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Δ^m−Deferred Statistical Convergence of Order α

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Abstract. In this paper, we introduce the concepts of Δ^m−deferred statistical convergence of order α and strong Δ_r^m−deferred Cesàro summability of order α of real sequences. Additionally, some inclusion relations about Δ^m−deferred statistical convergence of order α and strong Δ_r^m−deferred Cesàro summability of order α are given.

1. Introduction, Definitions and Preliminaries

The idea of statistical convergence was introduced by Fast [10] and the notion was associated with summability theory by Connor [3], Connor and Savaş [4], Fridy [11], Gökhan et al. [12], Işık [13], Kuçukaslan et al. [15, 17], Šalat [16] and many others.

The deferred Cesàro mean of sequences was introduced by Agnew [1] such as:

$$\left(D_{p,q}x\right)_n = \frac{1}{q(n)-p(n)}\sum_{p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = +\infty.$$

Throughout this work $\{p(n)\}$ and $\{q(n)\}$ will denote sequences of non-negative integers that satisfy the above conditions.

Let A be a subset of $\mathbb{N}$ and denote the set $\{k : p(n) < k \le q(n), k \in A\}$ by $A_{p,q}(n)$. The α−deferred density of A is defined by

$$\delta_{p,q}^{\alpha}(A) = \lim_{n\to\infty} \frac{1}{(q(n)-p(n))^{\alpha}} \left|A_{p,q}(n)\right|, \text{ provided the limit exists, } \alpha \in (0,1] \quad (1)$$

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The vertical bars in (1) indicate the cardinality of the set $A_{p,q}(n)$.

It can be clearly seen that every finite subset of $\mathbb{N}$ has zero $\alpha-$deferred density. Beside, it does not need to hold $\delta^{\alpha}_{p,q}(A^c) = 1 - \delta^{\alpha}_{p,q}(A)$ for $0 < \alpha < 1$ in general. Note that the $\alpha-$deferred density reduces to the $\alpha-$density given in [5] for $q(n) = n$, $p(n) = 0$. Additionally, if $\alpha = 1$ then the notion coincides with the natural density. It can be shown that the inequality $\delta^{\beta}_{p,q}(A) \leq \delta^{\alpha}_{p,q}(A)$ is satisfied for $0 < \alpha \leq \beta \leq 1$.

If $x = (x_k)$ is a sequence such that x_k satisfies property $P(k)$ for all k except a set of $\alpha-$deferred density zero, then we say that x_k satisfies $P(k)$ for almost all k according D_α and we denote this by a.a.k (D_α).

The notion of difference sequence spaces was introduced by Kızmaz [14] and generalized by Et and Çolak [7]. Later on Et and Nuray [8] improved it as follows

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}\right)$ and so $\Delta^m x_k = \sum\limits_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$, then there exists one and only one $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m}, \tag{2}$$
$$y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. We shall use the sequence which is defined in (2) to define the sequence in (4), (5), (6) and (7) (see [2, 9]).

The main goal of this work is to examine the relation between Δ^m-deferred statistical convergence of order α and strong Δ^m_r-deferred Cesàro summability of order α, where $\alpha \in (0,1]$ and $r \in \mathbb{R}^+$. Also we investigate some properties related these concepts.

Now we begin with three new definitions.

Definition 1.1. Let $\{p(n)\}$, $\{q(n)\}$ be two sequences of non-negative integers satisfying conditions given above, $m \in \mathbb{N}$ and $\alpha \in (0,1]$ be given. A sequence $x = (x_k)$ is said to be Δ^m-deferred statistically convergent of order α to L if there is a real number L such that for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))^\alpha} \left|\{p(n) < k \leq q(n) : |\Delta^m x_k - L| \geq \varepsilon\}\right| = 0, \tag{3}$$

i.e.

$$|\Delta^m x_k - L| < \varepsilon \ \ a.a.k\,(D_\alpha).$$

In this case, we write $\Delta^m(DS^{\alpha}_{p,q}) - \lim x_k = L$.

