Filomat 30:4 (2016), 1021–1027
DOI 10.2298/FIL1604021U

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Modules over Groups

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Abstract. For a finite group G, by the endomorphism ring of a module M over a commutative ring R, we define a structure for M to make it an RG–module so that we study the relations between the properties of R–modules and RG–modules. Mainly, we prove that $Rad_R M$ is an RG–submodule of M if M is an RG–module; also $Rad_R M \subseteq Rad_{RG} M$ where $Rad_A M$ is the intersection of the maximal A–submodule of module M over a ring A. We also verify that M is an injective (projective) R–module if and only if M is an injective (projective) RG–module.

1. Introduction

Let R be a commutative ring with unity and G a finite abelian group. Let us recall the group ring RG.

RG denote the set of all formal expressions of the form $\sum_{g\in G} m_g g$ where $m_g \in R$ and $m_g = 0$ for almost every g. For elements $m = \sum_{g\in G} m_g g$, $n = \sum_{g\in G} n_g g \in RG$, by writing $m = n$ we mean $m_g = n_g$ for all $g \in G$.

The sum in RG is componentwise as

$$m + n = \sum\nolimits_{g\in G} m_g g + \sum\nolimits_{g\in G} n_g g = \sum\nolimits_{g\in G} (m_g + n_g) g$$

Moreover, RG is a ring with the following multiplication;

$$\mu\eta = \sum_{g\in G} \left(r_g k_h\right)(gh) = \sum_{g\in G} \sum_{h\in G} \left(r_g k_{h^{-1}g}\right) g$$

where $\mu = \sum\limits_{g\in G} r_g g, \eta = \sum\limits_{h\in G} k_h h \in RG$.

Since G is finite, RG is a finite dimensional R–algebra. Finite dimensional R–algebras (especially semisimple ones) have been more extensively investigated than finite groups; as a result RG has historically been used as a tool of group theory. If G is infinite, however, the group theory and the ring theory is not considerably well-known compared to one another. In this case, the emphasis is given to the relations between the two.

2010 *Mathematics Subject Classification*. 13C11, 15D50,16S34, 20C05
Keywords. Gruop rings, Projective modules and Injective modules
Received: 07 July 2015; Accepted: 14 August 2015
Communicated by Gradimir Milovanović and Yilmaz Simsek
The present investigation was supported by the Scientific Research Project Administration of Akdeniz University
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Consider the cyclic subgroup $\langle x\rangle$ of G, where x is a nonidentity element. Since $R\langle x\rangle$ is in RG, we simply direct our attention to it. If x has finite order $n \geq 1$, then $1, x, ..., x^{n-1}$ are distinct powers of x and in view of the equation

$$(1-x)(1+x+...+x^{n-1}) = 1-x^n = 0, \tag{1}$$

$R\langle x\rangle$, and hence RG, has a proper divisor of zero. On the other hand, if x has infinite order, $R\langle x\rangle$ consists of all finite sums of the form $\sum a_i x^i$ since all powers of x are distinct. Therefore, elements of $R\langle x\rangle$ are polynomials in x divided by some sufficiently high power of x. Consequently, $R\langle x\rangle$ is contained in the Laurent polynomial ring $R\left[x, x^{-1}\right]$, which means it is an integral domain. In addition, the Laurent polynomial ring $R\left[x, x^{-1}\right]$ is isomorphic to the group ring of the group $\mathbb{Z}$ of integers over R. In fact, the Laurent polynomial ring in n variables is isomorphic to the group ring of the free abelian group of rank n.

In this paper, we impose a new structure on an R–module M to make it an RG–module, so that we study the relations between the properties of these classes. Furthermore, we will give an alternative proof for Generalized Maschke's Theorem, and using the relations between RG–modules and R–modules, we will get a sufficient condition for M to be a free R–module, in case M is a projective R–module.

