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On Properties of Multiplication Conditional Type Operators between L^p-Space

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Abstract. In this paper, first we give some necessary and sufficient conditions for multiplication conditional type operators between two L^p-spaces to have closed range. Then we investigate Fredholm ones when the underlying measure space is non-atomic. Finally we give some examples.

1. Introduction and Preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space. For any σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ with $1 \leq p \leq \infty$, the L^p-space $L^p(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on X by $L^0(\Sigma)$.

For a σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative function f as well as for all $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, where $E^{\mathcal{A}} f$, by the Radon-Nikodym theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is an idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. If there is no possibility of confusion we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. Let $f \in L^0(\Sigma)$. Then f is said to be conditionable with respect to E if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$. Throughout this paper we take u and w in $\mathcal{D}(E)$. This operator will play a major role in our discussion and we list here some of its useful properties:

- If g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.
- $|E(f)|^p \leq E(|f|^p)$.
- If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- $|E(fg)| \leq E(|f|^p)|^{\frac{1}{p}} E(|g|^{p'})|^{\frac{1}{p'}}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (Hölder inequality).
- For each $f \geq 0$, $S(E(f))$ is the smallest $\mathcal{A}$-measurable set such that $S(f) \subseteq S(E(f))$.

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A detailed discussion and verification of most of these properties may be found in [11]. We recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \mathcal{A}$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space (X, Σ, μ) with no atoms is called non-atomic measure space. It is well-known fact that every σ-finite measure space $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ can be partitioned uniquely as $X = (\bigcup_{n\in\mathbb{N}} A_n) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and B, being disjoint from each A_n, is non-atomic (see [14]).

Compositions of conditional expectation operators and multiplication operators appear often in the study of other operators such as multiplication operators and weighted composition operators. Specifically, in [10], S.-T. C. Moy characterized all operators on L^p of the form $f \to E(fg)$ for g in L^q with $E(|g|)$ bounded. Eleven years later, R. G. Douglas, [5], analyzed positive projections on L^1 and many of his characterizations are in terms of combinations of multiplications and conditional expectations. P.G. Dodds, C.B. Huijsmans and B. De Pagter, [1], extended these characterizations to the setting of function ideals and vector lattices. J. Herron presented some assertions about the operator EM_u on L^p spaces in [7]. Also, some results about multiplication conditional type operators can be found in [6, 8]. In [2–4] we investigated some classic properties of multiplication conditional type operators M_wEM_u on L^p spaces. In this paper, some necessary and sufficient conditions for closeness of range of multiplication conditional type operators between two L^p-spaces are given. Also, Fredholm ones are characterized when the underlying measure space is non-atomic.

Now we give a definition of multiplication conditional type operators on L^p-spaces.

Definition 1.1. Let (X, Σ, μ) be a σ-finite measure space and let $\mathcal{A}$ be a σ-subalgebra of Σ such that $(X, \mathcal{A}, \mu_{\mathcal{A}})$ is also σ-finite. Let E be the conditional expectation operator relative to $\mathcal{A}$. If $1 \le p, q < \infty$ and $u, w \in L^0(\Sigma)$ (the spaces of Σ-measurable functions on X) such that uf is conditionable and $wE(uf) \in L^q(\Sigma)$ for all $f \in \mathcal{D} \subseteq L^p(\Sigma)$, where $\mathcal{D}$ is a linear subspace, then the corresponding multiplication conditional type operator is the linear transformation $M_wEM_u : \mathcal{D} \to L^q(\Sigma)$ defined by $f \to wE(uf)$.

The results of [1] state that our results are valid for a large class of linear operators, since for finite measure space (X, Σ, μ), we have $L^\infty(\Sigma) \subseteq L^p(\Sigma) \subseteq L^1(\Sigma)$ and $L^p(\Sigma)$ is an order ideal of measurable functions on (X, Σ, μ).

2. Closed Range and Fredholm Multiplication Conditional Type Operators

In this section first we describe closed range multiplication conditional type operators M_wEM_u between two L^p-spaces. Let $1 \le p, q < \infty$ and $f \in L^p$ such that $wE(uf) \in L^q$. Then it is easily seen that

$$\|M_wEM_u(f)\|_q = \|EM_v(f)\|_q,$$

where $v = u(E(|w|^q))^{\frac{1}{q}}$. Thus without loss of generality we can consider the operator EM_v instead of M_wEM_u in our discussion about closedness of range. Also, we recall that for any operator T on a Banach space X, $\mathcal{N}(T) = \{x \in X : T(x) = 0\}$ and $\mathcal{R}(T) = \{T(x) : x \in X\}$ are called null space and range of T, respectively. Now in the next theorem we consider multiplication conditional type operator EM_u on L^p.

