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Weighted Lacunary Statistical Convergence of Double Sequences in Locally Solid Riesz Spaces

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Abstract. Recently, the notion of weighted lacunary statistical convergence is studied in a locally solid Riesz space for single sequences by Başarır and Konca [7]. In this work, we define and study weighted lacunary statistical τ-convergence, weighted lacunary statistical τ-boundedness of double sequences in locally solid Riesz spaces. We also prove some topological results related to these concepts in the framework of locally solid Riesz spaces and give some inclusion relations.

1. Introduction

The notion of Riesz space was first introduced by F. Riesz in 1928, at the International Mathematical Congress in Bologna, Italy [29]. Soon after, in the mid-thirties, H. Freudenthal [13] and L. V. Kantorovich [15] independently set up the axiomatic foundation and derived a number of properties dealing with the lattice structure of Riesz spaces. From then on the growth of the subject was rapid. In the forties and early fifties the Japanese school led by H. Nakano, T. Ogasawara and K. Yosida, and the Russian school, led by L. V. Kantorovich, A. I. Judin, and B. Z. Vulikh, made fundamental contributions. At the same time a number of books started to appear on the field. The general theory of topological Riesz spaces seems somehow to have been neglected. The recent book by D. H. Fremlin [12] is partially devoted to this subject. Riesz spaces play an important role in analysis, measure theory, operator theory and optimization. They also provide the natural framework for any modern theory of integration. Further, they have some applications in economics [2]. A Riesz space is an ordered vector space which is a lattice at the same time. A locally solid Riesz space is a Riesz space equipped with a linear topology that has a base consisting of solid sets. For recent results related this topic, we may refer, for example, [1, 4, 5, 7, 14, 18–20, 22, 32].

The idea of statistical convergence was initially given by Zygmund in 1935 [37]. The concept was formally introduced by Fast [11] and Steinhaus [35] and later on by Schoenberg [34], and also independently by Buck [9].

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense. A double sequence $x = (x_{k,l})$ is said to converge to the limit L in Pringsheim's sense (shortly, P-convergent to L) if for every $\varepsilon > 0$ there exists an integer N such that $\left|x_{k,l} - L\right| < \varepsilon$ whenever $k, l > N$ [26]. We shall write this as $\lim_{k,l\to\infty} x_{k,l} = L$, where k and l tending to infinity independent of each other and L is called the P-limit of

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x. A double sequence $x = (x_{k,l})$ of real or complex numbers is said to be bounded if $\|x\| = sup_{k,l\geq 0}|x_{k,l}| < \infty$. Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. We may refer to [3–5, 8, 10, 17, 19, 21–25, 28, 30–32] for further results related with the concept of double sequence.

Başarır and Konca [6] defined a new concept of statistical convergence for single sequences which is called weighted lacunary statistical convergence. In [16] they defined weighted almost lacunary statistical convergence in a real n-normed space. Recently, the notion of weighted lacunary statistical convergence for single sequences is studied in a locally solid Riesz space by Başarır and Konca [7]. In this paper, we define and study weighted lacunary statistical τ-convergence, weighted lacunary statistical τ-bounded of double sequences in a locally solid Riesz space. We also prove some topological results related to these concepts in the framework of locally solid Riesz spaces. Further, we establish some inclusion relations.

2. Definitions and Preliminaries

Before beginning of the presentation of the main results, we recall the following basic facts and notations, and some of the basic concepts of Riesz spaces.

A topological vector space (X, τ) is a vector space, which has a linear topology τ, such that the algebraic operations of addition and scalar multiplication in X are continuous. Continuity of addition means that the function $f : X \times X \to X$ defined by $f(x, y) = x + y$ is continuous on $X \times X$, and continuity of scalar multiplication means that the function $f : \mathbb{C} \times X \to X$ defined by $f(\lambda, x) = \lambda x$ is continuous on $\mathbb{C} \times X$.

Let X be a real vector space and $\leq$ be a partial order on this space. Then X is said to be an ordered vector space if it satisfies the following properties:

1. If $x, y \in X$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in X$.
2. If $x, y \in X$ and $y \leq x$, then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

If in addition X is a lattice with respect to the partial ordering, then X is said to be a Riesz space (or a vector lattice). For an element x of a Riesz space X, the positive part of x by $x^+ = x \vee \theta$ $=\sup\{x, \theta\}$, the negative part of x by $x^- = (-x) \vee \theta$ and the absolute value of x by $|x| = x \vee (-x)$, where θ is the zero element of X [36].

A subset S of a Riesz space X is said to be solid if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.

A linear topology τ on a Riesz space X is said to be locally solid if τ has a base at zero consisting of solid sets. A locally solid Riesz space (X, τ) is a Riesz space equipped with a locally solid topology τ [27].