2. Relations between R-modules and RG-modules

Let M be a module over a commutative ring R and $EndM$ denotes the endomorphism ring of M. We use the notation $Rad_A M$ for the intersection of maximal A–submodules of module M over a ring A.

Firstly, we define the structure of an R–module M by making it an RG–module using the endomorphism ring of M. We also study the properties of RG–modules.

Let τ be a group homomorphism from G to $End(M)$. So, for all $g \in G, m \in M$, we define the multiplication mg as

$$mg = \tau(g)(m).$$

With this multiplication, it is easy to check that M is an RG–module. The group homomorphism τ in the multiplication is called a representation of G for M over R.

If $\tau(g) = 1_{End(M)}$ for all $g \in G$, the structure of RG–module is the same with the structure of R–module.

The following is an example for the multiplication of an RG–module M.

Example 2.1. *Let $R = \mathbb{Z}, M = \mathbb{Z} \oplus \mathbb{Z}, G = C_2 = \{e, a\}$.*

i)Consider the R–homomorphism f from M to M such that $f(x, y) = (3x - 4y, 2x - 3y)$. Clearly, f is an endomorphism of M.

Let define a map τ from G to $EndM$ such that $\tau(e) = 1$ and $\tau(a) = f$. Hence, τ is a group homomorphism and so M is an RG–module. For any $m = (x, y) \in M = \mathbb{Z} \oplus \mathbb{Z}$,

$$\begin{aligned} ma &= f(a)(m) \\ &= (3x - 4y, 2x - 3y) \end{aligned}$$

ii) Consider the R–homomorphism f from M to M such that $f(x, y) = (x, -y)$. Clearly, f is an endomorphism of M.

Let define a map τ from G to $EndM$ such that $\tau(e) = 1$ and $\tau(a) = f$. Hence, τ is a group homomorphism and so M is an RG–module. For any $m = (x, y) \in M = \mathbb{Z} \oplus \mathbb{Z}$,

$$\begin{aligned} ma &= f(a)(m) \\ &= (x, -y). \end{aligned}$$

From now on, by the multiplication above we can consider an R–module M as an RG–module. The R–module structure and the RG–module structure of M have many different properties. In the following example, although a submodule N of an RG–module M is indecomposable as an RG–submodule, it is decomposable as an R–submodule.

Example 2.2. *Let $R = \mathbb{C}$, $M = \mathbb{C} \oplus \mathbb{C}$, $G = D_8 = \left\langle a, b : a^4 = b^2 = e,\ b^{-1}ab = a^{-1} \right\rangle$. Consider the R–homomorphisms f_1, f_2 from M to M such that*

$$f_1(x, y) = (-y, x),\ f_2(x, y) = (x, -y)$$

Clearly, f_1, f_2 are endomorphisms of M. Let define a map τ from G to EndM such that $\tau(e) = 1$ and $\tau(a) = f_1$ and $\tau(b) = f_2$. Hence, τ is a group homomorphism. For any $m = (x, y) \in M$,

$$ma = a(x, y) = f_1(x, y) = (-y, x)$$

$$(x, y)a^2 = (-x, -y),\ (x, y)a^3 = (y, -x)$$

$$mb = (x, y)b = f_2(x, y) = (x, -y)$$

$$(x, y)ba = (y, x),\ (x, y)ba^2 = (-x, y),\ (x, y)ba^3 = (-y, -x)$$

Moreover, M is a semisimple RG–module since RG is a semisimple ring. Now we claim that M is an irreducible RG–module. If there is a proper RG–submodule N and $N \neq M$, $\dim N = 1$, then $N = RG(\alpha, \beta)$ for $(\alpha, \beta) \in M$.