Theorem 2.1. Let $1 < p < \infty$ and let p' be conjugate component to p. Then

(a) If the operator EM_u from $L^p(\Sigma)$ into itself is injective and has closed range, then there exists $\delta > 0$ such that $v = (E(|u|^{p'}))^{\frac{1}{p'}} \ge \delta$ a.e., on S, where $S = S(v)$.

(b) If $S(E(u)) = S(E(|u|^{p'}))$ and there exists $\delta > 0$ such that $E(u) \ge \delta$ a.e. On $S = S(E(|u|^{p'}))$, then the operator EM_u has closed range on $L^p(\Sigma)$.

Proof. (a) Let $f \in L^p(\Sigma)$. Then

$$\begin{aligned}\|EM_u f\|_p^p &= \int_X |E(uf)|^p d\mu \\ &\le \int_X (E(|u|^{p'}))^{\frac{p}{p'}} E(|f|^p) d\mu \\ &= \int_X v^p |f|^p d\mu = \|M_v f\|_p^p.\end{aligned}$$

Since EM_u is injective and has closed range, then there exists $\delta > 0$ such that for $f \in L^p(\Sigma)$, $\|EM_u f\|_p \geq \delta \|f\|_p$. Thus

$$\begin{aligned}\|M_v f\|_{L^p(S)} &= \|M_v f\|_{L^p(X)} \\ &\geq \|EM_u f\|_p \\ &\geq \delta \|f\|_{L^p(X)} \\ &\geq \delta \|f\|_{L^p(S)}\end{aligned}$$

and so $\|M_v f\|_{L^p(S)} \geq \delta \|f\|_{L^p(S)}$, for all $f \in L^p(\Sigma)$. This mean's that M_v has closed range on $L^p(X)$. Thus there exists $\beta > 0$ such that $v \geq \beta$ a.e. on S.

(b) Let $f_n, g \in L^p(\Sigma)$ such that $\|E(uf_n) - g\|_p \to 0$, when $n \to \infty$. Since $E(u) \geq \delta$ a.e., on S, then $\frac{1}{E(u)} \leq \frac{1}{\delta}$ a.e., on S. This implies that $\frac{g}{E(u)} \chi_S \in L^p(S)$ and $E(u \frac{g}{E(u)} \chi_S) = g \in L^p(X, \mathcal{A}, \mu)$. Hence

$$\begin{aligned}\|E(uf_n) - E(u\frac{g}{E(u)}\chi_S)\|_p^p &= \int_X |E(uf_n) - E(u\frac{g}{E(u)})\chi_S|^p d\mu \\ &= \int_S |E(uf_n) - E(u\frac{g}{E(u)})\chi_S|^p d\mu \\ &= \int_S |E(uf_n) - g|^p d\mu \\ &\le \|E(uf_n) - g\|_p^p \to 0,\end{aligned}$$

when $n \to \infty$. So the operator EM_u has closed range on $L^p(\Sigma)$.

If $u \geq 0$, then easily we have $S(E(u)) = S(E(|u|^{p'}))$. Hence in the part (b) of Theorem 2.1, the condition $E(u) \geq \delta$ on S, is a sufficient condition for closedness of range of EM_u. In the next theorem we consider bounded operator $EM_u : L^p \to L^q$, when $1 < q < p < \infty$.

Theorem 2.2. Let $1 < q < p < \infty$ and let p', q' be conjugate component to p and q respectively. Then
(a) If the operator EM_u from $L^p(\Sigma)$ into $L^q(\Sigma)$ is injective and has closed range, then we have

1. $v = 0$ a.e. On B and the set $\{n \in \mathbb{N} : v(A_n) \neq 0\}$ is finite.

2. M_v from $L^p(\Sigma)$ into $L^q(\Sigma)$ has finite rank.

Where $v = (E|u|^{q'})^{\frac{1}{q'}}$ and $S = \{x \in X : v(x) \neq 0\}$.
(b) Let

1. The operator EM_u has closed range.

2. The operator EM_u has finite rank.

3. $v = 0$ a.e. On B and the set $N_v = \{n \in \mathbb{N} : v(A_n) \neq 0\}$ is finite.