The set of all Δ^m-deferred statistically convergent sequences of order α will be denoted by $\Delta^m(DS^{\alpha}_{p,q})$. If $m = 0$, then Δ^m-deferred statistical convergence of order α reduces to deferred statistical convergence of order α which was defined and studied by Çınar et al. [6]. If $m = 0$, $q(n) = n$ and $p(n) = 0$, then the concept coincides statistical convergence of order α and in the special case $m = 0$, $\alpha = 1$, $q(n) = n$ and $p(n) = 0$, Δ^m-deferred statistical convergence of order α coincides with the usual statistical convergence. Also in the special case $\alpha = 1$, $q(n) = n$ and $p(n) = 0$, Δ^m-deferred statistical convergence of order α coincides with Δ^m-statistical convergence which was defined and studied by Et and Nuray [8]. Therefore, Δ^m-deferred statistical convergence of order α is more general than all these notions.

The Δ^m-deferred statistical convergence of order α is well defined for $\alpha \in (0,1]$, but it is not well defined for $\alpha > 1$. For this let $m = 2$ and take a sequence $y = (y_k)$ such that $\Delta^2 x_k = y_k$ for the sequence $x = (x_k)$ as follows

$$x_k = \begin{cases} 0 & 1 \leq k \leq 3 \\ x_{k-1} + \frac{k-2}{2} & k = 2n, n \geq 2 \\ x_{k-1} + \frac{k-3}{2} & k = 2n+1, n \geq 2 \end{cases}$$

$$y_k = \begin{cases} 1 & k = 2n \\ 0 & k \neq 2n \end{cases} \quad n \in \mathbb{N} \,. \tag{4}$$

Then $\Delta^2(DS^{\alpha}_{p,q}) - \lim x_k = 0$ and $\Delta^2(DS^{\alpha}_{p,q}) - \lim x_k = 1$ which is impossible, where $q(n) = 4n^2$, $p(n) = 2n$ and $\alpha > 1$.

It is clear that $\Delta^m(c) \subset \Delta^m(DS^{\alpha}_{p,q})$ for each $0 < \alpha \leq 1$, but the converse of this is not true in general. For instance, let us take $y = (y_k)$ such that $\Delta^m x_k = y_k$ for some $x = (x_k)$ as follows:

$$y_k = \begin{cases} 2 & k = n^2 \\ 0 & k \neq n^2 \end{cases} \,. \tag{5}$$

Then we have

$$\frac{1}{(q(n) - p(n))^{\alpha}} \left|\left\{p(n) < k \leq q(n) : \left|y_k - 0\right| \geq \varepsilon\right\}\right| \leq \frac{\sqrt{q(n)} - \sqrt{p(n)} + 1}{(q(n) - p(n))^{\alpha}}.$$

Therefore, $x = (x_k)$ is Δ^m-deferred statistically convergent of order α to 0 for $\alpha > \frac{1}{2}$, but $x \notin \Delta^m(c)$, where $\Delta^m(c) = \{x = (x_k) : (\Delta^m x_k) \in c\}$.

Definition 1.2. Let $\{p(n)\}$ and $\{q(n)\}$ be two sequences of non-negative integers satisfying conditions given above, $m \in \mathbb{N}$ and $\alpha \in (0, 1]$ be given. A sequence $x = (x_k)$ is said to be Δ^m-deferred statistically Cauchy of order α if there is a natural number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |\Delta^m x_k - \Delta^m x_N| \geq \varepsilon\}\right| = 0$$

for every $\varepsilon > 0$.

This notion reduces to the concept of Δ^m-statistically Cauchy given in [8] for $q(n) = n$, $p(n) = 0$ and $\alpha = 1$.

Definition 1.3. Let $\{p(n)\}$ and $\{q(n)\}$ be two sequences of non-negative integers satisfying conditions given above, $m \in \mathbb{N}$, $r \in \mathbb{R}^+$ and $\alpha \in (0, 1]$ be given. A sequence $x = (x_k)$ is called strongly Δ^m_r-deferred Cesàro summable of order α to L if

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))^{\alpha}} \sum_{p(n)+1}^{q(n)} |\Delta^m x_k - L|^r = 0$$

and this is denoted by $\Delta^m - Dw^{\alpha}_r[p, q] - \lim x_k = L$.

The set of all strongly Δ^m_r-deferred Cesàro summable sequences of oder α will be denoted by $\Delta^m(Dw^{\alpha}_r[p, q])$.

