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# Positive Decreasing Solutions of Second Order Quasilinear Ordinary Differential Equations in the Framework of Regular Variation

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**Abstract.** This paper is concerned with asymptotic analysis of positive decreasing solutions of the second-order quasilinear ordinary differential equation

$$\text{(E)} \qquad (p(t)\varphi(|x'(t)|))' = q(t)\psi(x(t)),$$

with the regularly varying coefficients $p,\ q,\ \varphi,\ \psi$. An application of the theory of regular variation gives the possibility of determining the precise information about asymptotic behavior at infinity of solutions of equation (E) such that $\lim_{t\to\infty} x(t) = 0,\ \lim_{t\to\infty} p(t)\varphi(-x'(t)) = \infty$.

## 1. Introduction

In this paper we study the differential equation of the form

$$\text{(E)} \qquad (p(t)\varphi(|x'(t)|))' = q(t)\psi(x(t)), \quad t \geq a > 0,$$

under the following assumptions
(i) $\varphi, \psi : (0,\infty) \to (0,\infty)$ are continuous functions which are regularly varying at zero of index $\alpha > 0$ and $\beta \in (0,\alpha)$, respectively and function $\varphi$ is increasing;
(ii) $p, q : [a,\infty) \to (0,\infty)$ are continuous functions which are regularly varying at infinity of index $\eta > \alpha$ and $\sigma \in \mathbb{R}$, respectively.

By a solution of (E) we mean a function $x(t) : [T,\infty) \to \mathbb{R}$, $T \geq a$, which is continuously differentiable together with $p(t)\varphi(|x'(t)|)$ on $[T,\infty)$ and satisfies the equation (E) at every point of $[T,\infty)$.

### 1.1. Theory of regularly varying functions

In our analysis we shall extensively use the class of regularly varying functions introduced by J. Karamata in 1930 by the following

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**Definition 1.1.** *A measurable function $f : [a, \infty) \to (0, \infty)$, $a > 0$ is said to be regularly varying at infinity of index $\rho \in \mathbb{R}$ if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \textit{for all } \lambda > 0. \tag{1}$$

*A measurable function $f : (0, a) \to (0, \infty)$ is said to be regularly varying at zero of index $\rho \in \mathbb{R}$ if $f(1/t)$ is regularly varying at infinity of index $-\rho$ i.e. if*

$$\lim_{t\to 0+} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \textit{for all } \lambda > 0. \tag{2}$$

With $\mathrm{RV}(\rho)$ and $\mathcal{RV}(\rho)$ we denote, respectively, the set of regularly varying functions of index $\rho$ at infinity and at zero. If in particular, $\rho = 0$, the function $f$ is called *slowly varying* at infinity or at zero. With SV and $\mathcal{SV}$ we denote, respectively, the set of slowly varying functions at infinity and at zero. Saying only regularly or slowly varying function, we mean regularity at infinity.

It follows from the Definition 1.1 that any function $f(t) \in \mathrm{RV}(\rho)$ can be written as

$$f(t) = t^{\rho}\, g(t), \quad g(t) \in \mathrm{SV}, \tag{3}$$

and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. If in particular the function $g(t) \to k > 0$ as $t \to \infty$, it is called *trivial slowly varying*, denoted by $g(t) \in \mathrm{tr} - \mathrm{SV}$, and the function $f(t)$ is called *trivial regularly varying of index $\rho$*, denoted by $f(t) \in \mathrm{tr} - \mathrm{RV}(\rho)$. Otherwise, the function $g(t)$ is called *nontrivial slowly varying* , denoted by $g(t) \in \mathrm{ntr} - \mathrm{SV}$, and the function $f(t)$ is called *nontrivial regularly varying of index $\rho$*, denoted by $f(t) \in \mathrm{ntr} - \mathrm{RV}(\rho)$. Similarly for the set $\mathcal{RV}$.

For a comprehensive treatise on regular variation the reader is referred to N.H. Bingham et al. [1]. See also E. Seneta [14]. However, to help the reader we present here some elementary properties of regularly varying functions and a fundamental result, called *Karamata's integration theorem*, which will be used throughout the paper.

The symbol $\sim$ denotes the asymptotic equivalence of two positive functions, i.e.,

$$f(t) \sim g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

**Proposition 1.1.** ( Karamata's integration theorem) *Let $L(t) \in \mathrm{SV}$. Then,*

(i) *If $\alpha > -1$,*

$$\int_a^t s^{\alpha} L(s)\, ds \sim \frac{t^{\alpha+1} L(t)}{\alpha + 1}, \quad t \to \infty;$$

(ii) *If $\alpha < -1$, $\int_a^{\infty} t^{\alpha}\, L(t)\, dt < \infty$ and*

$$\int_t^{\infty} s^{\alpha}\, L(s)\, ds \sim -\frac{t^{\alpha+1} L(t)}{\alpha + 1}, \quad t \to \infty;$$

(iii) *If $\alpha = -1$, provided that $\int_a^{\infty} t^{-1}\, L(t)\, dt < \infty$,*

$$m_1(t) = \int_a^t s^{-1}\, L(s)\, ds \in \mathrm{SV}, \quad m_2(t) = \int_t^{\infty} s^{-1}\, L(s)\, ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m_i(t)} = 0, \quad i = 1, 2.$$

We shall also use the following results:

**Proposition 1.2.** *Let $g_1(t) \in \mathrm{RV}(\sigma_1)$, $g_2(t) \in \mathrm{RV}(\sigma_2)$, $g_3(t) \in \mathcal{RV}(\sigma_3)$. Then,*

(i) $(g_1(t))^\alpha \in \mathrm{RV}(\alpha\sigma_1)$ *for any* $\alpha \in \mathbb{R}$*;*

(ii) $g_1(t) + g_2(t) \in \mathrm{RV}(\sigma),\ \sigma = \max(\sigma_1, \sigma_2)$*;*

(iii) $g_1(t)g_2(t) \in \mathrm{RV}(\sigma_1 + \sigma_2)$*;*

(iv) $g_1(g_2(t)) \in \mathrm{RV}(\sigma_1\sigma_2)$*, if* $g_2(t) \to \infty$*, as* $t \to \infty$*;* $g_3(g_2(t)) \in \mathrm{RV}(\sigma_3\sigma_2)$*, if* $g_2(t) \to 0$*, as* $t \to \infty$*;*

(v) *for any* $\varepsilon > 0$ *and* $L(t) \in \mathrm{SV}$ *one has* $t^\varepsilon L(t) \to \infty,\quad t^{-\varepsilon}L(t) \to 0$*, as* $t \to \infty$*.*

**Proposition 1.3.** *If* $f(t) \sim t^\alpha l(t)$ *as* $t \to \infty$ *with* $l(t) \in \mathrm{SV}$*, then* $f(t)$ *is a regularly varying function of index* $\alpha$ *i.e.* $f(t) = t^\alpha l^*(t),\ l^*(t) \in \mathrm{SV}$*, where in general* $l^*(t) \neq l(t)$*, but* $l^*(t) \sim l(t)$ *as* $t \to \infty$*.*

**Proposition 1.4.** *A positive measurable function* $l(t)$ *belongs to* SV *if and only if for every* $\alpha > 0$ *there exists a non-decreasing function* $\Psi$ *and a non-increasing function* $\psi$ *with*

$$t^\alpha l(t) \sim \Psi(t) \quad \text{and} \quad t^{-\alpha} l(t) \sim \psi(t), \quad t \to \infty.$$

**Proposition 1.5.** *For the function* $f(t) \in \mathrm{RV}(\alpha), \alpha > 0$*, there exists* $g(t) \in \mathrm{RV}(1/\alpha)$ *such that*

$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as} \quad t \to \infty.$$

*Here* $g$ *is an asymptotic inverse of* $f$ (and it is determined uniquely to within asymptotic equivalence).