$$(\alpha, \beta)a = f_1(\alpha, \beta) = (-\beta, \alpha)$$

$$(\alpha, \beta)b = f_2(\alpha, \beta) = (\alpha, -\beta)$$

Since N is an RG–submodule of M, (α, β), $(-\beta, \alpha)$, $(\alpha, -\beta) \in N$. Moreover, $(\alpha, \beta)+\ (\alpha, -\beta) = (2\alpha, 0) \in N$ and $(2\alpha, 0) = (\alpha, \beta)r_1$ for some $0 \neq r_1 \in RG$. Hence $\beta = 0$. Also, $(\alpha, \beta) - (\alpha, -\beta) = (0, 2\beta) \in N$ and $(0, 2\beta) = r_2(\alpha, \beta)$ for some $0 \neq r_2 \in RG$. Hence $\alpha = 0$. So we get $\alpha = \beta = 0$. Thus, $N = \{0\}$ and M is an irreducible RG–module. Then M is a cyclic RG-module, ($m \in M$, $RGm = M$). On the other hand, $\dim_R M = 2$ and there are proper R–submodules in M.

It is clear that any RG–submodule of M is an R–submodule, but in generally the converse is not true. Now we study some properties of RG–modules. Obviously, for a group homomorphism τ from G to $End(M)$ we have $\tau(G) \subseteq End(M)$. Then we define τ–fully invariant submodule as:

Definition 2.3. *An R−submodule N of an RG–module M is called τ−fully invariant if for all $f \in \tau(G)$,*

$$f(N) \subseteq N.$$

Lemma 2.4. *Let N be an R–submodule of an RG–module M. Then $NG = \sum_{g \in G} Ng$ is a minimal RG–submodule containing N.*

Proof. Clearly, NG is an RG–submodule. So we show that NG is a minimal RG–submodule containing N. Assume that N_1 is an RG–submodule such that $N \subset N_1 \subset NG$. Take an element $n \in N$ and so for all $g \in G$,we get $ng \in N_1$ since N_1 is an RG–submodule containing N. This means that that $N_1 = NG$. □

Lemma 2.5. *Let N be a maximal R–submodule of an RG–module M. Then $NG = N$ or $NG = M$.*
Furthermore, if N is τ–fully invariant then $NG = N$. If N is not τ–fully invariant then $NG = M$.

Proof. Clearly, $N \subseteq NG \subseteq M$. If N is τ–fully invariant, then $f(N) \subseteq N$ for all $f \in \tau(G)$ and so $Ng \subseteq N$ for all $g \in G$. Therefore, $NG = N$. On the other hand, if N is not τ–fully invariant, then clearly $NG = M$ since N is maximal. □

Theorem 2.6. *Let M be a finitely generated RG–module and N the only maximal R–submodule of M. If N is not τ–fully invariant, then M is a cyclic RG–module.*

Proof. Since N is not τ–fully invariant, we get $N \neq NG$ and $NG = M$. So there exists $ng \in NG$, $ng \notin N$ for some $g \in G$, $n \in N$. Thus we have an RG–submodule $ngRG$ of M and $ngRG$ is not in N. On the other hand, $ngRG$ is also an R–submodule of M. Since N is the only maximal R–submodule of M, we get $ngRG = M$. □

Following [1, page 72], recall that a submodule K of an R–module M is essential (or large) in M, abbreviated $K \trianglelefteq M$, in case for every submodule L of M, $K \cap L = 0$ implies $L = 0$. Moreover, a submodule K of an R–module M is superfluous (or small) in M, abbreviated $K \ll M$, in case for every submodule L of M, $K + L = M$ implies $L = M$.

Lemma 2.7. *Let M be an RG–module. If N is an essential R–submodule of M, then NG is an essential RG–submodule of M.*

Proof. Let L be an RG–submodule of M such that $NG \cap L = 0$. Thus $N \cap L = 0$ and so $L = 0$ since N is an essential R–submodule of M. Hence NG is an essential RG–submodule of M. □

Lemma 2.8. *Let τ be a group homomorphism from G to $End(M)$. If N is a superfluous R–submodule of M, then $Ng = \tau(g)(N)$ is a superfluous RG–submodule of M.*