2. Main Results

In the present part we give the main results of this paper. For instance, in Theorem 2.6 we give the relation between the Δ^m-deferred statistical convergence of order α and the Δ^m-deferred statistical convergence of order β, and in Theorem 2.8, we give the relation between the strong Δ^m_r-deferred Cesàro summability of order α and the strong Δ^m_r-deferred Cesàro summability of order β. Also the fact that the strong Δ^m_r-deferred Cesàro summability of order α implies the Δ^m-deferred statistical convergence of order β for $\alpha \leq \beta$ is given in Theorem 2.10.

The proof of each of the following results is straightforward, so we choose to state these results without proof.

Theorem 2.1. *Let* $0 < \alpha \leq 1$ *and* $x = (x_k)$, $y = (y_k)$ *be two sequences of complex numbers. Then each of the following assertions is true*:

(*i*) *If* $\Delta^m(DS^\alpha_{p,q}) - \lim x_k = L$ *and* $c \in \mathbb{R}$, *then* $\Delta^m(DS^\alpha_{p,q}) - \lim cx_k = cL$,

(*ii*) *If* $\Delta^m(DS^\alpha_{p,q}) - \lim x_k = L_1$ *and* $\Delta^m(DS^\alpha_{p,q}) - \lim y_k = L_2$, *then* $\Delta^m(DS^\alpha_{p,q}) - \lim(x_k + y_k) = L_1 + L_2$,

(*iii*) *If* $\Delta^m - Dw^\alpha_r[p,q] - \lim x_k = L$ *and* $c \in \mathbb{R}$, *then* $\Delta^m - Dw^\alpha_r[p,q] - \lim cx_k = cL$,

(*iv*) *If* $\Delta^m - Dw^\alpha_r[p,q] - \lim x_k = L_1$ *and* $\Delta^m - Dw^\alpha_r[p,q] - \lim y_k = L_2$, *then* $\Delta^m - Dw^\alpha_r[p,q] - \lim(x_k + y_k) = L_1 + L_2$.

Theorem 2.2. *For each* $m \in \mathbb{N}$, $\Delta^m(DS^\alpha_{p,q}) \subset \Delta^{m+1}(DS^\alpha_{p,q})$ *and the inclusion is strict.*

Proof. Let $x = (x_k) \in \Delta^m(DS^\alpha_{p,q})$. Then there exists a real number L such that

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))^\alpha} \left|\{p(n) < k \leq q(n) : |\Delta^m x_k - L| \geq \varepsilon\}\right| = 0,$$

i.e.

$$|\Delta^m x_k - L| < \varepsilon \quad a.a.k\ (D_\alpha)$$

for every $\varepsilon > 0$. Since $\Delta^{m+1}x_k = \Delta^m x_k - \Delta^m x_{k+1}$ we can write that

$$\left|\Delta^{m+1}x_k\right| \leq |\Delta^m x_k - L| + |\Delta^m x_{k+1} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad a.a.k\ (D_\alpha),$$

which means $\Delta^{m+1}(DS^\alpha_{p,q}) - \lim x_k = 0$. So $x \in \Delta^{m+1}(DS^\alpha_{p,q})$. To see the strictness, let x be defined by $x = \left(k^{m+2}\right)$. Then it can be easily seen that $x \in \Delta^{m+1}(DS^\alpha_{p,q})$, but $x \notin \Delta^m(DS^\alpha_{p,q})$ for $q(n) = n$, $p(n) = 0$ and $\alpha = 1$. □

The following result is easily derivable from Theorem 2.2.

Corollary 2.3. *Let* $m_1, m_2 \in \mathbb{N}$ *with* $m_1 < m_2$. *Then* $\Delta^{m_1}(DS^\alpha_{p,q}) \subset \Delta^{m_2}(DS^\alpha_{p,q})$ *and the inclusion is strict.*

Theorem 2.4. *If* $x = (x_k)$ *is* Δ^m*−deferred statistically convergent of order* α, *then it is* Δ^m*−deferred statistically Cauchy of order* α.

Proof. Let $\Delta^m(DS^\alpha_{p,q}) - \lim x_k = L$ and $\varepsilon > 0$, then the inequality $|\Delta^m x_k - L| < \frac{\varepsilon}{2}$ is satisfied for $a.a.k\ (D_\alpha)$. If N is chosen so that $|\Delta^m x_N - L| < \frac{\varepsilon}{2}$ for $a.a.k\ (D_\alpha)$, then we obtain

$$|\Delta^m x_k - \Delta^m x_N| \leq |\Delta^m x_k - L| + |\Delta^m x_N - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } a.a.k\ (D_\alpha).$$

Hence x is Δ^m−deferred statistically Cauchy of order α. □

Theorem 2.5. *If x is a sequence for which there is a* Δ^m*−deferred statistically convergent of order* α *sequence* y *such that* $\Delta^m x_k = \Delta^m y_k$ *for* $a.a.k\ (D_\alpha)$, *then* x *is* Δ^m*−deferred statistically convergent of order* α.