Note, the same result holds for $t \to 0$ i.e. when $f(t) \in \mathcal{RV}(\alpha), \alpha > 0$.

Next result is proved in [13] and we are going to use it very often in our proofs. It helps us with manipulation of the asymptotic relations.

The symbol $\simeq$ denotes the asymptotic similarity of two positive functions, i.e.,

$$f(t) \simeq g(t),\quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = c > 0.$$

**Proposition 1.6.** *Let* $F : [a, \infty) \to (0, \infty)$ *be a measurable function and* $x_1, x_2$ *positive functions defined on* $[a, \infty)$ *such that* $x_i(t) \to \infty,\ t \to \infty,\ i = 1, 2$*. Then:*

$$F \in \mathrm{RV}(\rho),\ \rho \neq 0 \quad \textit{iff} \quad x_1(t) \simeq x_2(t),\ t \to \infty \Longrightarrow F(x_1(t)) \simeq F(x_2(t)),\ t \to \infty.$$

To avoid repetitions we state here basic conditions imposed of the functions $\varphi, \psi, p, q$. In what follows we always assume:

$$\varphi(s) \in \mathcal{RV}(\alpha),\quad \alpha > 0;\quad \psi(s) \in \mathcal{RV}(\beta),\quad \alpha > \beta > 0;\quad p(t) \in \mathrm{RV}(\eta),\quad \eta > \alpha;\quad q(t) \in \mathrm{RV}(\sigma),\quad \sigma \in \mathbb{R}. \tag{4}$$

Using notation (3), we will express $\varphi(s)$, $\psi(s)$, $p(t)$ and $q(t)$ as

$$\varphi(s) = s^\alpha L_1(s),\quad L_1(s) \in \mathcal{SV};\quad \psi(s) = s^\beta L_2(s),\quad L_2(s) \in \mathcal{SV}; \tag{5}$$

$$p(t) = t^\eta l_p(t),\quad l_p(t) \in \mathrm{SV};\qquad q(t) = t^\sigma l_q(t),\quad l_q(t) \in \mathrm{SV}. \tag{6}$$

By assumption (i), $\varphi(s)$ is increasing function, then $\varphi(s)$ has the inverse function, denoted by $\varphi^{-1}(s)$ and from (5) we conclude that

$$\varphi^{-1}(s) \in \mathcal{RV}(1/\alpha) \quad \Rightarrow \quad \varphi^{-1}(s) = s^{1/\alpha} L(s),\quad L(s) \in \mathcal{SV}. \tag{7}$$

We also need the additional requirements:

$$\varphi^{-1}(t^\lambda u(t)) \sim \varphi^{-1}(t^\lambda) u(t)^{\frac{1}{\alpha}},\quad t \to \infty,\quad \forall \lambda \in \mathbb{R}^-,\quad \forall u(t) \in SV \cap C^1(\mathbb{R}); \tag{8}$$

$$\psi(t^\lambda u(t)) \sim \psi(t^\lambda) u(t)^\beta,\quad t \to \infty,\quad \forall \lambda \in \mathbb{R}^-,\quad \forall u(t) \in SV \cap C^1(\mathbb{R}). \tag{9}$$

In other words, for each slowly varying $u(t) \in C^1(\mathbb{R})$, the slowly varying part $L(s)$ of $\varphi^{-1}(s)$ must satisfy $L(t^\lambda u(t)) \sim L(t^\lambda),\ t \to \infty$ for each $\lambda \in \mathbb{R}^-$ and the slowly varying part $L_2(s)$ of $\psi(s)$ must satisfy $L_2(t^\lambda u(t)) \sim L_2(t^\lambda),\ t \to \infty$ for each $\lambda \in \mathbb{R}^-$. It is easy to check that this is satisfied by e.g.

$$L(t) = \prod_{k=1}^{N} (\log_k t)^{\alpha_k},\ \alpha_k \in \mathbb{R}, \quad \text{but not by} \quad L(t) = \exp \prod_{k=1}^{N} (\log_k t)^{\beta_k},\ \beta_k \in (0, 1),$$

where $\log_k t = \log \log_{k-1} t$.

## 2. Classification of Positive Decreasing Solutions

In this section we classify the set of positive decreasing solutions of (E) according to their asymptotic behavior as $t \to \infty$.

It is easily seen (see[3]) that if $x(t)$ is a positive decreasing solution of (E), then there are positive constants $c_1$ and $c_2$, such that for all large $t$

$$c_1 \pi(t) \le x(t) \le c_2 \tag{10}$$

more precisely, the asymptotic behavior of any positive decreasing solution $x(t)$ of (E) falls into one of the following three types:

(I) $\lim_{t\to\infty} \frac{x(t)}{\pi(t)} = c > 0;$

(II) $\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} p(t)\varphi(-x'(t)) = \infty;$

(III) $\lim_{t\to\infty} x(t) = c > 0,$

where the function $\pi(t)$ is defined as

$$\pi(t) = \int_t^\infty \varphi^{-1}(p(s)^{-1})\, ds.$$

Using (6), (8) and (7) we have as $t \to \infty$

$$\pi(t) = \int_t^\infty \varphi^{-1}(s^{-\eta}\, l_p(s)^{-1})\, ds \sim \int_t^\infty \varphi^{-1}(s^{-\eta})\, l_p(s)^{-\frac{1}{\alpha}}\, ds = \int_t^\infty s^{-\frac{\eta}{\alpha}} L(s^{-\eta}) l_p(s)^{-\frac{1}{\alpha}}\, ds.$$

Applying Karamata's integration theorem (Proposition 1.1) to the last integral in the above relation we obtain

$$\pi(t) \sim \frac{\alpha}{\eta - \alpha}\, t^{1-\frac{\eta}{\alpha}}\, L(t^{-\eta})\, l_p(t)^{-\frac{1}{\alpha}}, \quad t \to \infty. \tag{11}$$

Clearly, $\pi(t) \in \mathrm{RV}(1 - \frac{\eta}{\alpha})$.

Solutions of type (I), (II), (III) is often called, respectively, *subdominant*, *intermediate* and *dominant* solutions.

It is known (see [3]) that the existence of positive solutions of type (I) and (III) for the equation (E) with continuous coefficients $p(t), q(t), \varphi(s)$ and $\psi(s)$ can be completely characterized by the convergence or divergence of integrals

$$W = \int_a^\infty q(t)\psi(\pi(t))dt, \quad Z = \int_a^\infty \varphi^{-1}\left(p(t)^{-1} \int_a^t q(s)ds\right) dt.$$

**Theorem 2.1.** *Let $p(t), q(t) \in C[a,\infty)$ and $\varphi(s), \psi(s) \in C[0,\infty)$.*

*(a) Equation (E) has positive solutions of type (I) if and only if $W < \infty$.*

*(b) Equation (E) has positive solutions of type (III) if and only if $Z < \infty$.*

*(c) Equation (E) has positive solutions of type (II) if $W = \infty$ and $Z < \infty$.*

Note that only the sufficient condition for the existence of intermediate solutions of (E) is given in Theorem 2.1(c). If we use the theory of regularly varying functions, i.e. we assume that coefficients of (E) are regularly varying functions and look for regularly varying solutions of (E), it is possible to establish the necessary and sufficient conditions for the existence of intermediate solutions of (E) and precisely determine their behavior at infinity. The present work was motivated by the recent progress in the asymptotic analysis of differential equations by means of regularly varying functions in the sense of Karamata, which was initiated by the monograph of Marić [4]. The special case of (E) with $\varphi(s) = |s|^{\alpha-1}s$ and $\psi(s) = |s|^{\beta-1}s$ is considered in [12]. Therefore, this paper is a generalization of [12]. For the related results on second order equations, see [5–8, 11], and for the results on first order systems, see [9, 10].