Every linear topology τ on a vector space X has a base $\mathcal{N}$ for the neighborhoods of θ satisfying the following properties:

(T1) Each $Y \in \mathcal{N}$ is a balanced set, that is, $\lambda x \in Y$ holds for all $x \in Y$ and every $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

(T2) Each $Y \in \mathcal{N}$ is an absorbing set, that is, for every $x \in X$, there exists $\lambda > 0$ such that $\lambda x \in Y$.

(T3) For each $Y \in \mathcal{N}$, there exists some $E \in \mathcal{N}$ with $E + E \subseteq Y$.

Throughout the paper, the symbol $\mathcal{N}_{sol}$ will stand for a base at zero consisting of solid sets and satisfying the conditions $(T1)$, $(T2)$, $(T3)$ in a locally solid topology.

Definition 2.1. ([3]) Let (p_n), $(\bar{p}_m)$ be sequences of positive numbers and $P_n = p_1 + p_2 + ... + p_n$, $\bar{P}_m = \bar{p}_1 + \bar{p}_2 + ... + \bar{p}_m$. Then the transformation given by

$$T_{n,m}(x) = \frac{1}{P_n \bar{P}_m} \sum_{k=1}^{n} \sum_{l=1}^{m} p_k \bar{p}_l x_{k,l}$$

is called the Riesz mean of double sequence $x = (x_{k,l})$. If P - $\lim_{\mathrm{n,m}} T_{n,m}(x) = L$, $L \in \mathbb{R}$, then the sequence $x = (x_{k,l})$ is said to be Riesz convergent to L. If $x = (x_{k,l})$ is Riesz convergent to L, then we write $\mathrm{P_R}$ - $\lim x = L$.

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $l_0 = 0$, $\bar{h}_s = l_s - l_{s-1} \to \infty$ as $s \to \infty$. Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \text{ and } l_{s-1} < l \le l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q}_s$ [31].

Using the notations of lacunary sequence and Riesz mean for double sequences, Başarır and Konca [17] have given a new definition:

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and let (p_k), $(\bar{p}_l)$ be sequences of positive real numbers such that $P_{k_r} := \sum_{k \in (0,k_r]} p_k$ and $\bar{P}_{l_s} := \sum_{l \in (0,l_s]} \bar{p}_l$. If the Riesz transformation of double sequences is RH-regular (it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit), then $\theta'_{r,s} = \{(P_{k_r}, \bar{P}_{l_s})\}$ is a double lacunary sequence, that is; $P_0 = 0$, $0 < P_{k_{r-1}} < P_{k_r}$ and $H_r = P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$ and $\bar{P}_0 = 0$, $0 < \bar{P}_{l_{s-1}} < \bar{P}_{l_s}$ and $\bar{H}_s = \bar{P}_{l_s} - \bar{P}_{l_{s-1}} \to \infty$ as $s \to \infty$.

Let $P_{k_{r,s}} = P_{k_r} \bar{P}_{l_s}$, $H_{r,s} = H_r \bar{H}_s$ and $I'_{r,s} = \{(k,l) : P_{k_{r-1}} < k \le P_{k_r} \text{ and } \bar{P}_{l_{s-1}} < l \le \bar{P}_{l_s}\}$, $Q_r = \frac{P_{k_r}}{P_{k_{r-1}}}$, $\bar{Q}_s = \frac{\bar{P}_{l_s}}{\bar{P}_{l_{s-1}}}$ and $Q_{r,s} = Q_r \bar{Q}_s$. If we take $p_k = 1$, $\bar{p}_l = 1$ for all k and l, then $H_{r,s}$, $P_{k_{r,s}}$, $Q_{r,s}$ and $I'_{r,s}$ reduce to $h_{r,s}$, $k_{r,s}$, $q_{r,s}$ and $I_{r,s}$.

Definition 2.2. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. The double number sequence $x = (x_{k,l})$ is said to be $S_{(R^2,\theta_{r,s})}$-P-convergent to L provided that for every $\varepsilon > 0$,

$$\mathrm{P} - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} \left|\left\{(k,l) \in I'_{r,s} : p_k \bar{p}_l \left|x_{k,l} - L\right| \ge \varepsilon\right\}\right| = 0.$$

In this case, we write $S_{(R^2,\theta_{r,s})}$-P-$\lim_{k,l\to\infty} x_{k,l} = L$.

Throughout the paper, we will use the limit notation in Pringsheim's sense P-$\lim_{r,s}$ instead of P-$\lim_{r,s\to\infty}$, for brevity.

