Some Additive and Multiplicative Results for Generalized Inverses

Jovana Nikolov Radenković

 Faculty of Sciences and Mathematics, University of Niš, Serbia

Abstract. In this paper we give necessary and sufficient conditions for \(A_1[1,3] + A_2[1,3] + \ldots + A_k[1,3] \subseteq (A_1 + A_2 + \ldots + A_k)[1,3]\) and \(A_1[1,4] + A_2[1,4] + \ldots + A_k[1,4] \subseteq (A_1 + A_2 + \ldots + A_k)[1,4]\) for regular operators on Hilbert space. We also consider similar inclusions for \([1,2,3]-\) and \([1,2,4]-\) inverses. We give some new results concerning the reverse order law for reflexive generalized inverses.

1. Introduction

Let \(H, K, L\) and \(I\) be complex Hilbert spaces and let \(\mathcal{B}(H, K)\) denote the set of all bounded linear operators from \(H\) to \(K\). For a given \(A \in \mathcal{B}(H, K)\), the symbols \(\mathcal{N}(A)\) and \(\mathcal{R}(A)\) denote the null space and the range of operator \(A\), respectively. The identity operator on \(H\) is denoted by \(I_H\). For an operator \(A\), by \(A^{-1}\) we denote the left (right) inverse of \(A\) and by \(\mathcal{B}_r^{-1}(H, K)\) the set of all left (right) invertible operators from the space \(\mathcal{B}(H, K)\). \(E_A\) and \(F_A\) stand for two orthogonal projectors \(E_A = I_K - AA^\dagger\) and \(F_A = I_H - A^\dagger A\). For given sets \(M, N\), by \(MN\) or \(M \cdot N\) we denote the set consisting of all products \(XY\), where \(X \in M\) and \(Y \in N\).

Recall that \(A \in \mathcal{B}(H, K)\) has a Moore-Penrose inverse if there exists an operator \(X \in \mathcal{B}(K, H)\) such that

\[
(1) \quad AXA = A \quad (2) \quad XAX = X \quad (3) \quad (AX)^* = AX \quad (4) \quad (XA)^* = XA. \tag{1}
\]

Moore-Penrose inverse of an operator \(A \in \mathcal{B}(H, K)\) exists if and only if \(A\) has a closed range and in this case it is unique and is denoted by \(A^\dagger\). If for \(A \in \mathcal{B}(H, K)\) there exists a Moore-Penrose inverse, we say that \(A\) is regular operator.

For \(A \in \mathcal{B}(H, K)\), let \(A[i, j, \ldots, k]\) denote the set of all operators \(X \in \mathcal{B}(K, H)\) which satisfy equations (i), (j), \(\ldots, (k)\) from among equations (1)–(4) of (1). In this case \(X\) is a \([i, j, \ldots, k]-\)inverse of \(A\) which we denote by \(A^{[i, j, \ldots, k]}\).

Let \(A_i \in \mathcal{B}(H, K), i = 1, k\). We give necessary and sufficient conditions for the following inclusions

\[
A_1[1,3] + A_2[1,3] + \ldots + A_k[1,3] \subseteq (A_1 + A_2 + \ldots + A_k)[1,3],
\]

\[
A_1[1,4] + A_2[1,4] + \ldots + A_k[1,4] \subseteq (A_1 + A_2 + \ldots + A_k)[1,4],
\]

\[
(A_1 + A_2 + \ldots + A_k)[1,3] \subseteq A_1[1,3] + A_2[1,3] + \ldots + A_k[1,3].
\]

2010 Mathematics Subject Classification. Primary 15A09
Keywords. additive; generalized inverse; \([1,3]-\)inverse; \([1,4]-\)inverse; reflexive generalized inverse; reverse order law
Received: 12 June 2015; Accepted: 17 June 2015
Communicated by Dragana Cvetković-Ilić
Supported by Grant No. 174007 of the Ministry of Science, Technology and Development, Republic of Serbia
Email address: jovana.nikolov@gmail.com (Jovana Nikolov Radenković)
Proof. (which is satisfied by (V)).

\[ A_1 + A_2 + \ldots + A_k \] \subseteq A_1 + A_2 + \ldots + A_k \]

and

\[ A_1, A_2, \ldots, A_k \] \subseteq (A_1 + A_2 + \ldots + A_k) \]

In [2] authors considered necessary and sufficient conditions for \( B[1, 2] \cdot A[1, 2] \subseteq (AB)[1, 2] \) in the case of bounded linear operators on Hilbert space. We derived necessary and sufficient conditions for the inclusion \( C[1, 2] \cdot B[1, 2] \cdot A[1, 2] \subseteq (ABC)[1, 2] \).