The main body of the paper is divided into six sections. In Section 2 we classify the set of decreasing positive solutions of (E). The main results are stated in Section 3, and proved in Section 5. In Section 4 we collect some preparatory results which will help us to simplify the proof of our main theorems. Finally, some illustrative examples are presented in Section 6.

## 3. Main Results

This section is devoted to the study of the existence and asymptotic behavior of intermediate regularly varying solutions of equation (E) with functions $\varphi, \psi, p, q$ satisfying (4). We seek such solutions $x(t)$ of (E) expressed in the form

$$x(t) = t^{\rho} l_x(t), \quad l_x(t) \in \mathrm{SV}. \tag{12}$$

In the view of (10), the regularity index $\rho$ of $x(t)$ must satisfy $1 - \frac{\eta}{\alpha} \le \rho \le 0$. Therefore, the class of intermediate regularly varying solutions of (E) is divided into three types of subclasses:

$$\mathrm{ntr} - \mathrm{RV}\left(1 - \frac{\eta}{\alpha}\right), \quad \mathrm{RV}(\rho),\ \rho \in \left(1 - \frac{\eta}{\alpha}, 0\right), \quad \mathrm{ntr} - \mathrm{SV}.$$

Our main results formulated below characterize completely the membership of each of the three subclasses of solutions and show that all members of each subclass enjoy one and the same asymptotic behavior as $t \to \infty$.

To state our main results, we will need the function

$$\Psi(y) = \int_0^y \frac{dv}{\psi(v)^{\frac{1}{\alpha}}},\ y > 0, \tag{13}$$

which is clearly increasing on $(0, \infty)$. From (5), (13) and Proposition 1.1 we get

$$\Psi(y) = \int_0^y v^{-\frac{\beta}{\alpha}} L_2(v)^{-\frac{1}{\alpha}}\, dv \sim \frac{\alpha}{\alpha - \beta}\, y^{1-\frac{\beta}{\alpha}}\, L_2(y)^{-\frac{1}{\alpha}} = \frac{\alpha}{\alpha - \beta} \frac{y}{\psi(y)^{\frac{1}{\alpha}}}, \quad y \to \infty, \tag{14}$$

implying $\Psi(y) \in \mathcal{RV}(\frac{\alpha-\beta}{\alpha})$ and $\Psi^{-1}(y) \in \mathcal{RV}(\frac{\alpha}{\alpha-\beta})$ with $\frac{\alpha-\beta}{\alpha} > 0$.

**Theorem 3.1.** *Suppose that* (4), (8) *and* (9) *hold. Equation (E) has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{RV}(1 - \frac{\eta}{\alpha})$ *if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - \beta - 1 \quad and \quad \int_a^{\infty} q(t)\, \psi(\pi(t))\, dt = \infty, \tag{15}$$

*in which case any such solution* $x(t)$ *has the asymptotic behavior* $x(t) \sim X_1(t),\ t \to \infty$, *where*

$$X_1(t) = \pi(t) \left( \frac{\alpha - \beta}{\alpha} \int_a^t q(s)\ \psi(\pi(s))\ ds \right)^{\frac{1}{\alpha-\beta}}. \tag{16}$$

**Theorem 3.2.** *Suppose that* (4), (8) *and* (9) *hold. Equation (E) has intermediate solutions* $x(t) \in \mathrm{RV}(\rho)$ *with* $\rho \in (1 - \frac{\eta}{\alpha}, 0)$ *if and only if*

$$\frac{\beta}{\alpha}\eta - \beta - 1 < \sigma < \eta - \alpha - 1 \tag{17}$$

*in which case* $\rho$ *is given by*

$$\rho = \frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta} \tag{18}$$

*and any such solution* $x(t)$ *has the asymptotic behavior* $x(t) \sim X_2(t),\ t \to \infty$, *where*

$$X_2(t) = \Psi^{-1}\left(\frac{\alpha}{\alpha - \beta}\ \frac{t^{2-\rho+\frac{1}{\alpha}}}{(-\rho)[\alpha(\rho-1)+\eta]^{\frac{1}{\alpha}}}\varphi^{-1}(t^{\alpha(\rho-1)})\, p(t)^{-\frac{1}{\alpha}}\, q(t)^{\frac{1}{\alpha}}\right). \tag{19}$$

**Theorem 3.3.** *Suppose that* (4), (8) *and* (9) *hold. Equation (E) has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{SV}$ *if and only if*

$$\sigma = \eta - \alpha - 1 \quad \textit{and} \quad \int_a^\infty \varphi^{-1}\left(p(t)^{-1}\int_a^t q(s)\, ds\right) dt < \infty, \tag{20}$$

*in which case any such solution* $x(t)$ *has the asymptotic behavior* $x(t) \sim X_3(t),\ t \to \infty$, *where*

$$X_3(t) = \Psi^{-1}\left(\int_t^\infty \varphi^{-1}\left(p(s)^{-1}\int_a^s q(r)\, dr\right) ds\right). \tag{21}$$

## 4. Preparatory Results

Our main tools in establishing necessary and sufficient condition for the existence and precise asymptotic forms of intermediate positive solutions of (E) will be Schauder-Tychonoff fixed point theorem combined with theory of regular variation. To that end, the closed convex subset $\mathcal{X}$ of $C[t_0, \infty)$, which should be chosen in such a way that $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ and send it into a relatively compact subset of $C[t_0, \infty)$, will be now found by means of regularly varying functions satisfying the integral asymptotic relation

$$x(t) \sim \int_t^\infty \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^s q(r)\psi(x(r))\, dr\right) ds, \quad t \to \infty. \tag{22}$$

Thus, the proof of the "if" part of our main results is performed in three steps:

(i) the analysis of the integral asymptotic relation (22),

(ii) the construction of intermediate solutions by means of fixed point techniques, and

(iii) the verification of the regularity of those solutions with the help of the generalized L'Hospital rule (see [2]):

**Lemma 4.1.** *Let* $f, g \in C^1[T, \infty)$. *Let*

$$\lim_{t\to\infty} g(t) = \infty \quad \textit{and} \quad g'(t) > 0 \ \ \textit{for all large } t. \tag{23}$$

*Then*

$$\liminf_{t\to\infty}\frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty}\frac{f(t)}{g(t)} \le \limsup_{t\to\infty}\frac{f(t)}{g(t)} \le \limsup_{t\to\infty}\frac{f'(t)}{g'(t)}.$$

*If we replace* (23) *with condition*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \textit{and} \quad g'(t) < 0 \ \ \textit{for all large } t,$$

*then the same conclusion holds.*

To simplify the "if" part of proof of our main results we now take the frst step and prove the next three Lemmas verifying that regularly varying functions $X_i(t),\ i = 1, 2, 3$ defined, respectively by (16), (19) and (21) satisfy the integral asymptotic relation (22).

