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Uniform Cantor Sets as Hyperbolic Boundaries

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Abstract. In this paper, we prove that the Gromov hyperbolic space $(\mathcal{X}, h)$ which was introduced by Z. Ibragimov and J. Simanyi in [3] is an asymptotically PT_{-1} space and extend the methods of [3] to the case of uniform Cantor sets, show that the uniform Cantor set is isometric to the Gromov hyperbolic boundary at infinity of some asymptotically PT_{-1} space.

1. Introduction

Hyperbolization is an important process to convert a geometry object into a metric space with non-positive curvature in the sense of Gromov. Several such processess are described by many authors in the past several years. For examples, in the paper [4], Ka-Sing Lau and Xiang-Yang Wang proved that, for an iterated function system $\{S_j\}_{j=1}^{N}$ of similitudes that satisfies the open set condition, there is a natural graph structure in the representing symbolic space to make it a hyperbolic graph in the sense of Gromov, and the Gromov hyperbolic boundary at infinity is homeomorphic to the self-similar set generated by $\{S_j\}_{j=1}^{N}$. The result of [4] has been generalized by Jun Jason Luo in [5] to the Moran set case. In [5], he proved that a Moran set is homeomorphic to the Gromov hyperbolic boundary at infinity of the representing symbolic space. In the complex dynamics context, V.Nekrashevych obtained that the Julia sets of postcritically finite rational maps arise as Gromov hyperbolic boundaries at infinity in [7]. In [3], Z. Ibragimov and J. Simanyi considered the hyperbolization of the ternary Cantor set and presented a new construction of the ternary Cantor set within the context of Gromov hyperbolic geometry and proved that the ternary Cantor set is isometric to the hyperbolic boundary of some Gromov hyperbolic space $(\mathcal{X}, h)$.

Recently, in order to generalize the well studied relation between the geometry of the classical hyperbolic space and the Möbius geometry of its Gromov hyperbolic boundary at infinity to CAT(-1) space case,

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R.Miao and V.Schroeder defined the asymptotically PT_{-1} space and proved that the asymptotically PT_{-1} space is a Gromov hyperbolic space in [6], but has better properties than the Gromov hyperbolic space. For examples, the asymptotically PT_{-1} space is boundary continuous and its Gromov hyperbolic boundary is a Ptolemy space under the visual metric. Since the asymptotically PT_{-1} space has better properties than the Gromov hyperbolic space, thus it is interesting to determine whether a Gromov hyperbolic space is an asymptotically PT_{-1} space.

This paper is motivated by the above result of Z. Ibragimov and J. Simanyi. We prove that the Gromov hyperbolic space $(\mathcal{X}, h)$ is an asymptotically PT_{-1} space and extend the methods of the paper [3] to the uniform Cantor set case. We prove that the uniform Cantor set is isometric to the hyperbolic boundary of some asymptotically PT_{-1} space, which generalizes the results of [3] to the uniform Cantor set case.

2. Uniform Cantor Set and Gromov Hyperbolic Space

Firstly, we define the uniform Cantor set, which is a class of more general Cantor type set, has abundant exotic fractal structure and has been the object of a series of papers [1, 2].

Let $E_0 = [0,1]$. Let $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers and $\mathbf{c} = \{c_k\}_{k=1}^{\infty}$ be a sequence of real numbers in $(0,1)$, such that $n_k c_k < 1$ for all k. Suppose that $\{E_k\}$ be a nested sequence of closed sets in $[0,1]$ satisfying the following conditions:

(1) For every $k \geq 1$, E_k is the union of disjoint closed intervals of the same length.

(2) For every component interval I in E_{k-1} contains n_k+1 component intervals of E_k. These n_k+1 intervals are of the same spacing $c_k|I|$, the leftmost one and I have the same left endpoint, and the rightmost one and I have the same right endpoint.

