$, $v'$ on a field $K$ are equivalent if they have the same valuation ring i.e. $O_v = O_{v'}$. The set of all valuations of $K$ which are inequivalent in pairs will be denoted by $V(K)$ as in [4].

Let $v, v' \in V(K)$. It is said that $v$ dominates $v'$ if $O_v \subseteq O_{v'}$ and $M_{v'} \subseteq M_v$ and it is written as $v \leq v'$. Then $V(K)$ is an ordered set with respect to this relation by [4]. $v \leq v'$ if and only if there exists a group homomorphism $s : G_v \to G_{v'}$ such that $v' = sv$ then one has: $\varphi_v(v') = Kers$. The homomorphism $s$ is an onto mapping and it is uniquely defined in [4].

If $v \in V(K)$, $G$ is an ordered group and $s : G_v \to G$ is an onto homomorphism of ordered groups then $v' = sv$ is a valuation on $K$ such that $G_{v'} = G$ and $v \leq v'$ from [4].

2010 *Mathematics Subject Classification*. 2010 Mathematics Subject Classification. Primary 2J10; Secondary 12F210
*Keywords*. Extensions of valuations, residual algebraic free extensions, valued fields
Received: 23 July 2015; Accepted: 28 October 2015
Communicated by Gradimir Milovanović and Yilmaz Simsek
*Email address:* figenoke@gmail.com ( Figen Öke)

Let $L/K$ be an arbitrary field extension, $v$ be a valuation on $K$ and $w$ be a valuation on $L$. It is said that $w$ is an extension of $v$ to $L$ or $v$ is the restriction of $w$ to $K$ if $w(t) = v(t)$ for every $t \in K$. Then $O_{r(w)} = O_w \cap K = O_v$ is satisfied from [4].

Let $w$ be an extension of $v$ to $K(x)$. $w$ is called residual transcendental (r.t.) extension of $v$ if $k_w/k_v$ is a transcendental extension. If $w$ is a r.t. extension of $v$ to $K(x)$ then there exists a minimal pair $(a,\delta) \in \overline{K} \times G_{\bar{v}}$ with respect to $K$ where $a$ is seperable over $K$. Let $f = Irr(a,K)$ be a minimal polynomial of $a$ respect to $K$ and $\gamma = w(f)$. Let $F = F_0 + F_1 f + ... + F_n f^n$, $\deg F_i < \deg F$, $i = 0,...,n$ be the $f$-expansion of $F$, for each $F \in K[x]$. Define

$$w(F) = \inf_i (v_a(F_i(a)) + i\gamma).$$

Let $e$ be the smallest non-zero positive integer such that $e\gamma \in G_{v_a}$, where $v_a$ is the restriction of $\bar{v}$ to $K(a)$. Then $G_w = G_{v_a} + Z\gamma$, $[G_w : G_{v0}] = e[G_{v_a} : G_{v_0}]$. There exists $h \in K[x]$ such that $\deg h < \deg f$, $v_a(h(a)) = e\gamma$. Then $r = f^e/h$ is an element of $O_w$ of the smallest order such that $r^* \in k_w$ is transcendental over $k_v$. Thus the field $k_{v_a}$ can be identified cannonically with the algebraic closure of $k_v$ in $k_w$ and $k_w = k_{v_a}(r^*)$ from [2].

$w$ is called residual algebraic (r.a.) extension of $v$ if $k_w/k_v$ is an algebraic extension. $w/v$ is called residual algebraic torsion (r.a.t) extension if $w/v$ r.a. extension and $G_w/G_v$ is a torsion group. In this case $G_v \subseteq G_w \subseteq G_{\bar{v}}$ is satisfied according to [3].