Proof. Let L be an RG–submodule of M and assume $L + \tau(g)(N) = M$. Then $(\tau(g))^{-1}(L) + N = M$ and so $M = (\tau(g))^{-1}(L)$ since N is a superflous R–submodule of M. This means that $L = M$ and so Ng is a superfluous RG–submodule of M. □

Lemma 2.9. *Let M be a finitely generated RG–module. If N is a superfluous R–submodule of M then NG is a superfluous RG–submodule of M.*

Proof. Assume that $NG = M$. Then we get

$$NG = \sum_{g \in G} Ng = Ne + Ng_1 + ... + Ng_k = M$$

where $G = \{e, g_1, ..., g_k\}$. Since N is a superfluous R–submodule of M, we get $Ng_1 + ... + Ng_k = M$. Then by Lemma 2.8, Ng_1 is a superfluous submodule of M and we get $Ng_2 + ... + Ng_k = M$ and so on. Since Ng_{k-1} is also a superfluous submodule of M, we get $Ng_k = M$, a contradiction. Therefore, $NG \neq M$.

On the other hand, $NG = N + Ng_1 + ... + Ng_n$ is a sum of homomorphic images of superfluous R–submodules of M. Hence NG is a superfluous R–submodule of M. Let L be an RG-submodule of M such that $NG + L = M$. L is also an R–submodule of M and $NG + L = M$. Thus $L = M$ and so NG is also a superfluous RG–submodule of M. □

Theorem 2.10. *Let M be an RG–module. Then $Rad_R M$ is an RG–submodule of M and $Rad_R M \subseteq Rad_{RG} M$.*

Proof. It is known that $Rad_R M$ is the sum of superfluous R–submodules of M and $Rad_R M$ is a fully invariant R–submodule of M and so $(Rad_R M)G = Rad_R M$. This means that $Rad_R M$ is an RG-submodule of M. On the other hand, by Lemma 2.9, we get

$$Rad_R M = \sum_{N << _R M} N \subseteq \sum_{N << _{RG} M} NG \subseteq Rad_{RG} M.$$

Hence, $Rad_R M \subseteq Rad_{RG} M$. □

3. Projectivity and Injectivity as *RG*–modules

In this section, we will show some relations about projectivity and injectivity between R–modules and RG–modules. Moreover, we will give an alternative proof for Generalized Maschke's Theorem at the end of the section.

Lemma 3.1. *Let M be a free RG–module and H be a subgroup of G. Then M is a free RH–module and a free R–module.*

Proof. Let $S = \{m_i : i \in I\}$ be an RG–basis of M and take an element m of M. Then m is written uniquely by S such that $m = \sum_{i\in I} r_i m_i$ as a finite sum where $r_i = \sum_{g_i \in G} g_i r_{g_i} \in RG$. Let the set $T = \{y_j : y_j \in G, j \in J\}$ be a right transversal for H in G. Then for any i, there is $j \in J$ such that $g_i \in Hy_j$ and so $g_i = h_{ji} y_j$ for some $h_{ji} \in H$. Then $r_i = \sum_{h_{ji}\in T} h_{ji} r_{h_{ji}} y_j$ where $r_{h_{ji}} = r_{g_i}$, and $m = \sum_{h_{ji}\in T} h_{ji} r_{h_{ji}} \left(y_j m_i\right)$ where m is written as a lineer combination of the elements in RH. Hence, we have a new set $S' = \{y_j m_i : i \in I, j \in J\}$.

We will show that S' is linearly independent. Suppose that $\sum_{i\in I, j\in J} (y_j m_i) r_{ji} = 0$ where $r_{ji} \in RH$ for some $i \in I, j \in J$. Since $y_j r_{ji} \in RG$ and S an RG–basis of M, it follows that $(y_j r_{ji}) = 0$ for all $i \in I, j \in J$. This implies that $r_{ji} = 0$ and so $S' = \{y_j m_i : i \in I, j \in J\}$ is linearly independent. Therefore, M is a free RH–module.