The set

$$E \doteq E(\mathbf{n}, \mathbf{c}) = \bigcap_{k=0}^{\infty} E_k$$

is called a uniform Cantor set. Obviously, if $c_k = \frac{1}{3}$ and $n_k = 1$ for all k, then $E(\mathbf{n}, \mathbf{c})$ is the ternary Cantor set.

Denote by N_k the number of components of E_k and by δ_k their common length. From the definition, we obtain

$$N_k = \prod_{i=1}^{k}(n_i+1) \quad \text{and} \quad \delta_k = \prod_{i=1}^{k} \frac{1-n_i c_i}{n_i+1} = \delta_{k-1}\frac{1-n_k c_k}{n_k+1}.$$

Secondly, we begin with a brief discussion of Gromov hyperbolic spaces. Let (X, d) be a metric space. For $x, y, z \in X$, the Gromov product of x and y with respect to z is defined by

$$(x|y)_z = \frac{1}{2}[d(x,z) + d(y,z) - d(x,y)].$$

Definition 2.1. *A metric space (X, d) is called Gromov δ-hyperbolic if there is a $\delta \geq 0$ such that*

$$(x|y)_v \geq \min\{(x|z)_v, (z|y)_v\} - \delta$$

for all $x, y, z, v \in X$.

Each Gromov hyperbolic space X has a Gromov hyperbolic boundary at infinity ∂X (also called the Gromov boundary or the hyperbolic boundary). Fix a base point $v \in X$. We say that a sequence $\{a_i\}$ of points in X converges to infinity if

$$\lim_{i,j\to\infty} (a_i|a_j)_v = \infty.$$

It is easy to see that this definition does not depend on the choice of a base point. We say that two sequences $\{a_i\}$ and $\{b_i\}$ converging to infinity are equivalent and write $\{a_i\} \sim \{b_i\}$ if

$$\lim_{i\to\infty}(a_i|b_i)_v = \infty.$$

Once again, one can show that $\sim$ is an equivalence relation on the sequences converging to infinity and that the definition of the equivalence does not depend on the choice of the base point $v \in X$.

Definition 2.2. *Let (X, d) be a Gromov δ-hyperbolic space. The Gromov boundary ∂X of X is defined to be the equivalence classes of sequences converging to infinity.*

The Gromov boundary supports a family of so-called visual metrics. A metric d on ∂X is called a visual metric if there is a $v \in X, C \geq 1$ and $\epsilon > 0$ such that for all $x, y \in \partial X$,

$$\frac{1}{C}\rho_{\epsilon,v}(x, y) \leq d(x, y) \leq C\rho_{\epsilon,v}(x, y),$$

where $\rho_{\epsilon,v}(x, y) = e^{-\epsilon(x|y)_v}$ and $(x|y)_v$ is the Gromov product on ∂X defined by

$$(x|y)_v = \inf\{\liminf_{i\to\infty}(a_i|b_i)_v : \{a_i\} \in x, \{b_i\} \in y\}.$$

Here we set $e^{-\infty} = 0$. The visual metric on the Gromov boundary of any Gromov hyperbolic is bounded and complete.

3. Hyperbolic Construction

Let $E = E(\mathbf{n}, \mathbf{c})$ be a uniform Cantor set. Let $F = [0, 1] \setminus E$ and $\mathcal{X}$ be the collection of all connected components of F, that is $\mathcal{X}$ is the collection of all open intervals which are removed. Hence

$$E = [0, 1] \setminus \bigcup_{I\in\mathcal{X}} I.$$

Our goal in this section is to construct a metric h on the set $\mathcal{X}$ such that $(\mathcal{X}, h)$ is a Gromov hyperbolic space.

Now, we consider the upper Hausdorff distance u_H on the set of all nonempty subsets of $[0, 1]$ defined by

$$u_H(A, B) = \sup\{|x - y| : x \in A, y \in B\}.$$

for $A, B \subset [0, 1]$. Note that if $A, B \in \mathcal{X}$ and $A = (a, b), B = (c, d)$ with $b < c$, then $u_H(A, B) = |d - a|$. Now, and in what follows, we use $l(A)$ to denote the Euclidean length of $A \in \mathcal{X}$ and $a \vee b = \max\{a, b\}$ for positive numbers $a, b \in \mathbb{R}$. For $A, B \in \mathcal{X}$, we obtain

$$u_H(A, B) \geq l(A) \vee l(B) \geq \sqrt{l(A) \cdot l(B)}, \tag{1}$$

where the first equality holds only if $A = B$ and the second equality holds only if $l(A) = l(B)$.