**Lemma 4.2.** *Suppose that* (15) *holds. Function* $X_1(t)$ *given by* (16) *satisfies the asymptotic relation* (22).

*Proof.* Let (15) hold. Since $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$, using (11), (5) and (6), by Proposition 1.2 we obtain that $q(t)\psi(\pi(t)) \in \mathrm{RV}(-1)$ so that $\int_{t_0}^t q(s)\,\psi(\pi(s))\,ds \in \mathrm{SV}$ by Proposition 1.1-(iii). In view of (11) and (16), we conclude that $X_1(t) \in \mathrm{ntr}-\mathrm{RV}(1-\frac{\eta}{\alpha})$. Using (11), we get

$$\int_{t_0}^t q(s)\,\psi(\pi(s))\,ds \sim \int_{t_0}^t s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,\pi(s)^\beta\,ds,\ t\to\infty. \tag{24}$$

This, combined with (16), gives the following expression for $X_1(t)$ :

$$X_1(t) \sim \pi(t)\left(\frac{\alpha-\beta}{\alpha}\int_{t_0}^t s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,\pi(s)^\beta\,ds\right)^{\frac{1}{\alpha-\beta}},\ t\to\infty. \tag{25}$$

Next, we integrate $q(t)\psi(X_1(t))$ on $[t_0,t]$. Since $X_1(t) = t^{1-\frac{\eta}{\alpha}}l_1(t)$, $l_1(t)\in\mathrm{SV}$, due to (9), we obtain

$$\int_{t_0}^t q(s)\psi(X_1(s))\,ds \sim \int_{t_0}^t q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,l_1(s)^\beta\,ds \sim \int_{t_0}^t s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,X_1(s)^\beta\,ds,\ t\to\infty. \tag{26}$$

Changing (25) in the last integral in (26), by simple calculation we have

$$\begin{aligned}\int_{t_0}^t q(s)\psi(X_1(s))\,ds &\sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\int_{t_0}^t s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,\pi(s)^\beta\left(\int_{t_0}^s r^{\beta(\frac{\eta}{\alpha}-1)}\,q(r)\,\psi(r^{1-\frac{\eta}{\alpha}})\,\pi(r)^\beta\,dr\right)^{\frac{\beta}{\alpha-\beta}} ds \\ &= \left(\frac{\alpha-\beta}{\alpha}\int_{t_0}^t s^{\beta(\frac{\eta}{\alpha}-1)}\,q(s)\,\psi(s^{1-\frac{\eta}{\alpha}})\,\pi(s)^\beta\,ds\right)^{\frac{\alpha}{\alpha-\beta}} \sim \left(\frac{\alpha-\beta}{\alpha}\int_{t_0}^t q(s)\,\psi(\pi(s))\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty,\end{aligned} \tag{27}$$

where we use (24) in the last step. Since $\int_{t_0}^t q(s)\psi(X_1(s))\,ds \in SV$, (6), (7) and (8) gives

$$\begin{aligned}&\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^t q(s)\,\psi(X_1(s))\,ds\right) = \varphi^{-1}\left(t^{-\eta}\,l_p(t)^{-1}\int_{t_0}^t q(s)\,\psi(X_1(s))\,ds\right) \\ &\sim \varphi^{-1}(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_{t_0}^t q(s)\,\psi(X_1(s))\,ds\right)^{\frac{1}{\alpha}} = t^{-\frac{\eta}{\alpha}}L(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_{t_0}^t q(s)\,\psi(X_1(s))\,ds\right)^{\frac{1}{\alpha}},\quad t\to\infty.\end{aligned} \tag{28}$$

Integrating (28) on $[t,\infty)$, we conclude via Proposition 1.1 that

$$\int_t^\infty \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^s q(r)\,\psi(X_1(r))\,dr\right)ds \sim \frac{\alpha}{\eta-\alpha}t^{1-\frac{\eta}{\alpha}}\,L(t^{-\eta})\,l_p(t)^{-\frac{1}{\alpha}}\left(\int_{t_0}^t q(s)\,\psi(X_1(s))\,ds\right)^{\frac{1}{\alpha}},\quad t\to\infty,$$

which, combined with (11) and (27), shows that $X_1(t)$ satisfies the asymptotic relation (22). This completes the proof of Lemma 4.2. □

**Lemma 4.3.** *Suppose that* (17) *holds and let* $\rho$ *be defined by* (18). *Function* $X_2(t)$ *given by* (19) *satisfies the asymptotic relation* (22).

*Proof.* Let (17) hold. Using (6), (7) and (14) by Proposition 1.2, we conclude that $X_2(t)\in\mathrm{RV}(\rho)$, with $\rho$ given by (18). Thus, $X_2(t)$ is expressed as $X_2(t) = t^\rho l_2(t)$, $l_2(t)\in\mathrm{SV}$. Then, we get

$$\begin{aligned}&\int_{t_0}^t q(s)\psi(X_2(s))\,ds = \int_{t_0}^t q(s)\,\frac{\psi(X_2(s))}{X_2(s)^\alpha}\,X_2(s)^\alpha\,ds \\ &\sim (-\rho)^\alpha[\alpha(\rho-1)+\eta]\int_{t_0}^t q(s)\,s^{-\sigma-\alpha-1+\eta}\,L(s^{\alpha(\rho-1)})^{-\alpha}\,l_p(s)\,l_q(s)^{-1}\,X_2(s)^\alpha\,ds \\ &= (-\rho)^\alpha[\alpha(\rho-1)+\eta]\int_{t_0}^t s^{\alpha(\rho-1)+\eta-1}\,L(s^{\alpha(\rho-1)})^{-\alpha}\,l_p(s)\,l_2(s)^\alpha\,ds,\ t\to\infty.\end{aligned} \tag{29}$$

Applying Proposition 1.1 on the last integral in (29) and then multiplying the result with $p(t)^{-1}$ we obtain

$$p(t)^{-1}\int_{t_0}^{t} q(s)\psi(X_2(s))\,ds \sim (-\rho)^{\alpha} t^{\alpha(\rho-1)}\,L(t^{\alpha(\rho-1)})^{-\alpha}\,l_2(t)^{\alpha},\ t\to\infty,$$

from which, applying Proposition 1.6, it readily follows that

$$\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^{t} q(s)\psi(X_2(s))\,ds\right) \sim (-\rho)\,\varphi^{-1}(t^{\alpha(\rho-1)})\,L(t^{\alpha(\rho-1)})^{-1}\,l_2(t) = (-\rho)\,t^{\rho-1}\,l_2(t), \quad t\to\infty,$$

where we use (7) and (8) in two last steps. Integration of the above relation on $[t,\infty)$ with application of Proposition 1.1 yields

$$\int_t^{\infty}\varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)\psi(X_2(r))\,dr\right)ds \sim (-\rho)\int_t^{\infty} s^{\rho-1}\,l_2(s)\,ds \sim t^{\rho}\,l_2(t) = X_2(t),\ t\to\infty.$$

This completes the proof of Lemma 4.3. □

**Lemma 4.4.** *Suppose that* (20) *holds. Function* $X_3(t)$ *given by* (21) *satisfies the asymptotic relation* (22).

*Proof.* Let (20) hold. Using first (6) and Proposition 1.1 (which is possible since $\sigma > -1$) and then (8) and (7) we get

$$\begin{aligned}\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^{t} q(s)ds\right) &= \varphi^{-1}\left(t^{-\eta}\,l_p(t)^{-1}\int_{t_0}^{t} s^{\sigma}l_q(s)\,ds\right) \sim \varphi^{-1}\left((\sigma+1)^{-1}t^{\sigma+1-\eta}l_p(t)^{-1}l_q(t)\right) \\ &\sim (\sigma+1)^{-\frac{1}{\alpha}}\,t^{\frac{\sigma+1-\eta}{\alpha}}\,L(t^{\sigma+1-\eta})l_p(t)^{-\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}},\ t\to\infty.\end{aligned} \tag{30}$$

Integration of (30) on $[t,\infty)$ and application of Proposition 1.1-(iii) since $\sigma = \eta-\alpha-1$ gives as $t\to\infty$

$$\int_t^{\infty}\varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)dr\right)ds \sim (\eta-\alpha)^{-\frac{1}{\alpha}}\int_t^{\infty} s^{-1}\,L(s^{-\alpha})\,l_p(s)^{-\frac{1}{\alpha}}\,l_q(s)^{\frac{1}{\alpha}}\,ds \in \mathrm{SV}. \tag{31}$$