4. $E(u) = 0$ a.e. On B and the set $N_{N(u)} = \{n \in \mathbb{N} : E(u)(A_n) \neq 0\}$ is finite.

Then

$$(3) \rightarrow (2) \rightarrow (1) \rightarrow (4).$$

(c) If $u \geq 0$, then in part (b) the cases (1), (2), (3) and (4) are equivalent.

Proof. (a) Let $f \in L^p(\Sigma)$. Then

$$\begin{aligned} \|EM_u f\|_q^q &= \int_X |E(uf)|^q d\mu \\ &\leq \int_X (E(|u|^{q'}))^{\frac{q}{q'}} E(|f|^q) d\mu \\ &= \int_X v^q |f|^q d\mu \\ &= \|M_v f\|_q^q. \end{aligned}$$

Since EM_u is injective and closed range, then there exists $\delta > 0$ such that for $f \in L^p(\Sigma)$, $\|EM_u f\|_q \geq \delta \|f\|_p$. Thus $\|M_v f\|_q \geq \|EM_u f\|_p \geq \delta \|f\|_p$ and so $\|M_v f\|_q \geq \delta \|f\|_p$, for all $f \in L^p(\Sigma)$. This mean's that M_v from $L^p(\Sigma)$ into $L^q(\Sigma)$ has closed range. Thus by [13] we have $v = 0$ a.e. On B and the set $\{n \in \mathbb{N} : v(A_n) \neq 0\}$ is finite. Also, M_v from $L^p(\Sigma)$ into $L^q(\Sigma)$ has finite rank.

(b) If $\mu(S) = 0$, then EM_u is the zero operator. So we assume that $\mu(S) > 0$. (3) $\rightarrow$ (2). If (3) holds, then $S = \cup_{n \in N_v} A_n = \cup_{i=1}^k A_{n_i}$ for some integer $k > 0$. Hence

$$EM_u(L^p(X, \Sigma, \mu)) \subseteq L^p(S, \mathcal{A}, \mu),$$

since for any $f \in L^p(X, \Sigma, \mu)$, $\sigma(E(uf)) \subseteq S$. This implies that EM_u has finite rank.

(2) $\rightarrow$ (1) is trivial.

(1) $\rightarrow$ (4). Suppose that EM_u has closed range. First we show that $E(u) = 0$ a.e. On B. Suppose on the contrary that $\mu(\{x \in B : E(u)(x) \neq 0\}) > 0$. Then we have $\mu(\{x \in B : E(u)(x) > \delta\}) > 0$ for some $\delta > 0$. Set $G = \{x \in B : E(u)(x) > \delta\}$ and define a function v on G by $v(x) = \frac{1}{E(u)(x)}$ for $x \in G$. For every $f \in L^p(G)$ and $g \in L^q(G)$,

$$M_v EM_u(f) = v(E(u)|_G) f = f, \quad M_{E(u)|_G} M_v(g) = g.$$

Thus M_v is the inverse operator of $EM_u|_{L^p(G)} = M_{E(u)|_G}$ and $EM_u|_{L^p(G)} = M_{E(u)|_G}$ is a bounded operator from $L^p(G)$ into $L^q(G)$ that has closed rang.

For any $E \in \mathcal{A}_G = \{A \cap G : A \in \mathcal{A}\}$ with $\mu(E) < \infty$, put $f = \frac{1}{E(u)(x)} \chi_E(x)$. Then $f \in L^p(G)$. Moreover, $M_{E(u)} f = \chi_E$ and so $\chi_E \in M_{E(u)}(L^p(G))$. Hence, $M_{E(u)}(L^p(G))$ contains all linear combinations of such χ_E's. Since $M_{E(u)}$ has closed range and all linear combinations of such χ_E's are dense in $L^q(G)$, then $M_{E(u)}(L^p(G)) = L^q(G)$. This implies that M_v maps $L^q(G)$ into $L^p(G)$, that is M_v is bounded from $L^q(G)$ into $L^p(G)$. Since G is non-atomic, by theorem 1.4 of [13] we have $v = 0$ a.e on G. But this is a contradiction.