3. Main Results

In this section, we define the concepts of weighted lacunary statistical τ-convergence, weighted lacunary statistical τ-bounded and weighted statistical τ-convergence for double sequences in the framework of locally solid Riesz spaces and prove some topological results related to these concepts. We also examine some inclusion relations.

Definition 3.1. Let (X, τ) be a locally solid Riesz space and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. Then, a double sequence $x = (x_{k,l})$ in X is said to be weighted lacunary statistically τ-convergent (or $S_{(R^2,\theta_{r,s})}(\tau)$-convergent) to the element $L \in X$ if for every τ-neighborhood U of zero, the set $K_U(H_{r,s}) = \{(k,l) \in \mathbb{N} \times \mathbb{N}, (k,l) \in I'_{r,s} : p_k \bar{p}_l (x_{k,l} - L) \notin U\}$ has weighted double lacunary τ-density zero or shortly $\delta_{(R^2,\theta_{r,s})}(K_U(H_{r,s})) = 0$, i.e.,

$$P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} |\{(k,l) \in I'_{r,s} : p_k \bar{p}_l (x_{k,l} - L) \notin U\}| = 0. \tag{1}$$

In this case, we write $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$. We denote the set of all weighted lacunary statistically τ-convergent double sequences by $S_{(R^2,\theta_{r,s})}(\tau)$.

If we take $p_k = 1$ and $\bar{p}_l = 1$ for all $k, l \in \mathbb{N}$ in (1), then we obtain the definition of lacunary statistical τ-convergence for double sequences (given as a special case in [19]), that is; the double sequence $x = (x_{k,l})$ is said to be lacunary statistically τ-convergent to $L \in X$, if for every (τ)-neighborhood U of zero

$$P - \lim_{r,s\to\infty} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : (x_{k,l} - L) \notin U\}| = 0. \tag{2}$$

Now, we give another new definition which is related with the Definition 3.1.

Definition 3.2. Let (X,τ) be a locally solid Riesz space, (p_k), $(\bar{p}_l)$ be sequences of positive numbers and $P_n = p_1 + p_2 + ... + p_n$, $\bar{P}_m = \bar{p}_1 + \bar{p}_2 + ... + \bar{p}_m$. Then a double sequence $x = (x_{k,l})$ in X is said to be weighted statistically τ-convergent (or $S_{R^2}(\tau)$-convergent) to the element $L \in X$ if for every τ-neighborhood U of zero, the set $K_U(P_n) = \{(k,l) \in \mathbb{N}\times\mathbb{N}, k \le P_n \text{ and } l \le P_m : p_k\bar{p}_l(x_{k,l} - L) \notin U\}$ has weighted τ-density zero or shortly, $\delta_{R^2}(K_U(P_n)) = 0$, i.e.,

$$P - \lim_{n,m\to\infty} \frac{1}{P_n\bar{P}_m} |\{k \le P_n \text{ and } l \le \bar{P}_m : p_k\bar{p}_l(x_{k,l} - L) \notin U\}| = 0. \tag{3}$$

In this case, we write $S_{R^2}(\tau)$-P-$\lim_{k,l} x_{k,l} = L$. We denote the set of all weighted statistically τ-convergent double sequences by $S_{R^2}(\tau)$.

We will investigate the relations between $S_{R^2}(\tau)$ and $S_{(R^2,\theta_{r,s})}$, later.

Now, we will introduce the notion of weighted double lacunary statistical τ-bounded in locally solid Riesz spaces. The notion of double lacunary statistical τ-bounded in locally solid Riesz spaces was introduced by Alotaibi et al. (see reference [4]).

Definition 3.3. Let (X,τ) be a locally solid Riesz space and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. A double sequence $x = (x_{k,l})$ in X is said to be weighted lacunary statistically τ-bounded or $S_{(R^2,\theta_{r,s})}(\tau)$-bounded if for every τ-neighborhood U of zero there exists some $\alpha > 0$ such that $M_U := \{(k,l) \in \mathbb{N}\times\mathbb{N} : \alpha p_k\bar{p}_l x_{k,l} \notin U\}$ has weighted double lacunary τ-density zero or $\delta_{(R^2,\theta_{r,s})}(M_U) = 0$, i.e.

$$P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} |\{(k,l) \in I'_{r,s} : \alpha p_k\bar{p}_l x_{k,l} \notin U\}| = 0.$$

Theorem 3.4. *Let (X,τ) be a Hausdorff locally solid Riesz space and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. Assume that $x = (x_{k,l})$ and $y = (y_{k,l})$ are two double sequences in X. Then the followings hold:*