2. Results

It is well-known that for \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) and \( B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) regular,

\[ B \in A[1, 3] \quad \iff \quad A^*AB = A^*, \]

\[ B \in A[1, 4] \quad \iff \quad BAA^* = A^*, \]

and that sets of all \((1, 3)\)- and \((1, 4)\)-inverses of \( A \) are described by

\[ A[1, 3] = \{ A^* + F_A V : V \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \}, \]

\[ A[1, 4] = \{ A^* + VE_A : V \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \}. \]

Also, for \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) and \( B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) regular,

\[ B \in A[1, 2, 3] \quad \iff \quad (A^*AB = A^* \land BAA^+ = B), \]

\[ B \in A[1, 2, 4] \quad \iff \quad (BAA^* = A^* \land A^*AB = B), \]

and sets of all \((1, 2, 3)\) - and \((1, 2, 4)\)-inverses of \( A \) are described by

\[ A[1, 2, 3] = \{ A^* + F_A VAA^+ : V \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \}, \]

\[ A[1, 2, 4] = \{ A^* + A^*AVE_A : V \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \}. \]

Theorem 2.1. Let \( A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) be such that \( A_i, i = 1, 2, \ldots, k \) and \( A = A_1 + A_2 + \ldots + A_k \) are regular operators. The following conditions are equivalent:

(i) \( A_1[1, 3] + A_2[1, 3] + \ldots + A_k[1, 3] \subseteq (A_1 + A_2 + \ldots + A_k)[1, 3] \),

(ii) \( A^*AF_{A_i} = 0, i = \overline{1, k} \). \( A^*A \sum_{i=1}^k A_i^+ = A^* \).

Proof. (ii) \( \Rightarrow \) (i): Suppose that (ii) holds. We need to prove that for arbitrary \( A^{(1,3)}_i \in A_i[1, 3], i = \overline{1, k} \), it follows that \( A^{(1,3)}_1 + A^{(1,3)}_2 + \ldots + A^{(1,3)}_k \in A[1, 3] \). Thus, given any \( V_i \in \mathcal{B}(\mathcal{K}, \mathcal{H}), i = \overline{1, k} \), we must show that

\[ A^*A \left( \sum_{i=1}^k A_i^+ + \sum_{i=1}^k F_A V_i \right) = A^* \]

which is satisfied by (ii).

(i) \( \Rightarrow \) (ii): If (i) holds, then for any \( V_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}), i = \overline{1, k} \), we have that (6) holds. Specially, for \( V_i = 0, \)

\[ i = \overline{1, k} \], by (6) we get that \( A^*A \sum_{i=1}^k A_i^+ = A^* \). Similarly, if for any \( i \in \{1, 2, \ldots, k\} \), we take that \( V_i = F_A \) and that \( V_j = 0, j \neq i \), by (6) we will get that \( A^*AF_{A_i} = 0 \). Hence, (ii) holds. \( \Box \)

In the following theorem we present the necessary and sufficient condition for

\[ A_1 + A_2 + \ldots + A_k \subseteq (A_1 + A_2 + \ldots + A_k)[1, 3] \]
Theorem 2.2. Let $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $A_i, i = 1, 2, \ldots, k, A = A_1 + A_2 + \ldots + A_k$ and $C = \begin{bmatrix} F_{A_1} & F_{A_2} & \ldots & F_{A_k} \end{bmatrix}$ are regular operators. The following conditions are equivalent:

(i) $(A_1 + A_2 + \ldots + A_k)[1, 3] \subseteq A_1[1, 3] + A_2[1, 3] + \ldots + A_k[1, 3],$

(ii) $CC^t F_A = F_A, CC^t (A^t - \sum_{i=1}^{k} A_i^t) = A^t - \sum_{i=1}^{k} A_i^t.$

Proof. (ii) $\Rightarrow$ (i): Suppose that (ii) holds. We need to prove that for arbitrary $V \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ there exist $V_i \in \mathcal{B}(\mathcal{K}, \mathcal{H}), i = 1, k$ such that

$$A^t + F_A V = \sum_{i=1}^{k} A_i^t + \sum_{i=1}^{k} F_{A_i} V_i,$$

i.e.