Now, we show that $N_{E(u)}$ is finite. Since $\mu(S) > 0$, it follows that $N_{E(u)} \neq \emptyset$. Put $w(x) = \frac{1}{E(u)(x)}$ for $x \in S$ and consider the operator M_w. put $f = \frac{1}{E(u)(x)}\chi_{A_n}(x)$, then $f \in L^p(G)$, by the same method of preceding paragraph we see that M_w maps $L^q(S)$ into $L^p(S)$ that is M_w is bounded from $L^q(S)$ into $L^p(S)$. So by theorem 1.4 of [13] we have

$$b = \sup_{n\in N_{E(u)}} \frac{1}{|E(u)(A_n)|^r \mu(A_n)} = \sup_{n\in N_{E(u)}} \frac{|w(A_n)|^r}{\mu(A_n)} < \infty,$$

where $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Since $N_{E(u)} \neq \emptyset$, then $b > 0$ and $|E(u)(A_n)|^r \mu(A_n) \geq \frac{1}{b}$. While Theorem 1.3 of [13] says $E(u) \in L^r(X, \mathcal{A}, \mu)$. So,

$$\Sigma_{n\in N_{E(u)}} \frac{1}{b} \leq \|E(u)\|_r^r < \infty.$$

This implies that $N_{E(u)}$ is finite.
(c) If $u \geq 0$, then by some properties of E we get that $S(E(u)) = S(v)$ and so $N_v = N_{E(u)}$. So (4) → (3) holds. This completes the proof.

It is clear that (by Theorem 2.2), there is no non-zero closed range multiplication conditional type operators EM_u with $u \geq 0$ from L^p into L^q, when $1 \leq q < p < \infty$ and the underlying measure space is non-atomic. In the next theorem we consider bounded operator $EM_u : L^p \to L^q$, when $1 < p < q < \infty$.

Theorem 2.3. Let $1 < p < q < \infty$ and let p', q' be conjugate component to p and q respectively.

(a) If the operator EM_u from $L^p(\Sigma)$ into $L^q(\Sigma)$ is injective and has closed range, then

1. The set $\{n \in \mathbb{N} : v(A_n) \neq 0\}$ is finite.

2. M_v from $L^p(\Sigma)$ into $L^q(\Sigma)$ has finite rank,

where $v = (E|u|^{q'})^{\frac{1}{q'}}$ and $S = \{x \in X : v(x) \neq 0\}$.

(b) Let

1. The operator EM_u has closed range.

2. The operator EM_u has finite rank.

3. The set $N_v = \{n \in \mathbb{N} : v(A_n) \neq 0\}$ is finite.

4. The set $N_{N(u)} = \{n \in \mathbb{N} : E(u)(A_n) \neq 0\}$ is finite.

Then

$$(3) \to (2) \to (1) \to (4).$$

(c) If $u \geq 0$, then in part (b) the cases (1), (2), (3) and (4) are equivalent.

Proof. (a) Let $f \in L^p(\Sigma)$. Then $\|EM_u f\|_q^q \leq \|M_v f\|_q^q$.
Since T is injective and closed range, then there exists $\delta > 0$ such that for $f \in L^p(\Sigma)$, $\|EM_u f\|_q \geq \delta\|f\|_p$. Thus $\|M_v f\|_q \geq \|EM_u f\|_p \geq \delta\|f\|_p$ and so $\|M_v f\|_q \geq \delta\|f\|_p$, for all $f \in L^p(\Sigma)$. This mean's that M_v from $L^p(\Sigma)$

into $L^q(\Sigma)$ has closed range. Thus by [13] the set $\{n \in \mathbb{N} : v(A_n) \neq 0\}$ is finite. Also, M_v from $L^p(\Sigma)$ into $L^q(\Sigma)$ has finite rank.

(b) Theorem 2.3 of [4] tells us that $v = 0$ a.e on B and so $E(u) = 0$ a.e on B. By the same method that we used in last theorem, it is easy to see that (3) → (2) → (1). Now, we show that (1) → (4). Suppose that $N_{N(u)} \neq \emptyset$. If we put $S = \sigma(E(u))$, then we can write $S = \cup_{n\in N_{E(u)}} A_n$. Define a function w on S by $w(x) = \frac{1}{E(u)(x)}$ for $x \in S$. By the same method that is used in the proof of last $L^p(S)$. Hence by Theorem 1.3 of [13] we have $w \in L^s(\mathcal{A})$, where $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$. While Theorem 1.4 of [13] says that $b = \sup_{n\in N_{E(u)}} \frac{|E(u)(A_n)|^s}{\mu(A_n)} < \infty$. Since $N_{N(u)} \neq \emptyset$ implies $b > 0$ and since $|w(A_n)|^s \mu(A_n) = \frac{\mu(A_n)}{|E(u)(A_n)|^s} \geq \frac{1}{b}$ for all $n \in N_{N(u)}$, it follows that

$$\Sigma_{n\in N_{E(u)}} \frac{1}{b} \leq \|w\|_s^s < \infty$$

this implies that $N_{E(u)}$ is finite.