According to the definition $\mathcal{X}$, it has a natural ordered $\preceq$. We say that $I \preceq J$ if I is to the left of J or $I = J$. Obviously, if $I \preceq J \preceq K$, then

$$u_H(I, K) \geq u_H(I, J) \vee u_H(J, K). \tag{2}$$

In order to obtain our result, we need the following lemma from [3].

Lemma 3.1 ([3]). *For all $I, J, K, L \in \mathcal{X}$, we have*

$$u_H(I, J)u_H(K, L) \leq u_H(I, K)u_H(J, L) + u_H(I, L)u_H(J, K). \tag{3}$$

Given $I, J \in \mathcal{X}$, define

$$h(I, J) = 2 \log \frac{u_H(I, J)}{\sqrt{l(I) \cdot l(J)}}. \tag{4}$$

When E is the ternary Cantor set, Z. Ibragimov and J. Simanyi proved that $(\mathcal{X}, h)$ is a Gromov hyperbolic metric space in the paper [3].

Definition 3.2. *A metric space (X, d) is called asymptotically* PT_{-1}*, if there exists some $\delta > 0$ such that for all quadruples $x_1, x_2, x_3, x_4 \in X$, we have*

$$e^{\frac{1}{2}(\rho_{13}+\rho_{24})} \leq e^{\frac{1}{2}(\rho_{12}+\rho_{34})} + e^{\frac{1}{2}(\rho_{14}+\rho_{23})} + \delta e^{\frac{1}{2}\rho},$$

where $\rho_{ij} = d(x_i, x_j)$ and $\rho = \max_{i,j} \rho_{ij}$.

Using Lemma 3.1, we obtain the following theorem, which provides an example of the asymptotically PT_{-1} space.

Theorem 3.3. *The metric space $(\mathcal{X}, h)$ is an asymptotically* PT_{-1} *space.*

Proof. Given arbitrary four points $I_1, I_2, I_3, I_4 \in \mathcal{X}$, we have

$$e^{\frac{\rho_{ij}}{2}} = e^{\frac{h(I_i, I_j)}{2}} = \frac{u_H(I_i, I_j)}{\sqrt{l(I_i) \cdot l(I_j)}}, i \neq j.$$

Direct calculation gives

$$u_H(I_i, I_j) = \sqrt{l(I_i) \cdot l(I_j)} e^{\frac{\rho_{ij}}{2}}, i \neq j.$$

By Lemma 3.1, we obtain

$$u_H(I_1, I_3)u_H(I_2, I_4) \leq u_H(I_1, I_2)u_H(I_3, I_4) + u_H(I_1, I_4)u_H(I_2, I_3).$$

That is

$$\begin{aligned} \sqrt{l(I_1) \cdot l(I_3)} e^{\frac{\rho_{13}}{2}} \sqrt{l(I_2) \cdot l(I_4)} e^{\frac{\rho_{24}}{2}} &\leq \sqrt{l(I_1) \cdot l(I_2)} e^{\frac{\rho_{12}}{2}} \sqrt{l(I_3) \cdot l(I_4)} e^{\frac{\rho_{34}}{2}} \\ &+ \sqrt{l(I_1) \cdot l(I_4)} e^{\frac{\rho_{14}}{2}} \sqrt{l(I_2) \cdot l(I_3)} e^{\frac{\rho_{23}}{2}}, \end{aligned} \tag{5}$$

i.e.