Proof. Assume that $\Delta^m x_k = \Delta^m y_k$ for $a.a.k\ (D_\alpha)$ and $\Delta^m(DS^\alpha_{p,q}) - \lim y_k = L$. Then for each n the following inclusion is satisfied:

$$\begin{aligned}\{p(n) < k \leq q(n) : |\Delta^m x_k - L| \geq \varepsilon\} \subseteq\ & \{p(n) < k \leq q(n) : \Delta^m x_k \neq \Delta^m y_k\} \\ & \cup \left\{p(n) < k \leq q(n) : \left|\Delta^m y_k - L\right| \geq \varepsilon\right\}.\end{aligned}$$

Hence we can write

$$\begin{aligned}\frac{1}{(q(n) - p(n))^\alpha} \left|\{p(n) < k \leq q(n) : |\Delta^m x_k - L| \geq \varepsilon\}\right| \leq \\ \frac{1}{(q(n) - p(n))^\alpha} \left|\{p(n) < k \leq q(n) : \Delta^m x_k \neq \Delta^m y_k\}\right| + \\ \frac{1}{(q(n) - p(n))^\alpha} \left|\left\{p(n) < k \leq q(n) : \left|\Delta^m y_k - L\right| \geq \varepsilon\right\}\right|.\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain that $x = (x_k)$ is Δ^m−deferred statistically convergent of order α to L. □

Theorem 2.6. *Let* $0 < \alpha \leq \beta \leq 1$. *Then* $\Delta^m(DS^{\alpha}_{p,q}) \subseteq \Delta^m(DS^{\beta}_{p,q})$ *and the inclusion is strict.*

Proof. The inclusion part of the proof is easy. To show that the inclusion is strict, let us define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} 1 & k = n^2 \\ 0 & k \neq n^2 \end{cases} \tag{6}$$

such that $\Delta^m x_k = y_k$ for some $x = (x_k)$. Then $x \in \Delta^m(DS^{\beta}_{p,q})$ for $\frac{1}{2} < \beta \leq 1$, but $x \notin \Delta^m(DS^{\alpha}_{p,q})$ for $0 < \alpha \leq \frac{1}{2}$, where $q(n) = 4n^2$ and $p(n) = n^2$. □

Theorem 2.7. *If* $\lim_n \frac{(q(n) - p(n))^{\alpha}}{n} > 0$, *then* $\Delta^m(S) \subset \Delta^m(DS^{\alpha}_{p,q})$, *where* $\Delta^m(S)$ *is the set of all* Δ^m-*statistically convergent sequences.*

Proof. Let $\Delta^m(S) - \lim x_k = L$ and $\lim_n \frac{(q(n) - p(n))^{\alpha}}{n} > 0$. For $\varepsilon > 0$, we have

$$\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\} \supseteq \{p(n) < k \leq q(n) : |\Delta^m x_k - L| \geq \varepsilon\},$$

therefore

$$\begin{aligned} \frac{1}{n} |\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\}| &\geq \frac{1}{n} \left|\{p(n) < k \leq q(n) : |\Delta^m x_k - L| \geq \varepsilon\}\right| \\ &= \frac{(q(n) - p(n))^{\alpha}}{n} \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |\Delta^m x_k - L| \geq \varepsilon\}\right|. \end{aligned}$$

Taking limit as $n \to \infty$ and using the fact that $\lim_n \frac{(q(n) - p(n))^{\alpha}}{n} > 0$, we get $\Delta^m(DS^{\alpha}_{p,q}) - \lim x_k = L$. □

The proof of the following two theorems are straightforward, so we state these results without proof.