In the sequel we consider the function u is $\mathcal{A}$-measurable. In this case $E(u) = u$ and $EM_u = M_u \mid_{L^p(\mathcal{A})}$. Therefore we get the following results.

Corollary 2.4. Let $1 < p < \infty$ and let p' be conjugate component to p. If $u \in L^0(\mathcal{A})$ and EM_u is injective on $L^p(\Sigma)$. Then the operator EM_u has closed range if and only if there exists $\delta > 0$ such that $|u| \geq \delta$ a.e., on S. Where $S = \{x \in X : u(x) \neq 0\}$.

Corollary 2.5. Let $1 < q < p < \infty$ and let p', q' be conjugate component to p and q respectively. If $u \in L^0(\mathcal{A})$ and EM_u is injective from $L^p(\Sigma)$ into $L^q(\Sigma)$. Then the followings are equivalent:

1. The operator M_u has closed range.
2. The operator M_u has finite rank.
3. $u = 0$ a.e. on B and the set $N_u = \{n \in \mathbb{N} : u(A_n) \neq 0\}$ is finite.

Corollary 2.6. Let $1 < p < q < \infty$ and let p', q' be conjugate component to p and q respectively. If $u \in L^0(\mathcal{A})$ and EM_u is injective from $L^p(\Sigma)$ into $L^q(\Sigma)$, then the followings are equivalent:

1. The operator EM_u has closed range.
2. The operator EM_u has finite rank.
3. The set $N_u = \{n \in \mathbb{N} : u(A_n) \neq 0\}$ is finite.

If the multiplication conditional type operator EM_u is bounded from $L^p(\Sigma)$ onto $L^p(\mathcal{A})$, then $\mu(Z(E(|u|^{p'})) = \{x \in X : E(|u|^{p'})(x) = 0\}) = 0$. Suppose that $F \subseteq Z(E(|u|^{p'}))$ with $F \in \mathcal{A}$ and $\mu(F) < \infty$. Then $\chi_F \in L^p(\mathcal{A}) = \mathcal{R}(EM_u)$ and there exists $f \in L^p(\Sigma)$ such that $EM_u f = \chi_F$. So by conditional-type Hölder inequality we have

$$\begin{aligned}\mu(F) &= \int_F |E(u.f)|^p d\mu \\ &\leq \int_F (E(|u|^{p'})^{\frac{p}{p'}}.|f|^p d\mu = 0.\end{aligned}$$

Hence $\mu(Z(E(|u|^{p'}))) = 0$.

In the next theorem we characterize Fredholm multiplication conditional type operators on L^p-spaces when the underlying measure space is non-atomic.

Theorem 2.7. Let $1 < p < \infty$ and (X, Σ, μ) be a non-atomic measure space. If EM_u is a bounded operator from $L^p(\Sigma)$ into $L^p(\mathcal{A})$, then EM_u is Fredholm if and only if EM_u is invertible.

Proof. Let EM_u be Fredholm, then EM_u has closed range. First we show that EM_u is surjective. Suppose on the contrary. Let $f_0 \in L^p(\mathcal{A}) \setminus \mathcal{R}(EM_u)$. Then there exists a bounded linear functional L_{g_0} on $L^p(\mathcal{A})$ for some $g_0 \in L^{p'}(\mathcal{A})$ $(p^{-1} + p'^{-1} = 1)$, which is defined as

$$L_{g_0}(f) = \int_X f\bar{g}_0 d\mu, \quad f \in L^p(\mathcal{A}),$$

such that

$$L_{g_0}(f_0) = 1, \quad L_{g_0}(\mathcal{R}(EM_u)) = 0.$$

Then there exists a positive constant δ such that

$$\mu(E_\delta = \{x \in X : f_0(x)g_0(x) > \delta\}) > 0.$$

Since the underlying measure space is non-atomic, then we can find a disjoint sequences $\{E_n\}_n$ such that $E_n \subseteq E_\delta$ and $0 < \mu(E_n) < \infty$. Let $g_n = g_0.\chi_{E_n}$. Clearly $g_n \in L^{p'}(\mathcal{A})$ and for every $f \in L^p(\mathcal{A})$ we have

$$\begin{aligned}\int_X \bar{f}(EM_u)^*(g_n)d\mu &= \int_X \bar{f}\bar{u}E(g_n)d\mu \\ &= \int_X g_0 E(\bar{u}\bar{f}.\chi_{E_n})d\mu \\ &= L_{g_0}(EM_u f) = 0.\end{aligned}$$