$$\begin{bmatrix} F_{A_1} & F_{A_2} & \ldots & F_{A_k} \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_k \end{bmatrix} = F_A V + A^t - \sum_{i=1}^{k} A_i^t.$$  \hspace{1cm} (7)

Hence, to show (i) we need to prove that the equation (7) is solvable for any $V \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ which holds if and only if

$$CC^t (F_A V + A^t - \sum_{i=1}^{k} A_i^t) = F_A V + A^t - \sum_{i=1}^{k} A_i^t.$$  \hspace{1cm} (8)

Obviously, (8) is satisfied by (ii).

(i) $\Rightarrow$ (ii): If (i) is satisfied, then (8) holds for any $V \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Taking $V = 0$ and $V = F_A$ in (8), we get that the both equalities from (ii) hold. \quad $\square$

Theorem 2.3. Let $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $A_i, i = 1, 2, \ldots, k, A = A_1 + A_2 + \ldots + A_k$ and $D = \begin{bmatrix} E_{A_1} & E_{A_2} & \ldots & E_{A_k} \end{bmatrix}$ are regular operators. The following conditions are equivalent:

(i) $(A_1 + A_2 + \ldots + A_k)[1, 4] \subseteq A_1[1, 4] + A_2[1, 4] + \ldots + A_k[1, 4],$

(ii) $DD^t E_A = E_A, (A^t - \sum_{i=1}^{k} A_i^t)DD^t = A^t - \sum_{i=1}^{k} A_i^t.$

Proof. For arbitrary matrix $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we have from (2) that $B \in A[1, 4] \Leftrightarrow B^* \in A^*[1, 3]$ , so it follows that $(A[1, 4])^* = A^*[1, 3]$. Now, condition (i) is equivalent to

$$((A_1 + A_2 + \ldots + A_k)[1, 4])^* \subseteq (A_1[1, 4])^* + (A_2[1, 4])^* + \ldots + (A_k[1, 4])^*$$

$$\Leftrightarrow (A_1 + A_2 + \ldots + A_k)[1, 3] \subseteq A_1'[1, 3] + A_2'[1, 3] + \ldots + A_k'[1, 3]$$

$$\Rightarrow (A_1' + A_2' + \ldots + A_k')[1, 3] \subseteq A_1'[1, 3] + A_2'[1, 3] + \ldots + A_k'[1, 3]$$

which is from the Theorem 2.2 equivalent to

$$CC^t F_{A'} = F_{A'}, CC^t ((A')^t - \sum_{i=1}^{k} (A_i')^t) = (A')^t - \sum_{i=1}^{k} (A_i')^t,$$  \hspace{1cm} (9)

where $C = \begin{bmatrix} F_{A_1'} & F_{A_2'} & \ldots & F_{A_k'} \end{bmatrix}$. Since $F_{A'} = E_A$ and $F_{A_i'} = E_{A_i}$, it is easy to see that (9) is equivalent to (ii).

In an analogous way, necessary and sufficient condition for the opposite inclusion can be derived from the Theorem 2.1.

Theorem 2.4. Let $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $A_i, i = 1, 2, \ldots, k$ and $A = A_1 + A_2 + \ldots + A_k$ are regular operators. The following conditions are equivalent:
The following conditions are equivalent:

(i) \( A_1[1,4] + A_2[1,4] + ... + A_k[1,4] \subseteq (A_1 + A_2 + ... + A_k)[1,4] \),

(ii) \( AA^*E_{A_i} = 0, \ i = \overline{1,k} \) \( (\sum_{i=1}^{k} A_i^*)AA^* = A^* \).

**Theorem 2.5.** Let \( A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) be such that \( A_i \), \( i = 1,2, \ldots, k \) and \( A = A_1 + A_2 + ... + A_k \) are regular operators. The following conditions are equivalent:

(i) \( A_1[1,2,3] + A_2[1,2,3] + ... + A_k[1,2,3] \subseteq (A_1 + A_2 + ... + A_k)[1,2,3] \),

(ii) \( A^*AF_{A_i} = 0, \ i = \overline{1,k}, \ A^*A\sum_{i=1}^{k} A_i^* = A^*, \ \sum_{i=1}^{k} A_i^*F_{A_i} = 0, \) and \( F_{A_i} = 0 \) or \( A_iA_i^*F_{A_i} = 0 \) for every \( i \in \{1, \ldots, k\} \).