From (21) and (31), by Proposition 1.2-(iv), we find that $X_3(t)\in \mathrm{ntr}-\mathrm{SV}$ and $\psi(X_3(t))\in \mathrm{ntr}-\mathrm{SV}$. Integrate $q(t)\,\psi(X_3(t))$ on $[t_0,t]$, applying Proposition 1.1 and using (6) we obtain

$$\int_{t_0}^{t} q(s)\psi(X_3(s))ds = \int_{t_0}^{t} s^{\sigma}l_q(s)\psi(X_3(s))ds \sim \frac{t^{\sigma+1}}{\sigma+1}\,l_q(t)\psi(X_3(t)) = \frac{t^{\eta-\alpha}}{\eta-\alpha}l_q(t)\psi(X_3(t)), \quad t\to\infty,$$

from which using Proposition 1.6, (8) and (7) follows that

$$\begin{aligned}&\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^{t} q(s)\,\psi(X_3(s))\,ds\right) \sim \varphi^{-1}\left((\eta-\alpha)^{-1}\,t^{-\alpha}\,l_p(t)^{-1}\,l_q(t)\psi(X_3(t))\right) \\ &\sim (\eta-\alpha)^{-\frac{1}{\alpha}}\,t^{-1}L(t^{-\alpha})\,l_p(t)^{-\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\psi(X_3(t))^{\frac{1}{\alpha}} \sim \varphi^{-1}\left(p(t)^{-1}\int_{t_0}^{t} q(s)ds\right)\psi(X_3(t))^{\frac{1}{\alpha}},\ t\to\infty.\end{aligned} \tag{32}$$

On the other hand, we rewrite (21) as

$$\Psi(X_3(t)) = \int_t^{\infty}\varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)dr\right)ds. \tag{33}$$

Since

$$\Psi(X_3(t)) = \int_0^{X_3(t)}\frac{dv}{\psi(v)^{\frac{1}{\alpha}}},$$

differentiation of (33) gives

$$X_3'(t) = -\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^{t} q(s)ds\right)\,\psi(X_3(t))^{\frac{1}{\alpha}}. \tag{34}$$

Integrating (34) on $[t, \infty)$ and combine with (32) we have

$$X_3(t) \sim \int_t^{\infty} \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)\,\psi(X_3(r))\,dr\right)ds,\ t \to \infty.$$

This completes the proof of Lemma 4.4. □

## 5. Proof of Main Results

**Proof of the "only if" part of Theorems 3.1, 3.2, 3.3**: Suppose that the equation (E) has an intermediate solution $x(t) \in \mathrm{RV}(\rho)$ with $\rho \in [1 - \frac{\eta}{\alpha}, 0]$ defined on $[t_0, \infty)$. Integration of equation (E) from $t_0$ to $t$ using (5), (6) and (12) gives

$$p(t)\varphi(-x'(t)) \sim \int_{t_0}^{t} q(s)\,\psi(x(s))\,ds = \int_{t_0}^{t} s^{\sigma+\beta\rho}\, l_q(s) l_x(s)^{\beta}\, L_2(x(s))\,ds, \quad t \to \infty, \tag{35}$$

implying the divergence of the last integral in (35) i.e. implying that $\sigma + \beta\rho \geq -1$. We distinguish the two cases:

$$(a)\quad \sigma + \beta\rho = -1, \quad (b)\quad \sigma + \beta\rho > -1.$$

Assume that $(a)$ holds. Multiplying (35) with $p(t)^{-1}$ we get

$$\varphi(-x'(t)) \sim p(t)^{-1}\,\xi(t), \quad t \to \infty, \quad \text{where} \quad \xi(t) = \int_{t_0}^{t} s^{-1} l_q(s)\, l_x(s)^{\beta} L_2(x(s))\,ds. \tag{36}$$

Clearly, $\xi(t) \in \mathrm{SV}$ and $\lim_{t\to\infty} \xi(t) = \infty$. From (36), using (6) and (8) we have

$$-x'(t) \sim \varphi^{-1}\left(p(t)^{-1}\xi(t)\right) = \varphi^{-1}(t^{-\eta}\, l_p(t)^{-1}\xi(t)) \sim \varphi^{-1}(t^{-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, \xi(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \tag{37}$$

Integrating (37) from $t$ to $\infty$, using (7) we find via Karamata's integration theorem that

$$x(t) \sim \frac{\alpha}{\eta - \alpha}\, t^{1-\frac{\eta}{\alpha}}\, L(t^{-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, \xi(t)^{\frac{1}{\alpha}} \in \mathrm{RV}\left(1 - \frac{\eta}{\alpha}\right), \quad t \to \infty. \tag{38}$$

Using (11) we rewrite (38) in the form

$$x(t) \sim \pi(t)\, \xi(t)^{\frac{1}{\alpha}}, \quad t \to \infty. \tag{39}$$

Assume that $(b)$ holds. Applying Proposition 1.1 to the last integral in (35) we have

$$p(t)\varphi(-x'(t)) \sim \frac{t^{\sigma+\beta\rho+1}}{\sigma + \beta\rho + 1} l_q(t)\, l_x(t)^{\beta} L_2(x(t)), \quad t \to \infty. \tag{40}$$

Multiplying (40) with $p(t)^{-1}$ and then using Proposition 1.6, (6), (8) and (7) we have

$$\begin{aligned} -x'(t) &\sim \varphi^{-1}\left(t^{\sigma+\beta\rho+1-\eta}(\sigma + \beta\rho + 1)^{-1}\, l_p(t)^{-1}\, l_q(t)\, l_x(t)^{\beta} L_2(x(t))\right) \\ &\sim (\sigma + \beta\rho + 1)^{-\frac{1}{\alpha}}\, t^{\frac{\sigma+\beta\rho+1-\eta}{\alpha}}\, L(t^{\sigma+\beta\rho+1-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}}, \quad t \to \infty. \end{aligned} \tag{41}$$

Integration of (41) on $[t,\infty)$ leads to

$$x(t) \sim (\sigma+\beta\rho+1)^{-\frac{1}{\alpha}} \int_t^\infty s^{\frac{\sigma+\beta\rho+1-\eta}{\alpha}} L(s^{\sigma+\beta\rho+1-\eta})\, l_p(s)^{-\frac{1}{\alpha}}\, l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}} L_2(x(s))^{\frac{1}{\alpha}}\, ds, \quad t\to\infty. \tag{42}$$

Since the above integral tends to zero as $t\to\infty$(note that $x(t)\to 0, t\to\infty$), we consider the following two cases separately:

$$(b.1)\quad \frac{\sigma+\beta\rho+1-\eta}{\alpha} < -1, \quad (b.2)\quad \frac{\sigma+\beta\rho+1-\eta}{\alpha} = -1.$$

Assume that $(b.1)$ holds. Applying Proposition 1.1 to the integral in (42), we get

$$x(t) \sim -\frac{\alpha}{\sigma+\beta\rho+1-\eta+\alpha} (\sigma+\beta\rho+1)^{-\frac{1}{\alpha}}\, t^{\frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha}} L(t^{\sigma+\beta\rho+1-\eta})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}},\ t\to\infty, \tag{43}$$

so that $x(t)\in \mathrm{RV}\left(\frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha}\right)$.