In particular, for $H = \{e\}$, M is a free $R\{e\}$–module which implies M is a free R–module. □

It is clear that converse of the lemma above is not true, in general.

Theorem 3.2. *Let M be an RG–module, G a finite group and $|G|$ invertible in R. Then M is a projective R–module if and only if M is a projective RG–module.*

Proof. Assume that M is a projective R–module. Let A, B be RG–modules and α, β be RG–homomorphisms. Then we should have the following diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow^{\beta} & & \\ A & \underset{\alpha}{\longrightarrow} & B & \longrightarrow & 0 \end{array}$$

Obviously, A, B are also $R-$modules, α, β are $R-$homomorphisms. Then there exists an R–homomorphism φ from M to A such that $\beta = \alpha\varphi$. Consider the following map $\bar{\varphi}$ from M to A

$$\bar{\varphi}(m) = \frac{1}{|G|}\sum_{g\in G} \varphi(mg)g^{-1}$$

for all $m \in M$. Then clearly, $\bar{\varphi}$ is an R–homomorphism. Moreover, for any $m \in M, h \in G$, we get

$$\begin{aligned} \bar{\varphi}(mh) &= \frac{1}{|G|}\sum_{g\in G} \varphi(mhg)g^{-1} = \frac{1}{|G|}\sum_{g'\in G} \varphi(mg')g'^{-1}h, \text{ where } g' = hg \\ &= (\frac{1}{|G|}\sum_{g'\in G} \varphi(mg')g'^{-1})h = \bar{\varphi}(m)h. \end{aligned}$$

Hence, $\bar{\varphi}$ is an RG–homomorphism. Furthermore,

$$\begin{aligned} \alpha\bar{\varphi}(m) &= \alpha(\frac{1}{|G|}\sum_{g\in G} \varphi(mg)g^{-1}) = \frac{1}{|G|}\sum_{g\in G} \alpha(\varphi(mg)g^{-1}) = \frac{1}{|G|}\sum_{g\in G} (\alpha\varphi(mg))g^{-1} \\ &= \frac{1}{|G|}\sum_{g\in G} \beta(mg)g^{-1} = \frac{1}{|G|}\sum_{g\in G} \beta(mgg^{-1}) = \frac{1}{|G|}\sum_{g\in G} \beta(m) = \frac{1}{|G|}|G|\beta(m) \\ &= \beta(m) \end{aligned}$$

Thus, $\bar{\varphi}$ is the desired RG–homomorphism and M is a projective RG–module.

Conversely, let M be a projective RG–module. Then there is a free RG–module F and an RG–module N such that $F = M \oplus N$. By Lemma 3.1, F is a free R–module and so M is a projective R–module. □

Theorem 3.3. *Let M be a finitely generated projective R–module. If there are a decomposition $G = AB$ for subgroups A, B of G such that RB is an indecomposable RB–module and RA is isomorphic to $\oplus_{i=1}^{n} R$ as a ring where n is order of A, then M is a free R–module.*

Proof. If M is a projective R–module then M is a projective RG–module and so there is a positive integer m and an RG–module N such that $\oplus_{i=1}^{m} RG \cong M \oplus N$.

By the hypothesis, RA is isomorphic to $\oplus_{i=1}^{n} R$ as a ring. Then we also get $RG = R(AB) = (RA)B$ by [6, page 458] so that RG is isomorphic to $\oplus_{i=1}^{n} RB$ as a ring. Finally, we get that $K = \oplus_{i=1}^{m} \left(\oplus_{i=1}^{n} RB\right) \cong M \oplus N$. So by Krull-Schmitt theorem, M is isomorphic to direct sum of a finite number of indecomposable RB–submodules of K. On the other hand, by the hypothesis, RB is an indecomposable RB–module and so M is isomorphic to direct sum of RBs. Hence, M is a free RB–module. So by Lemma 3.1, M is a free R–module. $\square$