Theorem 2.8. *Let* $\alpha, \beta \in (0, 1]$ *with* $\alpha \leq \beta$, $r \in \mathbb{R}^+$ *and* $m \in \mathbb{N}$. *Then* $\Delta^m(Dw^{\alpha}_r [p, q]) \subseteq \Delta^m(Dw^{\beta}_r [p, q])$ *and the inclusion is strict.*

Theorem 2.9. *Let* $\alpha \in (0, 1]$ *and* $0 < r < s < \infty$. *Then* $\Delta^m(Dw^{\alpha}_s [p, q]) \subseteq \Delta^m(Dw^{\alpha}_r [p, q])$.

Theorem 2.10. *Let* $\alpha, \beta \in (0, 1]$ *with* $\alpha \leq \beta$, $r \in \mathbb{R}^+$ *and* $m \in \mathbb{N}$. *If a sequence* $x = (x_k)$ *is strongly* Δ^m_r-*deferred Cesàro summable of order* α *to* L, *then it is* Δ^m-*deferred statistically convergent of order* β *to* L.

Proof. Let $x = (x_k)$ be strongly Δ^m_r-deferred Cesàro summable of order α to L. For the sequence $y = (y_k)$ such that $\Delta^m x_k = y_k$ and $\varepsilon > 0$, we can write

$$\begin{aligned} \sum_{p(n)+1}^{q(n)} \left|y_k - L\right|^r &= \sum_{\substack{p(n)+1 \\ |y_k - L| \geq \varepsilon}}^{q(n)} \left|y_k - L\right|^r + \sum_{\substack{p(n)+1 \\ |y_k - L| < \varepsilon}}^{q(n)} \left|y_k - L\right|^r \\ &\geq \sum_{\substack{p(n)+1 \\ |y_k - L| \geq \varepsilon}}^{q(n)} \left|y_k - L\right|^r \\ &\geq \left|\left\{p(n) < k \leq q(n) : \left|y_k - L\right|^r \geq \varepsilon\right\}\right| . \varepsilon^r \end{aligned}$$

which gives the following inequality

$$\frac{1}{(q(n)-p(n))^{\alpha}}\sum_{p(n)+1}^{q(n)}\left|y_k-L\right|^r \geq \frac{1}{(q(n)-p(n))^{\alpha}}\left|\left\{p(n)<k\leq q(n):\left|y_k-L\right|^r\geq\varepsilon\right\}\right|\varepsilon^r$$
$$\geq \frac{1}{(q(n)-p(n))^{\beta}}\left|\left\{p(n)<k\leq q(n):\left|y_k-L\right|^r\geq\varepsilon\right\}\right|\varepsilon^r.$$

Then taking limit as $n\to\infty$ we see that (x_k) is Δ^m-deferred statistically convergent of order β to L. □

Even if $y=(y_k)$ is a bounded and deferred statistically convergent sequence of order β such that $\Delta^m x_k = y_k$ for some $x=(x_k)$, the converse of Theorem 2.10 does not hold in general. To show this we must find a sequence that Δ^m-bounded (that is $x\in\Delta^m(\ell_\infty)$) and Δ^m-deferred statistically convergent of order β, but need not to be strongly Δ_r^m-deferred Cesàro summable of order α, for some α and β real numbers such that $0<\alpha\leq\beta\leq 1$. For this, let $p(n)=0$ and $q(n)=n$ for all $n\in\mathbb{N}$, take $r=1$ and consider a sequence $y=(y_k)$ defined by

$$y_k=\begin{cases}\frac{1}{\sqrt{k}}, & k\neq i^3\\ 1, & k=i^3\end{cases}. \tag{7}$$

It can be shown that $x\in\Delta^m(\ell_\infty)\cap\Delta^m(DS_{p,q}^{\alpha})$ for $\alpha\in\left(\frac{1}{3},1\right]$, but $x\notin\Delta^m(Dw_r^{\alpha}[p,q])$ for $\alpha\in\left(0,\frac{1}{2}\right)$ if $r=1$. So $x\in\Delta^m(DS_{p,q}^{\alpha})-\Delta^m(Dw_r^{\alpha}[p,q])$ for $\alpha\in\left(\frac{1}{3},\frac{1}{2}\right)$ if $r=1$.

The proof of the following result is straightforward, so we omit the proof.