Assume that $(b.2)$ holds. Then, (42) shows that $x(t)\in\mathrm{SV}$, that is $\rho=0$, and hence $\sigma=\eta-\alpha-1$. Since $\sigma+\beta\rho+1=\eta-\alpha$, (42) reduced to

$$x(t) \sim (\eta-\alpha)^{-\frac{1}{\alpha}} \int_t^\infty s^{-1} L(s^{-\alpha})\, l_p(s)^{-\frac{1}{\alpha}}\, l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} L_2(x(s))^{\frac{1}{\alpha}}\, ds \in \mathrm{SV},\ t\to\infty. \tag{44}$$

Let us now suppose that $x(t)$ is an intermediate solution of (E) belonging to $\mathrm{ntr}-\mathrm{RV}(1-\frac{\eta}{\alpha})$. Then, the case $(a)$ is the only possibility for $x(t)$, which means that $\rho=1-\frac{\eta}{\alpha}$, $\sigma=\frac{\beta}{\alpha}\eta-\beta-1$ and (39) is satisfied by $x(t)$. Differentiation of $\xi(t)$, defined in (36), using (5), (6) and (12) leads to

$$\xi'(t) = t^{-1}\, l_q(t)\, l_x(t)^\beta\, L_2(x(t)) \sim q(t)\,\psi(x(t)),\ t\to\infty.$$

Noting that $x(t)\sim \pi(t)\xi(t)^{\frac{1}{\alpha}},\ t\to\infty$ and using (9), one can transform the above relation into

$$\xi'(t) \sim q(t)\,\psi(\pi(t)\xi(t)^{\frac{1}{\alpha}}) \sim q(t)\psi(\pi(t))\xi(t)^{\frac{\beta}{\alpha}},\ t\to\infty.$$

So, we get the differential asymptotic relation for $\xi(t)$ :

$$\xi(t)^{-\frac{\beta}{\alpha}}\ \xi'(t) \sim q(t)\ \psi(\pi(t)),\ t\to\infty. \tag{45}$$

Integration of (45) on $[t_0,t]$ yields

$$\xi(t) \sim \left(\frac{\alpha-\beta}{\alpha} \int_{t_0}^t q(s)\,\psi(\pi(s))\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty. \tag{46}$$

Since $\lim_{t\to\infty}\xi(t)=\infty$, from (46) we have $\int_{t_0}^\infty q(t)\,\psi(\pi(t))\,dt=\infty$. Thus, the condition (15) is satisfied. Combining (46) with (39) gives $x(t)\sim X_1(t)$, $t\to\infty$, where $X_1(t)$ is given by (16). This proves the "only if" part of Theorem 3.1.

Next, suppose that $x(t)$ is an intermediate solution of (E) belonging to $\mathrm{RV}(\rho)$, $\rho\in(1-\frac{\eta}{\alpha},0)$. This is possible only when $(b.1)$ holds, in which case $x(t)$ must satisfy the asymptotic relation (43). Therefore,

$$\rho = \frac{\sigma+\beta\rho+1-\eta+\alpha}{\alpha} \quad\Rightarrow\quad \rho = \frac{\sigma+\alpha+1-\eta}{\alpha-\beta},$$

which justifies (18). An elementary calculation shows that

$$1-\frac{\eta}{\alpha} < \rho < 0 \quad\Rightarrow\quad \frac{\beta}{\alpha}\eta-\beta-1 < \sigma < \eta-\alpha-1,$$

which determines the range (17) of $\sigma$. Since $\sigma+\beta\rho+1-\eta+\alpha=\alpha\rho$ and $\sigma+\beta\rho+1=\alpha(\rho-1)+\eta$, (43) reduced to

$$\begin{aligned} x(t) &\sim \frac{t^{\rho}}{(-\rho)(\alpha(\rho-1)+\eta)^{\frac{1}{\alpha}}} L(t^{\alpha(\rho-1)})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}}\, L_2(x(t))^{\frac{1}{\alpha}} \\ &= \frac{t^{2-\rho+\frac{1}{\alpha}}}{(-\rho)(\alpha(\rho-1)+\eta)^{\frac{1}{\alpha}}} \varphi^{-1}(t^{\alpha(\rho-1)})\, p(t)^{-\frac{1}{\alpha}}\, q(t)^{\frac{1}{\alpha}}\, \psi(x(t))^{\frac{1}{\alpha}},\ t\to\infty, \end{aligned} \tag{47}$$

where we use (5), (6), (7) and (12) in the last step. From (47) using (14) we get

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta}\,\frac{x(t)}{\psi(x(t))^{\frac{1}{\alpha}}} \sim \frac{\alpha}{\alpha-\beta}\,\frac{t^{2-\rho+\frac{1}{\alpha}}}{(-\rho)(\alpha(\rho-1)+\eta)^{\frac{1}{\alpha}}}\varphi^{-1}(t^{\alpha(\rho-1)})\, p(t)^{-\frac{1}{\alpha}}\, q(t)^{\frac{1}{\alpha}},\ t\to\infty.$$

Thus, we conclude that $x(t)$ enjoys the asymptotic formula $x(t)\sim X_2(t),\ t\to\infty$, where $X_2(t)$ is given by (19). This proves the "only if" part of the Theorem 3.2.

Finally, suppose that $x(t)$ is an intermediate solution of (E) belonging to $\mathrm{ntr}-\mathrm{SV}$. From the above observation this is possible only when the case ($b$.2) holds, in which case $\rho=0$, $\sigma=\eta-\alpha-1$ and $x(t)=l_x(t)$ must satisfy the asymptotic behavior (44). Denote the right-hand side of (44) by $\mu(t)$. Then, $\mu(t)\to 0,\ t\to\infty$ and satisfies

$$\mu'(t) = -(\eta-\alpha)^{-\frac{1}{\alpha}}\, t^{-1}\, L(t^{-\alpha})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}} L_2(x(t))^{\frac{1}{\alpha}} = -(\eta-\alpha)^{-\frac{1}{\alpha}}\, t^{-1}\, L(t^{-\alpha})\, l_p(t)^{-\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, \psi(x(t))^{\frac{1}{\alpha}},\ t\to\infty,$$

where we use (5) in the last step. Since (44) is equivalent to $x(t)\sim\mu(t),\ t\to\infty$, from the above using (30) we obtain

$$\frac{\mu'(t)}{\psi(\mu(t))^{\frac{1}{\alpha}}} \sim -\varphi^{-1}\left(p(t)^{-1}\int_{t_0}^{t} q(s)ds\right),\ t\to\infty.$$

An integration of the last relation over $[t,\infty)$ gives

$$\int_0^{\mu(t)} \frac{dv}{\psi(v)^{\frac{1}{\alpha}}} \sim \Psi(\mu(t)) \sim \int_t^{\infty} \varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)dr\right)ds,\ t\to\infty,$$

or

$$x(t)\sim\mu(t)\sim\Psi^{-1}\left(\int_t^{\infty}\varphi^{-1}\left(p(s)^{-1}\int_{t_0}^{s} q(r)dr\right)ds\right),\ t\to\infty.$$

Since $\lim_{t\to\infty}\mu(t)=0$, from the above relation we have convergence of integral $\int_a^{\infty}\varphi^{-1}\left(p(t)^{-1}\int_a^t q(s)ds\right)dt$, so the condition (20) is satisfied. Thus, it has been shown that $x(t)\sim X_3(t),\ t\to\infty$, where $X_3(t)$ is given by (21). This completes the "only if" part of the proof of Theorem 3.3. □

**Proof of the "if" part of Theorems 3.1, 3.2, 3.3**: Suppose that (15), (17) or (20) holds. From Lemmas 4.2, 4.3 and 4.4 it is known that $X_i(t),\ i=1,2,3$ defined by (16),(19) and (21) satisfy the asymptotic relation (22). We preform the simultaneous proof for $X_i(t),\ i=1,2,3$ so the subscript $i=1,2,3$ will be deleted in the rest of proof. By (22) there exists $T_0>a$ such that

$$\frac{X(t)}{2} \le \int_t^{\infty}\varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^{s} q(r)\,\psi(X(r))\,dr\right)ds \le 2X(t),\ \ t\ge T_0. \tag{48}$$