1. *If $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$lim_{k,l}\, x_{k,l} = L_1$ and $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$lim_{k,l}\, x_{k,l} = L_2$ then $L_1 = L_2$.*
2. *If $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$lim_{k,l}\, x_{k,l} = L$, then $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$lim_{k,l}\, \alpha x_{k,l} = \alpha L$, $\alpha \in \mathbb{R}$.*
3. *If $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$lim_{k,l}\, x_{k,l} = L_1$ and $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$lim_{k,l}\, y_{k,l} = L_2$, then $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$lim_{k,l}\, (x_{k,l} + y_{k,l}) = L_1 + L_2$.*

Proof. 1. Suppose that $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L_1$ and $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L_2$. Let U be any τ-neighborhood of zero. Then, there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Choose any $E \in \mathcal{N}_{sol}$ such that $E + E \subseteq Y$. We define the following sets:

$$K_1 = \{(k,l) \in \mathbb{N}\times\mathbb{N}, (k,l) \in I'_{r,s} : p_k\bar{p}_l(x_{k,l} - L_1) \in E\},$$

$$K_2 = \{(k,l) \in \mathbb{N}\times\mathbb{N}, (k,l) \in I'_{r,s} : p_k\bar{p}_l(x_{k,l} - L_2) \in E\}.$$

Since $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L_1$ and $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L_2$, we have $\delta_{(R^2,\theta_{r,s})}(K_1) = \delta_{(R^2,\theta_{r,s})}(K_2) = 1$. Thus, $\delta_{(R^2,\theta_{r,s})}(K_1 \cap K_2) = 1$ and in particular $K_1 \cap K_2 \neq \emptyset$. Now, let $(k,l) \in K_1 \cap K_2$. Then

$$p_k\bar{p}_l(L_1 - L_2) = p_k\bar{p}_l(x_{k,l} - L_2) + p_k\bar{p}_l(L_1 - x_{k,l}) \in E + E \subseteq Y \subseteq U.$$

Hence, for every τ-neighborhood U of zero, we have $p_k\bar{p}_l(L_1 - L_2) \in U$. Since (p_k), $(\bar{p}_l)$ are sequences of positive reals and (X,τ) is Hausdorff, the intersection of all τ-neighborhoods U of zero is the singleton set $\{\theta\}$. Thus, we get $L_1 - L_2 = \theta$, i.e., $L_1 = L_2$.

2. Let U be an arbitrary τ-neighborhood of zero and $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Since $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$, we have

$$P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} |\{(k,l) \in I'_{r,s} : p_k\bar{p}_l(x_{k,l} - L) \in Y\}| = 1.$$

Since Y is balanced, $p_k\bar{p}_l(x_{k,l} - L) \in Y$ implies $\alpha p_k\bar{p}_l(x_{k,l} - L) \in Y$ for every $\alpha \in \mathbb{R}$ with $|\alpha| \le 1$. Hence

$$\{(k,l) \in I'_{r,s} : p_k\bar{p}_l(x_{k,l} - L) \in Y\} \subseteq \{(k,l) \in I'_{r,s} : \alpha p_k\bar{p}_l(x_{k,l} - L) \in Y\}$$

$$\subseteq \{(k,l) \in I'_{r,s} : \alpha p_k\bar{p}_l(x_{k,l} - L) \in U\}.$$

Thus, we obtain

$$P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} |\{(k,l) \in I'_{r,s} : \alpha p_k\bar{p}_l(x_{k,l} - L) \in Y\}| = 1,$$

for each τ-neighborhood U of zero. Now, let $|\alpha| > 1$ and $[|\alpha|]$ be the smallest integer greater than or equal to $|\alpha|$. There exists $E \in \mathcal{N}_{sol}$ such that $[|\alpha|]E \subseteq Y$. Since $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$, we have $\delta_{(R^2,\theta_{r,s})}(K) = 1$ where $K = \{(k,l) \in \mathbb{N}\times\mathbb{N}, (k,l) \in I'_{r,s} : p_k\bar{p}_l(x_{k,l} - L) \in E\}$. Then we have,

$$|\alpha p_k\bar{p}_l(x_{k,l} - L)| = |\alpha||p_k\bar{p}_l(x_{k,l} - L)| \leq [|\alpha|]|p_k\bar{p}_l(x_{k,l} - L)| \in [|\alpha|]E \subseteq Y \subseteq U.$$

Since the set Y is solid, we have $\alpha p_k\bar{p}_l(x_{k,l} - L) \in Y$ and this implies that $\alpha p_k\bar{p}_l(x_{k,l} - L) \in U$. Thus,

$$P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} |\{(k,l) \in I'_{r,s} : \alpha p_k\bar{p}_l(x_{k,l} - L) \in U\}| = 1,$$

for each τ-neighborhood U of zero. Hence, $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} \alpha x_{k,l} = \alpha L$ for every $\alpha \in \mathbb{R}$.