**Proof.** (ii) \( \Rightarrow \) (i): Suppose that (ii) holds. We need to prove that for arbitrary \( A_1^{(1,2,3)} \in A_i[1,2,3], \ i = \overline{1,k} \) it follows that \( A_1^{(1,2,3)} + A_2^{(1,2,3)} + ... + A_k^{(1,2,3)} \in A[1,2,3] \). Thus, given any \( V_i \in \mathcal{B}(\mathcal{K}, \mathcal{H}), i = \overline{1,k} \), we must show that

\[
A^*A\left(\sum_{i=1}^{k} A_i^* + \sum_{i=1}^{k} F_{A_i}V_iA_iA_i^*\right) = A^*,
\]

and

\[
\left(\sum_{i=1}^{k} A_i^* + \sum_{i=1}^{k} F_{A_i}V_iA_iA_i^*\right)(I - AA^*) = 0,
\]

which is satisfied by (ii).

(i) \( \Rightarrow \) (ii): If (i) holds, then for any \( V_i \in \mathcal{B}(\mathcal{K}, \mathcal{H}), i = \overline{1,k} \), we have that (10) and (11) hold. Specially, for \( V_i = 0, i = \overline{1,k} \), by (10) we get that \( A^*A\sum_{i=1}^{k} A_i^* = A^* \) and by (11) we get that \( \sum_{i=1}^{k} A_i^*F_{A_i} = 0 \). Similarly, if for any \( i \in \{1,2, \ldots, k\} \), we take that \( V_i = F_{A_i} \) and that \( V_i = 0, j \neq i \), by (10) we will get that \( A^*AF_{A_i} = 0 \) and by (11) we will get that \( F_{A_i} = 0 \) or \( A_iA_i^*F_{A_i} = 0 \). Hence, (ii) holds. \( \Box \)

**Theorem 2.6.** Let \( A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) be such that \( A_i \), \( i = 1,2, \ldots, k \) and \( A = A_1 + A_2 + ... + A_k \) are regular operators. The following conditions are equivalent:

(i) \( A_1[1,2,4] + A_2[1,2,4] + ... + A_k[1,2,4] \subseteq (A_1 + A_2 + ... + A_k)[1,2,4] \),

(ii) \( AA^*E_{A_i} = 0, \ i = \overline{1,k}, \ (\sum_{i=1}^{k} A_i^*)AA^* = A^*, \ \sum_{i=1}^{k} A_i^*A_i^* = 0, \) and \( E_{A_i} = 0 \) or \( E_{A_i}A_i^*A_i = 0 \) for every \( i \in \{1, \ldots, k\} \).

**Proof.** For arbitrary matrix \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) we have from (4) that \( B \in A[1,2,4] \Leftrightarrow B^* \in A^*[1,2,3] \), so it follows that \( (A[1,2,4])^* = A^*[1,2,3] \). Now, condition (i) is equivalent to

\[
(A_1[1,2,4])^* + \ldots + (A_k[1,2,4])^* \subseteq ((A_1 + A_2 + ... + A_k)[1,2,4])^*
\]

\[
\Leftrightarrow A_1[1,2,3] + \ldots + A_k[1,2,3] \subseteq (A_1 + A_2 + ... + A_k)[1,2,3]
\]

\[
\Leftrightarrow A_1[1,2,3] + \ldots + A_k[1,2,3] \subseteq (A_1^* + A_2^* + ... + A_k^*)[1,2,3]
\]

which is from the Theorem 2.5 equivalent to

\[
AA^*F_{A_i} = 0, \ i = \overline{1,k}, \ AA^*\sum_{i=1}^{k} (A_i^*)^* = A,
\]

\[
\sum_{i=1}^{k} (A_i^*)^*F_{A_i} = 0,
\]

\[
F_{A_i} = 0 \text{ or } A_i^*A_iF_{A_i} = 0, \text{ for every } i \in \{1, \ldots, k\}
\]

where \( A = A_1 + A_2 + ... + A_k \). Since \( F_{A_i} = E_{A_i} \) and \( F_{A_i} = E_{A_i} \), it is easy to see that (12) is equivalent to (ii).
Theorem 2.7. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ and $C \in \mathcal{B}(I, \mathcal{L})$ be such that $A, B, C, AB, BC$ and $ABC$ are regular operators. The following conditions are equivalent:

(i) $C[1,2] \cdot B[1,2] \cdot A[1,2] \subseteq (ABC)[1,2]$, 

(ii) $A = 0$ or $B = 0$ or $C = 0$, or $A \in \mathcal{B}^{-1}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}^{-1}(\mathcal{L}, \mathcal{H})$, or $B \in \mathcal{B}^{-1}(\mathcal{L}, \mathcal{H})$ and $C \in \mathcal{B}^{-1}(I, \mathcal{L})$, or $A \in \mathcal{B}^{-1}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}^{-1}(I, \mathcal{L})$.

Proof. (i) $\Rightarrow$ (ii): If (i) holds, then evidently $C^+B^+A^+ \in (ABC)[1,2]$, so

$$ABCC^+B^+A^+ABC = ABC$$

and

$$C^+B^+A^+ABCC^+B^+A^+ = C^+B^+A^+.$$  \hspace{1cm} (13)

Since, for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L}), A^+ + (I_H - A^+)XAA^+ \in A[1,2]$ and $B^+ + B^+BY(I_H - BB^+) \in B[1,2]$, we get that

$$ABCC^+(B^+ + B^+BY(I_H - BB^+))(A^+ + (I_H - A^+)XAA^+)ABC = ABC$$

holds for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L})$. For $X = 0$ in (15) we get

$$ABCC^+(B^+ + B^+BY(I_H - BB^+))A^+ABC = ABC.$$ \hspace{1cm} (16)

Substracting (16) from (15) we get that

$$ABCC^+(B^+ + B^+BY(I_H - BB^+))(I_H - A^+)XAA^+ABC = 0$$

holds for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L})$. If we take $Y = 0$ in (17) we get

$$ABCC^+B^+I_H - A^+)XAA^+ABC = 0.$$ \hspace{1cm} (18)

Substracting (18) from (17) we get that

$$ABCC^+B^+BY(I_H - BB^+)I_H - A^+)XABC = 0$$

holds for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, so it follows that

$$ABCC^+B^+ = 0 \text{ or } (I_H - BB^+)(I_H - A^+) = 0 \text{ or } ABC = 0 \text{ which is from (13) equivalent to}$$

$$(I_H - BB^+)(I_H - A^+) = 0 \text{ or } ABC = 0.$$ \hspace{1cm} (20)

Similarly, for any $Y \in \mathcal{B}(\mathcal{H}, \mathcal{L}), Z \in \mathcal{B}(\mathcal{L}, I), B^+ + (I_L - B^+)YBB^+ \in B[1,2]$ and $C^+ + C^+CZ(I_L - CC^+) \in C[1,2]$, so

$$ABC(C^+ + C^+CZ(I_L - CC^+))(B^+ + (I_L - B^+)YBB^+)A^+ABC = ABC.$$ \hspace{1cm} (21)

In analogous way as above, by taking $Y = 0$ and $Z = 0$ we will obtain

$$(I_L - CC^+)(I_L - B^+)B = 0 \text{ or } ABC = 0.$$ \hspace{1cm} (21)
Since for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $Z \in \mathcal{B}(\mathcal{L}, I)$, $A^\dagger + (I_H - A^\dagger A)XAA^\dagger \in A_{1,2}$ and $C^\dagger + C^\dagger CZ(I_L - CC^\dagger) \in C_{1,2}$, we have that

\begin{align*}
ABC(C^\dagger + C^\dagger CZ(I_L - CC^\dagger))B^\dagger \\
(A^\dagger + (I_H - A^\dagger A)XAA^\dagger) \cdot ABC = ABC.
\end{align*}

(22)

Subtracting $X = 0$ in (22) we get

\begin{equation}
ABC(C^\dagger + C^\dagger CZ(I_L - CC^\dagger))B^\dagger A^\dagger ABC = ABC.
\end{equation}

(23)

Subtracting (23) from (22) we get that

\begin{equation}
ABC(C^\dagger + C^\dagger CZ(I_L - CC^\dagger))B^\dagger (I_H - A^\dagger A)XAA^\dagger ABC = 0
\end{equation}