$$e^{\frac{\rho_{13}}{2}}e^{\frac{\rho_{24}}{2}} \le e^{\frac{\rho_{12}}{2}}e^{\frac{\rho_{34}}{2}} + e^{\frac{\rho_{14}}{2}}e^{\frac{\rho_{23}}{2}}. \tag{6}$$

So we obtain

$$e^{\frac{\rho_{13}+\rho_{24}}{2}} \le e^{\frac{\rho_{12}+\rho_{34}}{2}} + e^{\frac{\rho_{14}+\rho_{23}}{2}}. \tag{7}$$

Thus the metric space $(\mathcal{X}, h)$ is an asymptotically PT_{-1} space. $\square$

Remark 3.4. *In the paper [6], the authors proved that an asymptotic* PT_{-1} *space is a Gromov hyperbolic space and is boundary continuous and* $\rho_o(x,y) = e^{-(x|y)_o}, x, y \in \partial X$ *is a metric on* ∂X *such that* $(\partial X, \rho_o(x,y))$ *is a Ptolemy space.*

4. The Boundary at Infinity and Main Result

According to Theorem 3.3, the metric space $(\mathcal{X}, h)$ is an asymptotically PT_{-1} space, thus is a Gromov hyperbolic space. In this section, we will construct a visual metric d on the Gromov boundary $\partial\mathcal{X}$. Our goal is to prove that the space $(\partial\mathcal{X}, d)$ is isometric to the uniform Cantor set $(E, |\cdot|)$. By the definition of $\partial\mathcal{X}$, we know that $\partial\mathcal{X}$ is the collection of equivalence classes of sequence in $\mathcal{X}$ at infinity. Firstly, we fix a interval $V = (t, w) \in \mathcal{X}$ to be the base point. Notice that if the sequence $\{I_n\}$ converges at infinity, then $\lim_{j,k\to\infty}(I_j|I_k)_V = \infty$.

Our main result is the following theorem.

Theorem 4.1. *The uniform Cantor set* $(E, |\cdot|)$ *is isometric to the metric space* $(\partial\mathcal{X}, d)$.

In order to obtain our theorem, we need several lemmas. The following lemma shows that there is a bijective map between $\partial\mathcal{X}$ and the uniform Cantor set E.

Lemma 4.2. *Given* $a \in \partial\mathcal{X}$, *there exists unique* $x_a \in E$ *such that*

$$\lim_{n\to\infty} u_H(I_n, \{x_a\}) = 0 \text{ for each } \{I_n\} \in a.$$

Conversely, for each $x \in E$ *there exists* $a \in \partial\mathcal{X}$ *such that*

$$\lim_{n\to\infty} u_H(J_n, \{x_a\}) = 0 \text{ for each } \{J_n\} \in a.$$

Proof. Given $\{I_n\} \in a$, we obtain

$$\begin{aligned}(I_j|I_k)_V &= \frac{1}{2}[h(I_j, V) + h(I_k, V) - h(I_j, I_k)] \\ &= \log \frac{u_H(I_j, V)u_H(I_k, V)}{l(V)u_H(I_j, I_k)} \\ &\le \log \frac{(w \vee (1-t))^2}{(w-t)u_H(I_j, I_k)}.\end{aligned}$$

Since

$$\lim_{j,k\to\infty}(I_j|I_k)_V = \infty,$$

we have

$$\lim_{j,k\to\infty} u_H(I_j, I_k) = 0.$$

Now, for each n, we choose some point $x_n \in I_n$. Next, given $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$|x_j - x_k| \le u_H(I_j, I_k) < \epsilon \text{ whenever } j, k \ge n_0,$$

which shows that the sequence $\{x_n\}$ is a Cauchy sequence in $[0, 1]$. Because $[0, 1]$ is complete, thus the sequence $\{x_n\}$ converges to some point in $[0, 1]$, call it x_a. We claim that the point x_a is well-defined. In order to prove this claim, we choose a different sequence $\{y_n\}$, where $y_n \in I_n$, then

$$|y_n - a| \le |x_n - a| + |x_n - y_n| \le u_H(I_n, I_n) + |x_n - a|,$$

which implies that $\{y_n\}$ also converges to x_a and shows that the point x_a is well-defined. Finally, since

$$u_H(I_n, \{x_a\}) \le u_H(I_n, \{x_n\}) + u_H(\{x_a\}, \{x_n\}) \le u_H(I_n, I_n) + |x_n - x_a|,$$

we obtain that $\lim_{n\to\infty} u_H(I_n, \{x_a\}) = 0$, as required.