3. Let U be an arbitrary τ-neighborhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Choose E in $\mathcal{N}_{sol}$ such that $E + E \subseteq Y$. Since $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L_1$ and $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} y_{k,l} = L_2$, we have $S_{(R^2,\theta_{r,s})}(H_1) = 1 = S_{(R^2,\theta_{r,s})}(H_2)$ where

$$H_1 = \{(k,l) \in \mathbb{N}\times\mathbb{N}, (k,l) \in I'_{r,s} : p_k\bar{p}_l(x_{k,l} - L_1) \in E\},$$

$$H_2 = \{(k,l) \in \mathbb{N}\times\mathbb{N}, (k,l) \in I'_{r,s} : p_k\bar{p}_l(y_{k,l} - L_2) \in E\}.$$

Let $H = H_1 \cap H_2$. Hence, we have $\delta_{(R^2,\theta_{r,s})}(H) = 1$ and

$$p_k\bar{p}_l((x_{k,l} + y_{k,l}) - (L_1 + L_2)) = p_k\bar{p}_l(x_{k,l} - L_1) + p_k\bar{p}_l(y_{k,l} - L_2) \in E + E \subseteq Y \subseteq U.$$

Thus, we get

$$P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} |\{(k,l) \in I'_{r,s} : p_k\bar{p}_l((x_{k,l} + y_{k,l}) - (L_1 + L_2)) \in U\}| = 1.$$

Since U is arbitrary, we have $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} (x_{k,l} + y_{k,l}) = L_1 + L_2$. □

Theorem 3.5. *Let (X,τ) be a locally solid Riesz space and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. If a double sequence $x = (x_{k,l})$ is weighted lacunary statistically τ-convergent and $(p_{k,l})$ is bounded where $p_{k,l} := p_k\bar{p}_l$, then the sequence $x = (x_{k,l})$ is weighted lacunary statistically τ-bounded.*

Proof. Suppose that the double sequence $x = (x_{k,l})$ is weighted lacunary statistically τ-convergent to $L \in X$ and $(p_{k,l})$ is a bounded sequence. Let U be an arbitrary τ-neighborhood of zero. Then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Let us choose $E \in \mathcal{N}_{sol}$ such that $E + E \subseteq Y$. Since $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$, $\delta_{(R^2,\theta_{r,s})}(K) = 0$ where

$$K = \{(k,l) \in \mathbb{N}\times\mathbb{N}, (k,l) \in I'_{r,s} : p_k\bar{p}_l(x_{k,l} - L) \notin E\}.$$

Since E is absorbing, there exists $\lambda > 0$ such that $\lambda L \in E$. Let α be such that $0 < \alpha \leq 1$. Since $(p_{k,l})$ is bounded, then there exists a $M = \frac{\lambda}{\alpha} > 0$ such that $p_{k,l} = p_k\bar{p}_l \leq M$ for all $k, l \in \mathbb{N}$. Then, we can write $\alpha p_k\bar{p}_l \leq \lambda$ for all $k, l \in \mathbb{N}$. Since E is solid and $|\alpha p_k\bar{p}_l L| \leq |\lambda L|$, we have $\alpha p_k\bar{p}_l L \in E$. Since E is balanced, $p_k\bar{p}_l(x_{k,l} - L) \in E$ implies $\alpha p_k\bar{p}_l(x_{k,l} - L) \in E$. Then we have

$$\alpha p_k\bar{p}_l x_{k,l} = \alpha p_k\bar{p}_l(x_{k,l} - L) + \alpha p_k\bar{p}_l L \in E + E \subseteq Y \subseteq U$$

for each $(k,l) \in (\mathbb{N}\times\mathbb{N})\backslash K$. Thus,

$$P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} |(k,l) \in I'_{r,s} : \alpha p_k\bar{p}_l x_{k,l} \notin U| = 0.$$

Hence, the double sequence $(x_{k,l})$ is weighted lacunary statistically τ-bounded. □

Theorem 3.6. *Let (X,τ) be a locally solid Riesz space and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. If $x = (x_{k,l})$, $y = (y_{k,l})$ and $z = (z_{k,l})$ are sequences such that*

1. $x_{k,l} \leq y_{k,l} \leq z_{k,l}$ *for all* $k, l \in \mathbb{N}$.
2. $S_{(R^2,\theta_{r,s})}(\tau)$*-P-lim*$_{k,l} x_{k,l} = L = S_{(R^2,\theta_{r,s})}(\tau)$*-P-lim*$_{k,l}\ z_{k,l}$ *then* $S_{(R^2,\theta_{r,s})}(\tau)$*-P-lim*$_{k,l}\ y_{k,l} = L$.