(24)

holds for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Z \in \mathcal{B}(\mathcal{L}, I)$. If we take $Z = 0$ in (24) we get

\begin{equation}
ABCC^\dagger B^\dagger (I_H - A^\dagger A)XAA^\dagger ABC = 0
\end{equation}

(25)

Subtracting (25) from (24) we get that

\begin{equation}
ABCZ(I_L - CC^\dagger)B^\dagger (I_H - A^\dagger A)XABC = 0
\end{equation}

(26)

holds for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Z \in \mathcal{B}(\mathcal{L}, I)$, so it follows that

\begin{equation}
(I_L - CC^\dagger)B^\dagger (I_H - A^\dagger A) = 0 \quad \text{or} \quad ABC = 0.
\end{equation}

(27)

Suppose that $ABC \neq 0$. Then from (20),(21) and (27) we have

\begin{align*}
(I_H - BB^\dagger)(I_H - A^\dagger A) = 0, \\
(I_L - CC^\dagger)(I_L - B^\dagger B) = 0, \\
(I_L - CC^\dagger)B^\dagger (I_H - A^\dagger A) = 0.
\end{align*}

(28)

Equalities (28) are equivalent to

\begin{align*}
I_H - BB^\dagger - A^\dagger A - BB^\dagger A^\dagger A = 0, \\
I_L - CC^\dagger - B^\dagger B - CC^\dagger B^\dagger B = 0, \\
B^\dagger - CC^\dagger B^\dagger - B^\dagger A^\dagger A + CC^\dagger B^\dagger A^\dagger A = 0.
\end{align*}

(29)  (30)  (31)

Since for any $Z \in \mathcal{B}(\mathcal{L}, I)$, $C^\dagger + C^\dagger CZ(I_L - CC^\dagger) \in C_{1,2}$, we have that

\begin{align*}
(C^\dagger + C^\dagger CZ(I_L - CC^\dagger))B^\dagger A^\dagger ABC(C^\dagger + C^\dagger CZ(I_L - CC^\dagger))B^\dagger A^\dagger &= (C^\dagger + C^\dagger CZ(I_L - CC^\dagger))B^\dagger A^\dagger.
\end{align*}

which is from (28) equivalent to

\begin{equation}
C^\dagger B^\dagger A^\dagger ABC(C^\dagger + C^\dagger CZ(I_L - CC^\dagger))B^\dagger A^\dagger = (C^\dagger + C^\dagger CZ(I_L - CC^\dagger))B^\dagger A^\dagger.
\end{equation}

(32)

Since (32) holds for any $Z \in \mathcal{B}(\mathcal{L}, I)$, we get

\begin{equation}
(I_L - CC^\dagger)B^\dagger A^\dagger = 0 \quad \text{or} \quad C^\dagger B^\dagger A^\dagger ABC = C^\dagger C.
\end{equation}

(33)

Since for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $A^\dagger + (I_H - A^\dagger A)XAA^\dagger \in A_{1,2}$, we have that
\[ C^i B^i(A^i + (I_H - A^i A)XAA^i)ABC^i B^i(A^i + (I_H - A^i A)XAA^i) \]

\[ = C^i B^i(A^i + (I_H - A^i A)XAA^i) \]

which us from (28) equivalent to

\[ C^i B^i(A^i + (I_H - A^i A)XAA^i)ABCC^i B^i A^i \]

\[ = C^i B^i(A^i + (I_H - A^i A)XAA^i). \]  \hspace{1cm} (34)

Since (34) holds for any \( X \in B(K, H) \), using (14) we get

\[ ABCC^i B^i A^i = AA^i \quad \text{or} \quad C^i B^i(I_H - A^i A) = 0. \]  \hspace{1cm} (35)

Suppose now that the first equality in (33) is satisfied. From (31) we get that \( B^i - CC^i B^i = 0 \), which is from (30) equivalent to \( CC^i = I_L \), i.e. \( C \in B^{-1}_L(I, L) \). If first equality in (35) is satisfied, then from \( CC^i = I_L \) we get \( AB^i A^i = AA^i \) i.e. \( BB^i A^i A = A^i A \) which from (29) implies \( BB^i = I_H \), i.e. \( B \in B^{-1}_L(L, H) \). Since \( ABC \neq 0 \) by assumption and \( B \in B^{-1}_L(L, H) \) and \( C \in B^{-1}_L(I, L) \), (ii) obviously holds. If second equality in (35) is satisfied, then multiplying it by \( C \) from the left we get \( B^i(I_H - A^i A) = 0 \), i.e. \( BB^i(I_H - A^i A) = 0 \) which from (29) implies \( A^i A = I_H \), i.e. \( A \in B^{-1}_L(H, K) \). Since \( A \in B^{-1}_L(H, K) \) and \( C \in B^{-1}_L(I, L) \), (ii) holds.