Theorem 3.4. *Let M be an RG–module, G a finite group and $|G|$ invertible in R. Then M is an injective R–module if and only if M is an injective RG–module.*

Proof. Assume that M is an injective R–module. Let I be an ideal of a ring RG and α be RG–homomorphisms, i is the injection RG–map. Hence, both I and RG are R–modules, α is an R–homomorphism and i is the injection R–map. Since M is an injective R–module, there is an RG–homomorphism φ such that $\varphi i = \alpha$. i.e we have the following commutative diagram

$$\begin{array}{ccccc} & & M & & \\ & & \uparrow^{\alpha} & \nwarrow^{\varphi} & \\ 0 & \longrightarrow & I & \underset{i}{\longrightarrow} & RG \end{array}$$

Consider the following map $\bar{\varphi}$ from RG to M

$$\bar{\varphi}(m) = \frac{1}{|G|} \sum_{g \in G} \varphi(mg) g^{-1}$$

for $m \in M$. We have already proved that $\bar{\varphi}$ is an RG–homomorphism. Furthermore,

$$\begin{aligned} \bar{\varphi} i(m) &= \bar{\varphi}(m) = \frac{1}{|G|} \sum_{g \in G} \varphi(mg) g^{-1} = \frac{1}{|G|} \sum_{g \in G} \varphi(i(mg)) g^{-1} = \frac{1}{|G|} \sum_{g \in G} \alpha(mg) g^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} \alpha(mgg^{-1}) = \frac{1}{|G|} \sum_{g \in G} \alpha(m) = \frac{1}{|G|} |G| \alpha(m) = \alpha(m). \end{aligned}$$

Thus, $\bar{\varphi}$ is the desired RG–homomorphism and M is an injective RG–module.

Assume that M is an injective RG–module. Let I be an ideal of a ring R and f an R–homomorphism, i the injection R–map.

$$\begin{array}{ccccc} & & M & & \\ & & \uparrow f & & \\ 0 & \longrightarrow & I & \underset{i}{\longrightarrow} & R \end{array}$$

On the other hand, IG is an ideal of RG and consider the following map $\bar{f}$ such that

$$\bar{f}(\sum_{g \in G} r_g g) = \sum_{g \in G} f(r_g) g.$$

Clearly, $\bar{f}$ is an RG–homomorphism by

$$\bar{f}(\sum_{g \in G} r_g gh) = \sum_{g \in G} f(r_g) gh = (\sum_{g \in G} f(r_g) g) h = \bar{f}(\sum_{g \in G} r_g g) h$$

where $r_g \in I$. $\bar{f}(\sum_{g\in G} r_g g) = m(\sum_{g\in G} r_g g)$ for some $m \in M$ since M is injective RG–module. Moreover, for $x \in I$, $xe \in IG$. Then $\bar{f}(xe) = f(x)e$ and also

$$\bar{f}(xe) = mxe = mex = mx = f(x)e = f(x).$$

Thus the desired R–homomorphism g from R to M is defined as $g(r) = mr$ for $r \in R$. So, M is an injective R–module. □

Theorem 3.5. *Let R be a ring, G a finite group and $|G|$ invertible in R. Then RG is semisimple if and only if R is semisimple.*

Proof. Let RG be semisimple, G a finite group and $|G|$ invertible in R. For any R–module M, M is an RG–module by $\tau : G \longrightarrow End(M)$, $g \mapsto 1$ for all $g \in G$. By Theorem 3.4, any injective RG–module M is an injective R–module. Therefore, every right module over R is injective and so R is semisimple.

Conversely, let R be semisimple, G a finite group and $|G|$ invertible in R. For any RG–module M, M is an R–module. Since R is semisimple, M is an injective R–module. By Theorem 3.4, any injective R–module M is an injective RG–module. Therefore, every module over RG is injective and so RG is semisimple. □

Acknowledgement

The third author was supported by the Scientific Research Project Administration of Akdeniz University. The authors are deeply grateful to the referees for their useful pointed comments and suggestions.

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