Let such a $T_0$ be fixed. In addition, since $X(t)\to 0$ as $t\to\infty$ and (2) holds uniformly on each compact $\lambda-$set on $(0,\infty)$([1],Theorem 1.2.1) we have

$$\frac{\lambda^{\beta}}{2}\psi(X(t)) \le \psi(\lambda X(t)) \le 2\lambda^{\beta}\psi(X(t)),\quad \text{for all sufficiently large } t. \tag{49}$$

Also, since $Q(t) = p(t)^{-1} \int_{t_0}^{t} q(s)\psi(X(s))\,ds \to 0$ as $t \to \infty$ and (2) holds uniformly on each compact $\lambda-$set on $(0, \infty)$ we have

$$\frac{\lambda^{1/\alpha}}{2}\varphi^{-1}(Q(t)) \le \varphi^{-1}(\lambda Q(t)) \le 2\lambda^{1/\alpha}\varphi^{-1}(Q(t)), \quad \text{for all sufficiently large } t. \tag{50}$$

Applying Proposition 1.4 to the function $\psi(s) \in \mathcal{RV}(\beta),\ \beta > 0$ we see that there exists a constant $A > 1$ such that

$$\psi(s_1) \le A\psi(s_2) \quad \text{for each} \quad 0 \le s_1 \le s_2 < a. \tag{51}$$

Now we choose positive constants $m$ and $M$ such that

$$m^{1-\frac{\beta}{\alpha}} \le \frac{1}{4(2A)^{1/\alpha}}, \quad M^{1-\frac{\beta}{\alpha}} \ge 4(2A)^{1/\alpha}. \tag{52}$$

Define the integral operator $\mathcal{F}$ by

$$\mathcal{F}x(t) = \int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^{s} q(r)\psi(x(r))\,dr\right) ds, \quad t \ge T_0, \tag{53}$$

and let it act on the set

$$\mathcal{X} := \{x(t) \in C[T_0, \infty) : mX(t) \le x(t) \le MX(t),\ t \ge T_0\}. \tag{54}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0, \infty)$.

Let $x(t) \in \mathcal{X}$. Using first (51) and (54) , and then (49) we get

$$\mathcal{F}x(t) \le \int_t^\infty \varphi^{-1}\left(\frac{A}{p(s)}\int_{T_0}^{s} q(r)\psi(MX(r))\,dr\right) ds \le \int_t^\infty \varphi^{-1}\left(\frac{2AM^\beta}{p(s)}\int_{T_0}^{s} q(r)\psi(X(r))\,dr\right) ds, \quad t \ge T_0,$$

from which, using (50), (48) and (52), follows that

$$\mathcal{F}x(t) \le 2(2AM^\beta)^{1/\alpha}\int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^{s} q(r)\psi(X(r))\,dr\right) ds \le 4(2AM^\beta)^{1/\alpha}X(t) \le MX(t), \quad t \ge T_0.$$

On the other hand, using (54), (51) and (49) we obtain

$$\mathcal{F}x(t) \ge \int_t^\infty \varphi^{-1}\left(\frac{1}{Ap(s)}\int_{T_0}^{s} q(r)\psi(mX(r))\,dr\right) ds \ge \int_t^\infty \varphi^{-1}\left(\frac{m^\beta}{2Ap(s)}\int_{T_0}^{s} q(r)\psi(X(r))\,dr\right) ds, \quad t \ge T.$$

From the above using (50) and (52) we conclude

$$\mathcal{F}x(t) \ge \frac{1}{2}\left(\frac{m^\beta}{2A}\right)^{\frac{1}{\alpha}}\int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^{s} q(r)\psi(X(r))\,dr\right) ds \ge \frac{1}{4}\left(\frac{m^\beta}{2A}\right)^{\frac{1}{\alpha}} X(t) \ge mX(t), \quad t \ge T_0.$$

This shows that $\mathcal{F}x(t) \in \mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

Furthermore it can be verified that $\mathcal{F}$ is a continuous map and that $\mathcal{F}(\mathcal{X})$ is relatively compact in $C[T_0, \infty)$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x(t) \in \mathcal{X}$ of $\mathcal{F}$, which satisfies integral equation

$$x(t) = \int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^{s} q(r)\psi(x(r))\,dr\right) ds, \quad t \ge T_0.$$

Differentiating the above twice shows that $x(t)$ is a solution of (E) on $[T_0, \infty)$. It is clear from (54) that $x(t)$ is an intermediate solution of (E).

Therefore, the existence of three types of intermediate solutions of (E) has been established. The proof of our main results will be completed with the verification that the intermediate solutions of (E) constructed above are actually regularly varying functions.

We defined the function

$$J(t) = \int_t^\infty \varphi^{-1}\left(\frac{1}{p(s)}\int_{T_0}^s q(r)\psi(X(r))\,dr\right)ds, \quad t \geq T_0,$$

and put

$$l = \liminf_{t\to\infty} \frac{x(t)}{J(t)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{J(t)}.$$

Since $x \in \mathcal{X}$, it is clear that

$$0 < \liminf_{t\to\infty} \frac{x(t)}{X(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{X(t)} < \infty.$$

By Lemmas 4.2, 4.3 and 4.4 we have

$$J(t) \sim X(t), \quad t \to \infty. \tag{55}$$

If we denote with

$$f(t) = \frac{1}{p(t)}\int_{T_0}^t q(s)\psi(x(s))ds \quad \text{and} \quad g(t) = \frac{1}{p(t)}\int_{T_0}^t q(s)\psi(X(s))ds,$$

using (7) and Lemma 4.1 we see that

$$L \leq \limsup_{t\to\infty} \frac{x'(t)}{J'(t)} = \limsup_{t\to\infty} \frac{\varphi^{-1}(f(t))}{\varphi^{-1}(g(t))} = \limsup_{t\to\infty} \frac{f(t)^{\frac{1}{\alpha}}L(f(t))}{g(t)^{\frac{1}{\alpha}}L(g(t))} \leq \limsup_{t\to\infty}\left(\frac{f(t)}{g(t)}\right)^{\frac{1}{\alpha}} \limsup_{t\to\infty} \frac{L\left(\frac{f(t)}{g(t)}g(t)\right)}{L(g(t))}. \tag{56}$$

Using (49) and (51) we obtain $m_1 = \dfrac{m^\beta}{2A} \leq \dfrac{f(t)}{g(t)} \leq 2AM^\beta = M_1$ implying by Uniform convergence theorem ([1],Theorem 1.2.1) that

$$\left|\frac{L\left(\frac{f(t)}{g(t)}g(t)\right)}{L(g(t))} - 1\right| \leq \sup_{\lambda\in[m_1,M_1]}\left|\frac{L(\lambda g(t))}{L(g(t))} - 1\right| \longrightarrow 0, \quad t \to \infty. \tag{57}$$

In the view of (57), from (56) it follows

$$L \leq \limsup_{t\to\infty}\left(\frac{f(t)}{g(t)}\right)^{\frac{1}{\alpha}} = \left(\limsup_{t\to\infty} \frac{\int_{T_0}^t q(s)\,\psi(x(s))\,ds}{\int_{T_0}^t q(s)\,\psi(X(s))\,ds}\right)^{\frac{1}{\alpha}}. \tag{58}$$