Suppose now that the second equality in (33) is satisfied. If the second equality in (35), i.e. \( C^i B^i = C^i B^i A^i A \) is satisfied, then we get \( C^i B^i BC = C^i C \), i.e. \( CC^i B^i B = CC^i \) which using (30) implies \( BB^i = I \), i.e. \( B \in B^{-1}_L(L, H) \). Substituting \( C^i B^i = C^i B^i A^i A \) in (31) we get

\[ B^i - CC^i B^i - B^i A^i A + CC^i B^i = 0. \]

Multiplying last equality by \( B \) from the right and using \( BB^i = I \) we get \( B^i A^i AB = I \), i.e. \( A^i ABB^i = BB^i \).

Now from (29) we obtain \( A^i A = I \), i.e. \( A \in B^{-1}_L(H, K) \). Since \( A \in B^{-1}_L(H, K) \) and \( B \in B^{-1}_L(L, H) \) and \( ABC \neq 0 \) by assumption, (ii) holds. Now, suppose that first equality in (35) is satisfied. Multiplying (31) by \( BC \) from the right we get

\[ B^i BC - CC^i B^i BC - B^i A^i ABC + CC^i B^i A^i ABC = 0. \]  \hspace{1cm} (36)

Substituting \( C^i B^i A^i ABC = C^i C \) in (36) and using \( CC^i B^i B = B^i BC^i \) we get

\[ -B^i A^i ABC + C = 0. \]  \hspace{1cm} (37)

Multiplying (37) by \( B \) from the left we get

\[ A^i ABC = BC. \]  \hspace{1cm} (38)

Substituting (38) in \( C^i B^i A^i ABC = C^i C \) we get \( C^i B^i BC = C^i C \), i.e. \( CC^i B^i B = CC^i \). Now from (30) we get \( B^i B = I \), i.e. \( B \in B^{-1}_L(L, H) \). Multiplying (31) by \( AB \) from the left and using \( CC^i B^i B = B^i BC^i \) we get

\[ ABCC^i B^i = ABCC^i B^i A^i A. \]  \hspace{1cm} (39)

Substituting \( ABCC^i B^i A^i = AA^i \) in (39) we get

\[ ABCC^i B^i = AA^i. \]  \hspace{1cm} (40)

Multiplying (40) by \( B \) from the left and using \( BB^i = I \) we get

\[ ABCC^i B^i = AB. \]  \hspace{1cm} (41)

Multiplying (41) by \( B^i A^i \) from the right and using \( ABCC^i B^i A^i = AA^i \) we get

\[ ABB^i A^i = AA^i, \]
i.e. 
\[ A^TABB^t = A^T A. \]

Now from (29) we have \( BB^t = I \), i.e. \( B \in B_{\mathcal{B}}(L, H) \). Since \( B \in B_{\mathcal{B}}(L, H) \), it follows that \( B \) is invertible, i.e. \( B \in B_{\mathcal{B}}^{-1}(L, H) \). Now, \( B[1, 2] = \{ B^{-1} \} \) and it is easy to see that \((AB)[1, 2] = B^{-1} A[1, 2] \). Now, \( C[1, 2] : B[1, 2] \cdot A[1, 2] \subseteq (ABC)[1, 2] \) is equivalent to \( C[1, 2] : (AB)[1, 2] \subseteq (ABC)[1, 2] \). From [2] we have that this is satisfied if and only if \( AB \in B_{\mathcal{B}}^{-1}(L, \mathcal{K}) \) or \( C \in B_{\mathcal{B}}^{-1}(I, L) \). If \( AB \in B_{\mathcal{B}}^{-1}(L, \mathcal{K}) \) then 
\[ N(AB) = \{0\}. \] (42)

Since 
\[
\begin{align*}
N(A) &= N(A) \cap \mathcal{H} \\
&= N(A) \cap \mathcal{R}(B) \\
&= N(A) \cap \mathcal{R}(B) \cup N(B) \\
&= N(AB) \\
&= \{0\},
\end{align*}
\]

it follows that \( A^T A = I \), i.e. \( A \in B_{\mathcal{B}}^{-1}(H, \mathcal{K}) \). Since \( A \in B_{\mathcal{B}}^{-1}(H, \mathcal{K}) \) and \( B \in B_{\mathcal{B}}^{-1}(L, H) \), (ii) is satisfied.