Proof. Let U be an arbitrary τ-neighborhood of zero, then there exists $Y \in \mathcal{N}_{sol}$ such that $Y \subseteq U$. Choose $E \in \mathcal{N}_{sol}$ such that $E + E \subseteq Y$. From the condition (2), we have $\delta_{(R^2,\theta_{r,s})}(K_1) = 1 = \delta_{(R^2,\theta_{r,s})}(\tau)(K_2)$, where

$$K_1 = \{(k,l) \in \mathbb{N}\times\mathbb{N}, (k,l) \in I'_{r,s} : p_k\bar{p}_l(x_{k,l} - L) \in E\},$$

$$K_2 = \{(k,l) \in \mathbb{N}\times\mathbb{N}, (k,l) \in I'_{r,s} : p_k\bar{p}_l(z_{k,L} - L) \in E\}.$$

Also, we get $\delta_{(R^2,\theta_{r,s})}(K_1 \cap K_2) = 1$ and from (1) we have

$$p_k\bar{p}_l(x_{k,l} - L) \leq p_k\bar{p}_l(y_{k,l} - L) \leq p_k\bar{p}_l(z_{k,l} - L)$$

for all $k, l \in \mathbb{N}$. This implies that for all $(k,l) \in K_1 \cap K_2$, we get

$$|p_k\bar{p}_l(y_{k,l} - L)| \leq |p_k\bar{p}_l(z_{k,l} - L)| + |p_k\bar{p}_l(x_{k,l} - L)| \in E + E \subseteq Y.$$

Since Y is solid, we have $p_k\bar{p}_l(y_{k,l} - L) \in Y \subseteq U$. Thus,

$$P-\ \lim_{r,s\to\infty}\frac{1}{H_{r,s}}|\{(k,l) \in I'_{r,s} : p_k\bar{p}_l(y_{k,l} - L) \in U\}| = 1$$

for each τ-neighborhood U of zero. Hence, $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l}\ y_{k,l} = L$. This completes the proof of the theorem. □

Theorem 3.7. *Let $x = (x_{k,l})$ be a double sequence in a locally solid Riesz space (X,τ) and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. If liminf$_r$ $Q_r > 1$ and liminf$_s$ $\bar{Q}_s > 1$ then $S_{R^2}(\tau) \subseteq S_{(R^2,\theta_{r,s})}(\tau)$.*

Proof. Suppose that $\liminf_r Q_r > 1$ and $\limsup_s \bar{Q}_s > 1$, then there exists a $\delta > 0$ such that $Q_r \geq 1+\delta$ and $\bar{Q}_s \geq 1+\delta$ for sufficiently large values of r and s, which implies that $\frac{H_r}{P_{k_r}} = 1 - \frac{P_{k_{r-1}}}{P_{k_r}} = 1 - \frac{1}{Q_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\bar{H}_s}{\bar{P}_{l_s}} = 1 - \frac{P_{l_{s-1}}}{\bar{P}_{l_s}} = 1 - \frac{1}{\bar{Q}_s} \geq \frac{\delta}{1+\delta}$. Let $S_{(R^2)}(\tau)$-P-$\lim_{k,l}\ x_{k,l} = L$ and let U be an arbitrary (τ)-neighborhood of zero. Then for all $r > r_0$ and $s > s_0$, we have

$$\begin{aligned}
&\frac{1}{P_{k_r}\bar{P}_{l_s}}\left|\{k \leqslant P_{k_r} \text{ and } l \leqslant \bar{P}_{l_s} :\ p_k\bar{p}_l\ (x_{k,l} - L) \notin U\}\right| \\
&\quad\geqslant \frac{1}{P_{k_r}\bar{P}_{l_s}}|\{P_{k_{r-1}} < k \leqslant P_{k_r} \text{ and } \bar{P}_{l_{s-1}} < l \leqslant \bar{P}_{l_s} : p_k\bar{p}_l(x_{k,l} - L) \notin U\}| \\
&\quad= \frac{H_{r,s}}{P_{k_r}\bar{P}_{l_s}}\left(\frac{1}{H_{r,s}}\left|\{(k,l) \in I'_{r,s} :\ p_k\bar{p}_l\ (x_k - \zeta) \notin U\}\right|\right) \\
&\quad\geqslant \left(\frac{\delta}{1+\delta}\right)^2 \frac{1}{H_{r,s}}\left|\{(k,l) \in I'_{r,s} :\ p_k\bar{p}_l\ (x_{k,l} - L) \notin U\}\right|.
\end{aligned}$$