If $w$ is a r.a. extension of $v$ to $K(x)$ and $G_w/G_v$ is not a torsion group then $w$ is called a residual algebraic free (r.a.f.) extension of $v$. If $w$ is a r.a.f. extension of $v$ to $K(x)$ then $rankw = rankv + 1$ and $w = w_1 \circ w_2$ where $w_1$ is a valuation of $K(x)$ and $w_2$ is a valuation of $k_{w_1}$. If $w_1$ is trivial on $K$ then it is defined by a monic irreducible polynomial $f \in K[x]$ or $w_1$ is the valuation at infinity. $k_{w_1} = K(a)$ where $a$ is the suitable root of $f$ or $k_{w_1} = K$ if $w_1$ is the valuation at infinity. Then $w$ is defined for each polynomial $F \in K[x]$, $F = F_0 + F_1 f + ... + F_n f^n$, $\deg F_i < \deg f$, $i = 0,...,n$ as;

$$w(F) = \inf_i (i, v_1(F_i(a))),$$

where $v_1$ is an extension of $v$ to $k_{w_1} = K(a)$ and $Q\text{x}G_{\bar{v}}$ is ordered lexicographically from [3].

If $w_1$ is the r.t extension of $v$ to $K(x)$ then $k_{w_1}$ has a valuation $w_2$ which is trivial on $k_v$. Hence $w_1$ is defined by a minimal pair $(a,\delta) \in \overline{K}\text{x}G_{\bar{v}}$. Since $w_2$ is trivial over $k_{v_a}$ then it is defined by an irreducible polynomial $G \in k_{v_a}[Y]$ or it is the valuation at infinity. A monic polynomial $g \in K[x]$ such that $w_1(g) = me\gamma$, $\deg g = me \deg f$ and $(g/h^m)^* = G$ is called a lifting polynomial of $G$ in $K[x]$. If $g$ is the lifting polynomial in $K[x]$ of $G \neq Y$ where $Y = r^*$ then $w$ is defined as follows: Let $F \in K[x]$, $F = F_0 + F_1 g + ... + F_n g^n$, $\deg F_i < \deg g$, $i = 0,...,n$ then

$$w(F) = \inf_i ((w_1(F_i), 0) + i(w(g), 1)),$$

where $G_{\bar{v}}\text{x}Q$ is ordered lexicographically, $k_w = k_{v_b}$ where $b$ is a suitable root of $g$ from [3].

## 3. Results

Let $v = v_1 \circ v_2$ be a valuation on $K$ such that $rankv = 2$. Then $v \leqslant v_1$ and there exists a group homomorphism $s : G_v \to G_{v_1}$ such that $sv = v_1$. Here $v_1$ is a valuation on $K$ and $v_2$ is a valuation on $k_v$. According to the general theory of composite valuations, there exists the exact sequence of groups:

$$0 \to G_{v_2} \xrightarrow{\rho} G_w \xrightarrow{s} G_{v_1} \to 0$$

where $\rho$ and $s$ are defined in a canonical way from [4].

We want to define a new kind extension $w$ of $v$ to $K(x)$ such that $rankw = 3$. Since $rankw = 3$ then $w = w_1 \circ w_2 \circ w_3$ is composite of valuations $w_1$, $w_2$ and $w_3$ here $rankw_1 = rankw_2 = rankw_3 = 1$. In this case there are different posibilities: Since $O_{r(w)} = O_w \cap K = O_v$ where $r$ is the restriction map; $r : V(K(x)) \to V(K)$ then $O_{w_1} \cap K = K$ or $O_{w_1} \cap K = O_{v_1}$ is satisfied. If $O_{w_1} \cap K = K$ then $w_1$ is trivial over $K$, $k_{w_1}$ is an algebraic extension of $K$ and $w_2 \circ w_3$ is an extension of $v$ to $k_{w_1}$. If $O_{w_1} \cap K = O_{v_1}$ then $w_1$ is a r.t. extension of $v_1$ to $k_{w_1}$ is a simple transcendental extension of $k'_{v_1}$ where $k'_{v_1}$ is an algebraic extension of $k_{v_1}$. There are two posibilities $O_{w_2} \cap k_{v_1} = k_{v_1}$ or $O_{w_2} \cap k_{v_1} = O_{v_2}$ when $O_{w_1} \cap K = O_{v_1}$. If $O_{w_2} \cap k_{v_1} = O_{v_2}$ then $w_2$ is a r.t. extension of $v_2$ to $k_{w_1}$. In this case $w_1 \circ w_2$ is a r.t. extension of $v = v_1 \circ v_2$ and this kind extensions are defined in [7]. If $O_{w_2} \cap k_{v_1} = O_{v_2}$ then $w_1 \circ w_2$ is a r.t. extension of $v = v_1 \circ v_2$ and $w_3$ is trivial over $k_v$, $w = w_1 \circ w_2 \circ w_3$ is a r.a.f extension of second kind of $v = v_1 \circ v_2$ to $K(x)$ and it can be obtained by using the definitions given in [3] and [7].