Now, let us consider the case \( ABC = 0 \). It is obvious that \((ABC)[1, 2] = \{0\} \). From [2] we have that \((AB)[1, 2] \subseteq B[1, 2] A[1, 2] \) always holds so \( C[1, 2] (AB)[1, 2] \subseteq C[1, 2] B[1, 2] A[1, 2] \subseteq (ABC)[1, 2] \). From [2] this is satisfied if and only if 
\[
\begin{align*}
AB &= 0, \quad (43) \\
\text{or } C &= 0, \quad (44) \\
\text{or } AB &\in B_{\mathcal{B}}^{-1}(L, \mathcal{K}), \quad (45) \\
\text{or } C &\in B_{\mathcal{B}}^{-1}(I, L). \quad (46)
\end{align*}
\]

Case 1. Suppose that (43) is satisfied. Let \( A[1, 2] \in A[1, 2] \) and \( B[1, 2] \in B[1, 2] \) be arbitrary. Then 
\[ C^T B[1, 2] A[1, 2] \subseteq (ABC)[1, 2] = \{0\} \], i.e. 
\[ C^T B[1, 2] A[1, 2] = 0. \]

Since for arbitrary \( Z \in B(L, I) \), 
\[ C^T + C^T C Z (L - CC^T) \in C[1, 2] \] we have that 
\[ (C^T + C^T C Z (L - CC^T) B[1, 2] A[1, 2] = 0, \] (47)

is satisfied for any \( Z \in B(L, I) \). Since \( C^T B[1, 2] A[1, 2] = 0 \) from (47) we get that 
\[ C^T C Z B[1, 2] A[1, 2] = 0 \] (48)

holds for any \( Z \in B(L, I) \). From (48) we have that either \( C = 0 \) or \( B[1, 2] A[1, 2] = 0 \). If \( C \neq 0 \) it follows that \( B[1, 2] A[1, 2] = 0 \). Now we have \( B[1, 2] A[1, 2] = 0 \) so from [2] we have that \( A = 0 \) or \( B = 0 \) or \( A \in B_{\mathcal{B}}^{-1}(H, \mathcal{K}) \) or \( B \in B_{\mathcal{B}}^{-1}(H, \mathcal{K}) \). If \( A \in B_{\mathcal{B}}^{-1}(H, \mathcal{K}) \) then from \( AB = 0 \) it follows that \( B = 0 \). If \( B \in B_{\mathcal{B}}^{-1}(H, \mathcal{K}) \) then from \( AB = 0 \) we get that \( A = 0 \).

Case 2. In this case we have \( C = 0 \).

Case 3. Suppose that (45) is satisfied. Then from \( ABC = 0 \) it follows that \( C = 0 \).

Case 4. Suppose that (46) holds. Then from \( ABC = 0 \) it follows that \( AB = 0 \) which is the same as Case 1. So \( AB = 0 \) implies \( A = 0 \) or \( B = 0 \) or \( C = 0 \).

(ii) \( \Rightarrow (i) \): If \( A \) or \( B \) or \( C \) is zero, it is evident that (i) holds.

Suppose that \( A \in B_{\mathcal{B}}^{-1}(H, \mathcal{K}) \) and \( B \in B_{\mathcal{B}}^{-1}(L, H) \). Then since \( B \in B_{\mathcal{B}}^{-1}(L, H) \) it follows from [2] that \( C[1, 2] B[1, 2] \subseteq (BC)[1, 2] \). Since \( A \in B_{\mathcal{B}}^{-1}(H, \mathcal{K}) \) from [2] it follows that \((BC)[1, 2] A[1, 2] \subseteq (ABC)[1, 2] \). Now we have 
\[ C[1, 2] B[1, 2] A[1, 2] \subseteq (BC)[1, 2] A[1, 2] \subseteq (ABC)[1, 2]. \]

The rest can be proved in the same manner. \( \Box \)
References