This implies that $g_n \in \mathcal{N}((EM_u)^*)$ and so $\mathcal{N}((EM_u)^*)$ is infinite dimensional. Therefore the codimension of $\mathcal{R}(EM_u)$ is not finite. This is a contradiction, therefore T is surjective. Let $0 \neq f \in L^p(\Sigma)$ such that $EM_u(f) = 0$. Since $S(f) \subseteq S(E(|f|))$, $\mu(S(f)) > 0$ and $S(E(|f|))$ is $\mathcal{A}$-measurable, then there exists $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ and $\mu(S(f) \cap A) > 0$. Also since the underlying measure space is non-atomic, we can find a disjoint sequence of $\mathcal{A}$-measurable subsets $\{A_n\}_n$ of A with $\mu(S(f) \cap A_n) > 0$. Clearly $f_n = f.\chi_{S(f)\cap A_n} \in L^p(\Sigma)$ and $EM_u(f_n) = \chi_{A_n}.EM_u(f) = 0$. This means $\mathcal{N}(EM_u)$ is infinite dimensional, that is a contradiction. Thus EM_u should be injective and so is invertible. The converse is obvious.

Finally, in the next proposition we show that if there exists a non-zero function in the null space of EM_u such that it's support is non-atomic set, then EM_u can not be Fredholm.

Proposition 2.8. Let EM_u be bounded from $L^p(\Sigma)$ into $L^q(\Sigma)$. If $\mathcal{N}(EM_u) \cap L^p(B) \neq \emptyset$, then EM_u can not be Fredholm.

Proof. By the methods that we used in the proof of Theorem 2.7, it is easy to prove.

In the sequel we present some examples of conditional expectations and corresponding multiplication conditional type operators.

Examples. (a) This example deals with a weighted conditional expectation operator which is in the form of an integral operator. Let $X = (0, 1] \times (0, 1]$, $d\mu = dxdy$, Σ the Lebesgue measurable subsets of X and let $\mathcal{A}$

be the σ-algebra generated by the family of the sets of the form $A \times (0,1]$ where A is a Lebesgue measurable subset of $(0,1]$. For f in $L^2((0,1]^2)$, we have $(E^{\mathcal{A}}f)(x,y) = \int_0^1 f(x,t)dt$. Then for every non-zero measurable function u such that $\int_0^1 u(t)dt \neq 0$, the by Theorem 2.2 the operator EM_u has not closed range as an operator from $L^p(X)$ into $L^q(X)$ when $1 \leq q < p < \infty$.

Let $X = (0,\infty) \times (0,\infty)$ and $\mathcal{A}$ be the σ-algebra generated by the family of the sets of the form $A \times (0,\infty)$ where A is a Lebesgue measurable subset of $(0,\infty)$. Then

$$Tf(x,y) = (E^{\mathcal{A}}M_u(f))(x,y) = \int_0^\infty u(x,t)f(x,t)dt.$$

Hence for every function $f : (0,\infty) \to \mathbb{R}$ we can define the function f' on X as $f'(x,y) = f(y)$. So,

$$\begin{aligned} Tf(x,y) &= (E^{\mathcal{A}}M_u(f))(x,y) \\ &= (E^{\mathcal{A}}M_u(f'))(x,y) \\ &= \int_0^\infty u(x,t)f(t)dt \\ &= T'(f)(x). \end{aligned}$$

This implies that T is an integral transform and specially by taking $u(x,y) = e^{-xy}$ we obtain one of the most important classical integral transforms that is widely used in analysis, namely the Laplace integral transform. We refer to [9] for some applications of integral transforms, especially Laplace integral transforms.

(b) Let $X = [-1,1]$, $d\mu = \frac{1}{2}dx$ and $\mathcal{A} = \langle\{(-a,a) : 0 \leq a \leq 1\}\rangle$ (σ-algebra generated by symmetric intervals). Then

$$E^{\mathcal{A}}(f)(x) = \frac{f(x) + f(-x)}{2}, \quad x \in X,$$

whenever $E^{\mathcal{A}}(f)$ is defined. If $u(x) = x^2$ for $-1 \leq \frac{1}{2}$ and $u(x) = 0$ elsewhere. Then the operator EM_u is not Fredholm on $L^p(\Sigma)$.

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