If $O_{w_2} \cap k_{v_1} = k_{v_1}$ then $w_2$ is trivial over $k_{v_1}$ and $k_{w_2}$ is an algebraic extension of $k_{v_1}$. In this case $w_3$ is an extension of $v_1$ to $k_{w_2}$. This kind extension was not defined before and it can not be obtained by using the extensions known before. Using the above investigations it can be given the following theorem for the existence of the extension of $v$ as desired:

**Theorem 3.1:** Let $v = v_1 \circ v_2$ be a valuation of $K$ with *rankv* = 2 and $w$ be an extension of $v$ to $K(x)$ with *rankw* = 3. Then there exist extensions $w_1$ and $u_1$ of $v_1$ to $K(x)$ such that $u_1$ is a r.a.f extension of second kind of $v_1$ to $K(x)$ and $w \leqslant u_1 \leqslant w_1$ is satisfied.

**Proof:** Since $v = v_1 \circ v_2$ is a valuation of $K$ with *rankv* = 2 then $v \leqslant v_1$. There exists a homomorphism of ordered groups; $s : G_v \to G_{v_1}$ such that $sv = v_1$. Since $w = w_1 \circ w_2 \circ w_3$ is an extension of $v$ to $K(x)$ it can be assumed that $w_1$ is non-trivial over $K$. Then $O_{w_1} \cap K = O_{v_1}$ and $w_1$ is a r.t. extension of $v_1$ to $K(x)$, so $k_{w_1} = k'_{v_1}(r^*)$ where $k'_{v_1}$ is an algebraic extension of $k_{v_1}$ and $r^*$ is transcendental over $k_{v_1}$. Define $i' : G_{v_1} \to G_{w_1} \times Q$ (ordered lexicographically) such that $i'(c) = (c, 0)$ for each $c \in G_{v_1}$. $i'$ is an one to one group homomorphism. Then $G_{v_1}$ is isomorphic to a subgroup of $G_{w_1} \times Q$. There exists an onto homomorphism of ordered groups; $z_1 : G_w \to G_{w_1} \times Q$, so $u_1 = z_1 w$ is a residual algebraic free extension of first kind of $v_1$ to $K(x)$ with value group $G_{u_1} \cong G_{w_1} \times Q$ according to [6]. $u_1 = w_1 \circ w_2$ and the residue field of $u_1$ is an algebraic extension of $k_{v_1}$. Similarly, define $i'' : G_{v_1} \to G_{v_1} \times Q \times G_{w_3} \cong G_{u_1} \times G_{w_3}$ (ordered lexicographically), such that $i''(c) = (c, 0, 0)$ for each $c \in G_{v_1}$, here $w_3$ is an extension of $v_2$ to $k_{u_1}$. There exists an onto homomorphism of ordered groups $z_2 : G_{u_1} \cong G_{w_1} \times Q \to G_{w_1}$, then it can be defined an onto homomorphism of ordered groups $z : G_w \to G_{w_1}$ satisfying $z = z_2 z_1$. Therefore $w, u_1, w_1 \in V(K(x))$ such that $z_1 w = u_1$, $z_2 u_1 = w_1$, $zw = w_1$ and $w \leqslant u_1 \leqslant w_1$. Moreover according the theory of composite valuations there exists the exact sequence of groups;

$0 \to G_{w_3} \xrightarrow{\rho_1} G_{w_2} \circ G_{w_3} \xrightarrow{\rho_2} G_w \xrightarrow{z_1} G_{u_1} \xrightarrow{z_2} G_{w_1} \to 0$

where $\rho_1$, $\rho_2$, $z_1$, $z_2$ are defined in a canonical way.