Theorem 2.11. *Let $\alpha\in(0,1]$ and $r\in\mathbb{R}^+$. Then $\Delta^m(Dw_r^{\alpha}[p,q])\subseteq\Delta^{m+1}(Dw_r^{\alpha}[p,q])$ for all $m\in\mathbb{N}$.*

In the following theorem we investigate inclusion properties related Δ^m-deferred statistical convergence of order α under some particular conditions. We would like to state that $\{p(n)\}$, $\{q(n)\}$, $\{p'(n)\}$ and $\{q'(n)\}$ are sequences of non-negative integers satisfying

$$p(n)\leq p'(n)<q'(n)\leq q(n) \text{ for all } n\in\mathbb{N}.$$

Theorem 2.12. *Let $\{p(n)\}$, $\{q(n)\}$, $\{p'(n)\}$ and $\{q'(n)\}$ be given, $\alpha\in(0,1]$ and $m\in\mathbb{N}$. If $\lim_{n\to\infty}(\frac{q'(n)-p'(n)}{q(n)-p(n)})^{\alpha}>0$ then $\Delta^m(DS_{p,q}^{\alpha})$ convergence of a sequence $x=(x_k)$ implies $\Delta^m(DS_{p',q'}^{\alpha})$ convergence.*

Proof. We have

$$\{p(n)<k\leq q(n):|\Delta^m x_k-L|\geq\varepsilon\}\supseteq\{p'(n)<k\leq q'(n):|\Delta^m x_k-L|\geq\varepsilon\},$$

so

$$\frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\leq q(n):|\Delta^m x_k-L|\geq\varepsilon\}\right|\geq$$
$$\frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p'(n)<k\leq q'(n):|\Delta^m x_k-L|\geq\varepsilon\}\right|=$$
$$(\frac{q'(n)-p'(n)}{q(n)-p(n)})^{\alpha}\frac{1}{(q'(n)-p'(n))^{\alpha}}\left|\{p'(n)<k\leq q'(n):|\Delta^m x_k-L|\geq\varepsilon\}\right|.$$

Taking limit as $n\to\infty$, $\lim_{n\to\infty}\frac{1}{(q'(n)-p'(n))^{\alpha}}\left|\{p'(n)<k\leq q'(n):|\Delta^m x_k-L|\geq\varepsilon\}\right|=0$ which means $x\in\Delta^m(DS_{p',q'}^{\alpha})$. □

References

[1] R.P. Agnew, On deferred Cesàro means, Ann. Math. 33 (1932) 413–421.
[2] Ç. A. Bektaş, M. Et, R. Çolak, Generalized difference sequence spaces and their dual spaces, J. Math. Anal. Appl. 292 (2004) 423–432.
[3] J.S. Connor, The statistical and strong $p-$Cesàro convergence of sequences, Analysis 8 (1988) 47–63.
[4] J.S. Connor, E. Savaş, Lacunary statistical and sliding window convergence for measurable functions, Acta Math. Hungar. 145 (2015) 416–432.
[5] R. Çolak, Statistical convergence of order α, Modern Methods in Analysis and its Applications, New Delhi, India, Anamaya Pub. (2010) 121–129.
[6] M. Çınar, M. Et, F. Temizsu , M. Karakaş , Deferred statistical convergence of order α, submitted.
[7] M. Et, R. Çolak, On some generalized difference sequence spaces, Soochow J. Math. 21 (1995) 377–386.
[8] M. Et, F. Nuray, Δ^m-statistical convergence, Indian J. Pure Appl. Math. 32 (2001) 961–969.
[9] M. Et, Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences, Appl. Math. Comput. 219 (2013) 9372–9376.
[10] H. Fast, Sur la convergence statistique, Coll. Math. 2 (1951) 241–244.
[11] J.A. Fridy, On statistical convergence, Analysis 5 (1985) 301–313.
[12] A. Gökhan, M. Et, M. Mursaleen, Almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers, Math. Comp. Modell. 49 (2009) 548–555.
[13] M. Işık, Strongly almost $(w, \lambda, q)-$summable sequences, Math. Slovaca 61 (2011) 779–788.
[14] H. Kızmaz, On certain sequence spaces, Canadian Math. Bull. 24 (1981) 169–176.
[15] M. Küçükaslan, M. Yılmaztürk, On deferred statistical convergence of sequences, submitted.
[16] T. Šalát, On statistically convergent sequences of real numbers, Math. Slovaca 30 (1980) 139–150.
[17] M. Yılmaztürk, M. Küçükaslan, On strongly deferred Cesàro summability and deferred statistical convergence of the sequences, Bitlis Eren Univ. J. Sci. Technology 3 (2011) 22–25.