Now let $\{J_n\}$ be another sequence converging at infinity in a. Then we claim that $\lim_{n\to\infty} u_H(J_n, \{x_a\}) = 0$. Since $\{I_n\}, \{J_n\} \in a$, according to the definition of boundary, we have

$$\lim_{n\to\infty}(I_n|J_n)_V = \infty.$$

Note that

$$\begin{aligned}
(I_n|J_n)_V &= \frac{1}{2}[h(I_n, V) + h(J_n, V) - h(I_n, J_n)] \\
&= \log \frac{u_H(I_n, V)u_H(J_n, V)}{l(V)u_H(I_n, J_n)} \\
&\le \log \frac{(w \vee (1-t))^2}{(w-t)u_H(I_n, J_n)}.
\end{aligned}$$

Thus, we obtain $\lim_{n\to\infty} u_H(I_n, J_n) = 0$. Since

$$u_H(J_n, \{x_a\}) \le u_H(I_n, \{x_a\}) + u_H(J_n, I_n),$$

we obtain that $\lim_{n\to\infty} u_H(J_n, \{x_a\}) = 0$. This shows that x_a is well defined and unique.

We claim that $x_a \in E$. Assume by contrary that $x \in [0, 1] \setminus E$. Thus there is a element $I = (e_-, e_+) \in \mathcal{X}$ such that $x_a \in I$. Since

$$0 < |x_a - e_-| \wedge |x_a - e_+| \le u_H(I_n, \{x_a\}) \to 0 \quad \text{as} \quad n \to \infty,$$

which yields a contraction. Thus $x_a \in E$, which shows that the first part of the lemma holds.

Now, we address the second part. At first, we show that for each $x \in E$, there is a sequence $\{I_n\}$ in $\mathcal{X}$ converging at infinity and such that

$$\lim_{n\to\infty} u_H(I_n, \{x\}) = 0.$$

Since $x \in E = \bigcap_{k=0}^{\infty} E_k$. For every integer number $k > 1$, let $I_k(x)$ denote the connected component of E_k that contains x and $F_k = [0,1] \setminus E_k$. According to the definition, if $0, 1 \notin I_k(x)$, there are two connected components in F_k on the two sides of $I_k(x)$. Let $I_k^R(x)$ denote the connected component of F_k on the right side of $I_k(x)$ and $I_k^L(x)$ denote the connected component of F_k on the left side of $I_k(x)$. Define

$$J_k(x) = \begin{cases} I_k^R(x) & \text{when } |I_k^L(x)| \geq |I_k^R(x)|, \\ I_k^L(x) & \text{when } |I_k^R(x)| > |I_k^L(x)|. \end{cases} \tag{8}$$

If $I_k(x)$ contains 0 (or 1), let $J_k(x)$ denote the unique connected component in F_k on the right (or left) side of $I_k(x)$. Thus according to the definitions of $I_k(x)$ and $J_k(x)$, we have

$$|I_k(x)| = \delta_k \text{ and } |J_k(x)| = c_k\delta_{k-1}.$$

Since $0 < c_k < 1$ and

$$\delta_k = \delta_{k-1}\frac{1 - n_k c_k}{n_k + 1} \leq \frac{\delta_{k-1}}{n_k + 1} \leq \frac{\delta_{k-1}}{2} < \delta_{k-1},$$

we obtain

$$\begin{aligned} u_H(\{x\}, J_k(x)) &\leq |I_k(x)| + |J_k(x)| \\ &= \delta_k + c_k\delta_{k-1} \\ &\leq 2\delta_{k-1} \\ &\dots \\ &\leq \frac{2\delta_1}{2^{k-1}} \to 0 \end{aligned} \tag{9}$$