**Definition of** $w = w_1 \circ w_2 \circ w_3$

In this section we will obtain the all kind r.a.f. extensions of the valuation $v = v_1 \circ v_2$ on $K$ to $K(x)$ as desired. Firstly; we can assume that $K$ is an algebraic closed field. Let $v = v_1 \circ v_2$ be a valuation on $K$ with *rankv* = 2 and $a \in K$. Each polynomial $F \in K[x]$ is uniquely written as: $F = a_0 + a_1(x-a) + ... + a_k(x-a)^k + ... + a_n(x-a)^n$, where $a_0, a_1, ..., a_n \in K$. Denote $w_1(x-a) = d$ and $p_{w_1}(\frac{x-a}{d}) = t$. If $k$ is a positive integer satisfying the equality; $w_1(F) = \inf_i(w_1(a_i(x-a)^i)) = w_1(a_k) + kd$ then the equality; $p_{w_1}(\frac{F}{a_k d^k}) = t^k + \frac{a_{k+1}}{a_k} t^{k+1} + ... + \frac{a_n}{a^k} t^{n-k}$ is hold. Because $w_1(\frac{a_i}{a_k}\frac{(x-a)^i}{d^k}) > 0$ for $i < k$ and then $p_{w_1}(\frac{a_i}{a_k}\frac{(x-a)^i}{d^k}) = 0$. Therefore it is obtained that $w_2(p_{w_1}(\frac{F}{a_k d^k})) = w_2(p_{w_1}(\frac{(x-a)^k}{d^k})) = w_2(t^k) = k$. $w_2(p_{w_1}(\frac{x-a}{d})) = 1$ and then $u_1(x-a) = (w_1(x-a), w_2(p_{w_1}(\frac{x-a}{d})) = (d, 1))$,

Hence;

$$u_1(F) = (w_1(a_k(x-a)^k)), k) = \inf_i(w_1(a_i(x-a)^i, i)).$$

Then it is obtained that; $p_{w_2}(\frac{p_{w_1}(F/a_k d^k)}{t^k}) = p_{w_2}(\frac{p_{w_1}(x-a)^k/d^k)}{t^k})$

and so; $w_3(p_{w_2}(\frac{p_{w_1}(F/a_k d^k)}{t^k})) = w_3(p_{w_2}(\frac{p_{w_1}(\frac{F/(x-a)^k}{a_k})}{t^k})$.

Using the above conclusions for each $F = a_0 + a_1(x-a) + ... + a_k(x-a)^k + ... + a_n(x-a)^n \in K[x]$,

$$(w_1 \circ w_2 \circ w_3)(F) = (w_1(a_k(x-a)^k), 0, 0) + (0, k, 0) + (0, 0, w_3(p_{u_1}(a_k)))$$

$$= \inf_i((w_1(a_i), 0, 0) + i(d, 1, 0) + (0, 0, w_3(p_{v_1}(a_i))) = \inf_i((v_1(a_i), 0, 0) + i(d, 1, 0), v_2(p_{v_1}(a_i)))$$

is obtained.

Now, let $(K, v)$ be an arbitrary valued field, $\overline{K}$ be its algebraic closure, $\overline{v}$ be a fixed extension of $v$ to $\overline{K}$. If $w$ is an extension of $v$ to $K(x)$ then denote $\overline{w}$ the common extension of $\overline{v}$ and $w$ to $\overline{K}(x)$. Since $w_1$ is a r.t. extension of $v_1$ and $w_2$ is trivial over $k_{v_1}$ then $w_1$ is defined by a minimal pair $(a, \delta) \in \bar{K} \times G_{\bar{v}_1}$, $k_{w_1} = k'_{v_1}(r^*)$, where $k'_{v_1}$ is a finite extension of $k_{v_1}$, $r^* = Y$ is transcendental over $k'_{v_1}$ and $w_2$ is defined by an irreducible polynomial $G \in k_{v_1}[Y]$ or is the valuation at infinity. Let $g \in K[x]$ be the lifting polynomial of $G \neq Y$. Each polynomial $F \in K[x]$ is uniquely written as; $F = F_0 + F_1 g + ... + F_n g^n$, $F_i \in K[x]$, $\deg F_i < \deg g$, $i = 0, ..., n$ and then $u_1(F) = (w_1 \circ w_2)(F) = \inf_i (w_1(F_i g^i), i) = (w_1(F_k g^k), k)$, where $u_1(g) = (w_1(g), 1)$, $k$ is the positive integer satisfying that equality. The equalities $w_2(p_{w_1}(\frac{F}{F_k g^k})) = k$, $p_{w_2}(\frac{p_{w_1}(\frac{F}{F_k h^{mk}})}{G^k}) = p_{w_2}(\frac{p_{w_1}(\frac{g^k}{h^{mk}})}{G^k})$ are satisfied.