Since $S_{R^2}(\tau)$-P-$\lim_{k,l}\ x_{k,l} = L$, then the inequality above implies that $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l}\ x_{k,l} = L$. Hence, $S_{R^2}(\tau) \subseteq S_{(R^2,\theta_{r,s})}(\tau)$. □

Theorem 3.8. *Let (X,τ) be a locally solid Riesz space and $x = (x_{k,l})$ be a double sequence in X. For any double lacunary sequence $\theta_{r,s} = \{(k_r, l_s)\}$, if limsup$_r$ $Q_r < \infty$ and limsup$_s$ $\bar{Q}_s < \infty$, then $S_{(R^2,\theta_{r,s})}(\tau) \subseteq S_{R^2}(\tau)$.*

Proof. If $\limsup_r Q_r < \infty$ and $\limsup_s \bar{Q}_s < \infty$, then there exists a $K > 0$ such that $Q_r \leq K$ for all $r \in \mathbb{N}$ and $\bar{Q}_s \leq K$ for all $s \in \mathbb{N}$. Let $x \in S_{(R^2,\theta_{r,s})}(\tau)$ with $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$ and U be an arbitrary τ-neighborhood of zero. We write

$$N_{r,s} := \left|\left\{(k,l) \in I'_{r,s} :\ p_k\bar{p}_l\,(x_{k,l} - L) \notin U\right\}\right|. \tag{4}$$

By (4) and from the definition of weighted lacunary statistical convergence for double sequences, for a given $\varepsilon > 0$, there exist positive integers r_0 and s_0 such that $\frac{N_{r,s}}{H_{r,s}} < \frac{\varepsilon}{K^2}$ for all $r > r_0$ and s_0. Now, let

$$M := \max\{N_{r,s} : 1 \leq r \leq r_0 \text{ and } 1 \leq r \leq r_0\} \tag{5}$$

and let n and m be any integers satisfying $k_{r-1} < n \leq k_r$ and $l_{s-1} < m \leq l_s$. Thus, we have the following

$$\begin{aligned}
&\tfrac{1}{P_n\bar{P}_m}\left|\{k \leq P_n \text{ and } l \leq \bar{P}_m : p_k\bar{p}_l\,(x_{k,l} - L) \notin U\}\right| \\
&\leq \tfrac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\left|\{k \leq P_{k_r} \text{ and } l \leq \bar{P}_{l_s} : p_k\bar{p}_l\,(x_{k,l} - L) \notin U\}\right| \\
&= \tfrac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\sum_{t,u=1,1}^{r_0,s_0} N_{t,u} + \tfrac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\sum_{(r_0<t\leq r)\cup(s_0<u\leq s)} N_{t,u} \\
&\leq \tfrac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \tfrac{\varepsilon}{K^2}\tfrac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\sum_{(r_0<t\leq r)\cup(s_0<u\leq s)} H_{t,u} \\
&\leq \tfrac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \tfrac{\varepsilon}{K^2}\tfrac{P_{k_r}\bar{P}_{l_s}}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} = \tfrac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \tfrac{\varepsilon}{K^2}Q_r\bar{Q}_s \\
&\leq \tfrac{Mr_0s_0}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} + \varepsilon.
\end{aligned}$$

Since $P_{k_{r-1}} \to \infty$ and $\bar{P}_{l_{s-1}} \to \infty$ as $r, s \to \infty$, it follows that

$$\frac{1}{P_n\bar{P}_m}\left|\{k \leqslant P_n \text{ and } l \leqslant \bar{P}_m :\ p_k\bar{p}_l\,(x_{k,l} - L) \notin U\}\right| \to 0$$

as $n, m \to \infty$. Therefore $x \in S_{R^2}(\tau)$ with $S_{R^2}(\tau)$-P-$\lim_{k,l} x_{k,l} = L$. □

The following corollary is a result of Theorem 3.7 and Theorem 3.8.

Corollary 3.9. *Let (X,τ) be a locally solid Riesz space and $x = (x_{k,l})$ be a double sequence in X. For any lacunary sequence $\theta_{r,s} = \{(k_r, l_s)\}$, if $1 < \liminf_r Q_r \leq \limsup_r Q_r < \infty$ and $1 < \liminf_s \bar{Q}_s \leq \limsup_s \bar{Q}_s < \infty$, then $S_{(R^2,\theta_{r,s})}(\tau) = S_{R^2}(\tau)$ and $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = S_{R^2}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$.*

Theorem 3.10. *Let (X,τ) be a locally solid Riesz space and $x = (x_{k,l})$ be a double sequence in X. For any double lacunary sequence $\theta_{r,s} = \{(k_r, l_s)\}$, the following statements are true:*