as $k \to \infty$, which shows that $\lim_{k\to\infty} u_H(\{x\}, J_k(x)) = 0$ as required. Since

$$u_H(J_n(x), J_k(x)) \leq u_H(\{x\}, J_n(x)) + u_H(\{x\}, J_k(x)),$$

we obtain

$$\lim_{n,k\to\infty} u_H(J_n(x), J_k(x)) = 0.$$

Thus

$$(J_n(x)|J_k(x))_V = \log \frac{u_H(J_n(x), V)u_H(J_k(x), V)}{l(V)u_H(I_n(x), J_k(x))} \to \infty,$$

which shows that the sequence $\{J_k(x)\}$ converges at infinity.

Finally, we let $a \in \mathcal{X}$ to be the equivalence class of sequences converging at infinity and equivalent to $\{I_n\}$. Then it follows from the first part that

$$\lim_{n\to\infty} u_H(a, J_n) = 0 \quad \text{for} \quad \text{each} \quad J_n \in a.$$

This completes the proof of the lemma. □

According to the above lemma, we define a map $f : \partial\mathcal{X} \to E$, given by $f(a) = x_a$. Using the map f, we define a metric d on $\partial\mathcal{X}$ by setting $d(a,b) = |f(a) - f(b)| = |x_a - x_b|$. Our result is the following lemma.

Lemma 4.3. *The metric d is a visual metric on $\partial\mathcal{X}$.*

Proof. Fix $V = (t, w) \in \mathcal{X}$. For any $a, b \in \partial\mathcal{X}$,

$$(a|b)_V = \inf\{\liminf_{n\to\infty}(I_n|J_n)_V : I_n \in a, J_n \in b\}.$$

Since

$$(I_n|J_n)_V = \log \frac{u_H(I_n, V)u_H(J_n, V)}{l(V)u_H(I_n, J_n)},$$

by Lemma 4.2, we have

$$\lim_{n\to\infty} u_H(I_n, \{x_a\}) = 0 \quad \text{and} \quad \lim_{n\to\infty} u_H(J_n, \{x_b\}) = 0.$$

Especially, since

$$|u_H(I_n, J_n) - |x_a - x_b|| \le u_H(I_n, \{x_a\}) + u_H(J_n, \{x_b\}),$$

we obtain

$$\lim_{n\to\infty} u_H(I_n, J_n) = |x_a - x_b| = d(a, b).$$

Since

$$|u_H(V, I_n) - u_H(V, \{x_a\})| \le u_H(I_n, \{x_a\}),$$

we obtain

$$\lim_{n\to\infty} u_H(I_n, V) = u_H(V, \{x_a\}).$$

Similarly

$$\lim_{n\to\infty} u_H(J_n, V) = u_H(V, \{x_b\}).$$

Thus, we have

$$(a|b)_V = \log \frac{u_H(V, \{x_a\})u_H(V, \{x_b\})}{l(V)d(a, b)},$$

which implies that

$$d(a, b) = e^{-(a|b)_V} \frac{u_H(V, \{x_a\})u_H(V, \{x_b\})}{l(V)}.$$

Since $l(V) = w - t$ and $w - t \le u_H(V, \{x\}) \le w \vee 1 - t$ for all $x \in E$, we obtain

$$(w - t)e^{-(a|b)_V} \le d(a, b) \le \frac{(w \vee 1 - t)^2}{w - t} e^{-(a|b)_V} \le \frac{1}{w - t} e^{-(a|b)_V},$$

which shows that d is a visual metric on $\partial\mathcal{X}$. This completes the proof of the lemma. □

Proof of Theorem 4.1. According to the above Lemma 4.2 and Lemma 4.3, the map $f : \partial\mathcal{X} \to E$, given by $f(a) = x_a$ is an isometric map. This completes the proof of the theorem 4.1. □

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