Hence $w_3(p_{w_2}(\frac{p_{w_1}(\frac{F}{F_k h^{mk}})}{G^k}) = w_3(p_{u_1}(\frac{F}{g^k}))$ and $p_{u_1}(F_k(x)) = p_{u_1}(F_k(b))$, where $b$ is a suitable root of $g \in K[x]$. Then we have;

$$w(F) = (u_1(F), w_3(p_{u_1}(F/g^k)) = (w_1(F_k), 0, 0) + k(w_1(g), 1, 0) + (0, 0, w_3(p_{u_1}(F/g^k))).$$

Therefore;

$$w(F) = (w_1 \circ w_2 \circ w_3)(F) = \inf_i((w_1(F_i), 0, 0) + i(w_1(g), 1, 0)) + (0, 0, w_3(p_{u_1}(F_k(b))))$$

$$= \inf_i((w_1(F_i), 0, 0) + i(w_1(g), 1, 0) + (0, 0, v'_2(p_{u_1}(F_i(b))))$$

where $w_3 = v'_2$ is an extension of $v_2$ to $k_{u_1} = k_{v_b}$.

Let $v_2$ be a valuation defined by $r* = Y$. Then each $F \in K[x]$ is uniquely written as; $F = F_0 + F_1 f + ... + F_k f^k + ... + F_n f^n$ and $u_1$ is defined as; $u_1(F) = (w_1 \circ w_2)(F) = (w_1(F_k f^k), [\frac{k}{e}]) = \inf_i(w_1(F_i f^i), [\frac{i}{e}])$, $w_1(f) = w_1(h^{1/e})$, $w_1(F) = w_1(F_k f^k) = w_1(F_k h^{k/e})$,

$p_{w_1}(F/F_k h^{k/e}) = \sum_{t=0}^{n-k} p_{w_1}(\frac{F_{k+t}}{F_k} h^{t/e})(r^*)^{\frac{k+t}{e}}$, $p_{w_2}(p_{w_1}(\frac{F}{F_k h^{k/e}})/r^{k/e}) = p_{w_2}(\frac{p_{w_1}(\frac{F/f^k}{F_k})}{r^{k/e}})$ and so $w_3(p_{w_2}(\frac{p_{w_1}(\frac{F}{F_k h^{k/e}})}{r^{k/e}})) = w_3(p_{u_1}(\frac{F}{f^k}))$ $= w_3(p_{u_1}(F_k(a))) = w_3(p_{w_2}(\frac{p_{w_1}(\frac{F/f^k}{F_k})}{r^k}))$. Then $w = w_1 \circ w_2 \circ w_3$ is defined as:

$$w(F) = \inf_i((w_1(F_i), 0, 0)+(w_1(f^i), [\frac{i}{e}], 0)+(0, 0, w_3(p_{u_1}(F_i(a)))) = \inf_i((w_1(F_i), 0, 0)+(w_1(f^i), [\frac{i}{e}], 0)+(0, 0, v'_2(p_{u_1}F_i(a)))),$$

where $v'_2$ is an extension of $v_2$ to $k_{u_1} = k_{w_2}$ and $[\frac{i}{e}]$ means the integral part of a real number. If $w_2$ is a valuation at infinity i.e. if it is defined by $r*^{-1}$ then

$$w(F) = \inf_i((w_1(F_i), 0, 0) + (w_1(f^i), -[\frac{i}{e}], 0) + (0, 0, v'_2(p_{u_1}(F_i(a))).$$

**Theorem 3.2:** Let $v = v_1 \circ v_2$ be a valuation on $K$ with *rankv* = 2 and let $w = w_1 \circ w_2 \circ w_3$ be an extension of $v$ to $K(x)$ such that *rankw*$_3$ and $w_2$ is trivial over the residue field $k_{v_1}$ of $v_1$. Then $w$ is equal to one of the valuations defined in this section.

**Proof:** The proof is obtained using the above considerations.

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