1. *If $p_k \leq 1$ and $\bar{p}_l \leq 1$ for all $k, l \in \mathbb{N}$, then $S_{\theta_{r,s}}(\tau) \subseteq S_{(R^2,\theta_{r,s})}(\tau)$ and $S_{\theta_{r,s}}(\tau)$- P -$\lim_{k,l} x_{k,l} = S_{(R^2,\theta_{r,s})}(\tau)$- P-$\lim_{k,l}$ $x_{k,l} = L$.*
2. *If $1 \leq p_k$ and $1 \leq \bar{p}_l$ for all $k, l \in \mathbb{N}$ and $(\frac{H_r}{h_r})$, $(\frac{\bar{H}_s}{\bar{h}_s})$ are upper bounded, then $S_{(R^2,\theta_{r,s})}(\tau) \subseteq S_{\theta_{r,s}}(\tau)$ with $S_{(R^2,\theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = S_{\theta_{r,s}}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$.*

Proof. 1. If $p_k \leq 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l \leq 1$ for all $l \in \mathbb{N}$, then $H_r \leq h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s \leq \bar{h}_s$ for all $s \in \mathbb{N}$. So, there exist M_1 and M_2 constants such that $0 < M_1 \leq \frac{H_r}{h_r} \leq 1$ for all $r \in \mathbb{N}$ and $0 < M_2 \leq \frac{\bar{H}_s}{\bar{h}_s} \leq 1$ for all $s \in \mathbb{N}$.

Let $x = (x_{k,l})$ be a double sequence in a locally solid Riesz space (X, τ) and assume that $S_{\theta_{r,s}}(\tau)$-P-$\lim_{k,l} x_{k,l} = L$. Let U be an arbitrary τ-neighborhood of zero, then we have

$$\frac{1}{H_{r,s}} \left| \left\{ (k,l) \in I'_{r,s} : p_k \bar{p}_l \left(x_{k,l} - L \right) \notin U \right\} \right|$$

$$= \frac{1}{H_r \bar{H}_s} \left| \left\{ P_{k_{r-1}} < k \leq P_{k_r} \text{and} \bar{P}_{l_{s-1}} < l \leq \bar{P}_{l_s} : p_k \bar{p}_l \left(x_{k,l} - L \right) \notin U \right\} \right|$$

$$\leq \frac{1}{M_1 M_2} \frac{1}{h_r \bar{h}_s} \left| \left\{ k_{r-1} < k \leq k_r \text{and} l_{s-1} < l \leq l_s : \left(x_{k,l} - L \right) \notin U \right\} \right|$$

$$= \frac{1}{M_{1,2}} \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \left(x_{k,l} - L \right) \notin U \right\} \right|$$

where $M_{1,2} = M_1 M_2$. Hence, we obtain the result by taking the P-limit as $r, s \to \infty$.

2. If $1 \leq p_k$ and $1 \leq \bar{p}_l$ for all $k, l \in \mathbb{N}$, then $h_r \leq H_r$ and $\bar{h}_s \leq \bar{H}_s$ for all $r, s \in \mathbb{N}$. Since $(\frac{H_r}{h_r})$, $(\frac{\bar{H}_s}{\bar{h}_s})$ are upper bounded, there exist N_1 and N_2 constants such that $1 \leq \frac{H_r}{h_r} \leq N_1 < \infty$ and $1 \leq \frac{H_s}{h_s} \leq N_2 < \infty$ for all $r, s \in \mathbb{N}$. Assume that $x \in S_{(R^2, \theta_{r,s})}(\tau)$ with $S_{(R^2, \theta_{r,s})}(\tau)$-$P$-$\lim_{k,l} x_{k,l} = L$. Let U be an arbitrary τ-neighborhood of zero. Then we have

$$\frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \left(x_{k,l} - L \right) \notin U \right\} \right|$$

$$= \frac{1}{h_r \bar{h}_s} \left| \left\{ k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s : \left(x_{k,l} - L \right) \notin U \right\} \right|$$

$$\leq N_1 N_2 \frac{1}{H_r \bar{H}_s} \left| \left\{ P_{k_{r-1}} < k \leq P_{k_r} \text{ and } \bar{P}_{l_{s-1}} < l \leq \bar{P}_{l_s} : p_k \bar{p}_l \left(x_{k,l} - L \right) \notin U \right\} \right|$$

$$= \frac{1}{N_{1,2}} \frac{1}{H_{r,s}} \left| \left\{ (k,l) \in I'_{r,s} : p_k \bar{p}_l \left(x_{k,l} - L \right) \notin U \right\} \right|$$

where $N_{1,2} = N_1 N_2$. Hence, the result is obtained by taking the P-limit as $r, s \to \infty$. □

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