Similarly, using (5) and Lemma 4.1 we have

$$\begin{aligned}\limsup_{t\to\infty} \frac{\int_{T_0}^t q(s)\,\psi(x(s))\,ds}{\int_{T_0}^t q(s)\,\psi(X(s))\,ds} &\leq \limsup_{t\to\infty} \frac{\psi(x(t))}{\psi(X(t))} = \limsup_{t\to\infty} \frac{x(t)^\beta L_2(x(t))}{X(t)^\beta L_2(X(t))} \\ &\leq \limsup_{t\to\infty}\left(\frac{x(t)}{X(t)}\right)^\beta \limsup_{t\to\infty} \frac{L_2\left(\frac{x(t)}{X(t)}X(t)\right)}{L_2(X(t))}.\end{aligned} \tag{59}$$

Since $m \le \dfrac{x(t)}{X(t)} \le M,\ t \ge T_0$, using Uniform convergence theorem we conclude

$$\left|\frac{L_2\left(\frac{x(t)}{X(t)}X(t)\right)}{L_2(X(t))} - 1\right| \le \sup_{\lambda\in[m,M]}\left|\frac{L_2(\lambda X(t))}{L_2(X(t))} - 1\right| \longrightarrow 0, \quad t \to \infty. \tag{60}$$

In the view of (60), from (55) and (59) it follows

$$\limsup_{t\to\infty} \frac{\int_{T_0}^t q(s)\,\psi(x(s))\,ds}{\int_{T_0}^t q(s)\,\psi(X(s))\,ds} \le \left(\limsup_{t\to\infty}\frac{x(t)}{X(t)}\right)^\beta = \left(\limsup_{t\to\infty}\frac{x(t)}{J(t)}\right)^\beta = L^\beta. \tag{61}$$

From (58) and (61), it follows that $L \le L^{\frac{\beta}{\alpha}}$, implying that $0 < L \le 1$ because $\alpha > \beta$. If we argue similarly by taking the inferior limits instead of the superior limits, we are led to the inequality $l \ge l^{\frac{\beta}{\alpha}}$, which implies that $l \ge 1$. Thus we conclude that $l = L = 1$, i.e. $\lim_{t\to\infty} x(t)/J(t) = 1$ . This combined with (55) shows that $x(t) \sim X(t),\ t \to \infty$, which yields that $x(t)$ is a regularly varying function whose regularity index $\rho$ is $1 - \frac{\eta}{\alpha}$, $\frac{\sigma+\alpha+1-\eta}{\alpha-\beta}$, or 0 according as $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$, $\frac{\beta}{\alpha}\eta - \beta - 1 < \sigma < \eta - \alpha - 1$, or $\sigma = \eta - \alpha - 1$. Thus, the if part of Theorems 3.1, 3.2, 3.3 has been proved. □

**Remark.** As special case of Theorems 3.1, 3.2, 3.3, when $\varphi(s) = |s|^{\alpha-1}s$ and $\psi(s) = |s|^{\beta-1}s$, Corollaries 6.1, 6.2, 6.3 proved in [12] could be obtain.

## 6. Examples

**Example 6.1.** Consider the equation

$$(p(t)\,\varphi(|x'(t)|))' = q(t)\psi(x(t)), \quad t \ge t_0 > e, \tag{62}$$

where $p(t) = t^{2\alpha}(\log t)^{-\frac{2\alpha}{3}} \in \mathrm{RV}(2\alpha)$, $\varphi(s) = s^\alpha \in \mathcal{RV}(\alpha)$ and $\psi(s) = s^\beta \log s \in \mathrm{RV}(\beta)$, $\alpha > \beta > 0$. So that $\eta = 2\alpha > \alpha$ and the functions $\varphi^{-1}(s)$ and $\psi(s)$ satisfy additional requirements (8) and (9), respectively. Since, $\varphi^{-1}(p(t)^{-1}) = \left(\frac{\sqrt[3]{\log t}}{t}\right)^2$, applying Proposition 1.1 we have $\pi(t) \sim \frac{\sqrt[3]{(\log t)^2}}{t}$, $t \to \infty$.

(i) Suppose that

$$q(t) \sim \frac{\alpha}{3}\, t^{\beta-1}\, \frac{r(t)\,(\log t)^{\frac{\alpha}{3}-\beta-1}}{\log\frac{\log t}{t}}, \quad t \to \infty, \tag{63}$$

where $r(t)$ is continuous function on $[t_0, \infty)$ such that $\lim\limits_{t\to\infty} r(t) = 1$. Here, $q(t) \in \mathrm{RV}(\beta - 1)$. Therefore, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and

$$\int_{t_0}^t q(s)\psi(\pi(s))ds \sim \frac{\alpha}{3}\int_{t_0}^t (\log s)^{\frac{\alpha-\beta}{3}-1}\frac{ds}{s} \sim \frac{\alpha}{\alpha-\beta}(\log t)^{\frac{\alpha-\beta}{3}} \longrightarrow \infty, \quad t \to \infty,$$

implying that (15) holds. Therefore, by Theorem 3.1 there exist nontrivial regularly varying solutions of index $1 - \frac{\eta}{\alpha}$ of (62) and any such solution $x(t)$ has asymptotic behavior

$$x(t) \sim \frac{\log t}{t}, \quad t \to \infty.$$

If in (63) instead of " ∼ " one has " = " and in particular

$$r(t) = \left(1 - \frac{1}{\log t}\right)^{\alpha-1}\left(1 + \frac{2}{\log t}\right),$$

then (62) possesses an exact solution $x(t) = \frac{\log t}{t}$.

(ii) Suppose that

$$q(t) \sim \frac{2\alpha}{3^{\alpha+1}}\, t^{\frac{2\alpha+\beta}{3}-1}\, \frac{r(t)}{(\log t)^{\frac{\alpha+\beta}{3}} \log \sqrt[3]{\frac{\log t}{t}}}, \quad t \to \infty, \tag{64}$$

where $r(t)$ is continuous function on $[t_0, \infty)$ such that $\lim_{t\to\infty} r(t) = 1$. It is clear that $q(t)$ is regularly varying function of index

$$\sigma = \frac{2\alpha+\beta}{3} - 1 \in \left(\frac{\beta}{\alpha}\eta - \beta - 1, \eta - \alpha - 1\right) = (\beta - 1, \alpha - 1)$$

and that $\rho = \dfrac{\sigma + \alpha + 1 - \eta}{\alpha - \beta} = -\dfrac{1}{3}$. By Theorem 3.2 there exist regularly varying solutions of index $\rho$ of (62) and any such solution $x(t)$ has asymptotic behavior

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta} t^{\frac{\beta-\alpha}{3\alpha}} (\log t)^{\frac{\alpha-\beta}{3\alpha}} \left(\log \sqrt[3]{\frac{\log t}{t}}\right)^{-\frac{1}{\alpha}}, \quad t \to \infty.$$

In the view of (14) we have

$$x(t)^{\frac{\alpha-\beta}{\alpha}} (\log x(t))^{-\frac{1}{\alpha}} \sim \left(\sqrt[3]{\frac{\log t}{t}}\right)^{\frac{\alpha-\beta}{\alpha}} \left(\log \sqrt[3]{\frac{\log t}{t}}\right)^{-\frac{1}{\alpha}}, \quad t \to \infty,$$

implying that

$$x(t) \sim \sqrt[3]{\frac{\log t}{t}}, \quad t \to \infty.$$

Observe that in (64) instead " $\sim$ " one has " $=$ " and

$$r(t) = \left(1 - \frac{3}{2\log t} + \frac{2}{\log^2 t}\right)\left(1 - \frac{1}{\log t}\right)^{\alpha-1},$$

then $x(t) = \sqrt[3]{\frac{\log t}{t}}$ is an exact solution.

(iii) Suppose that

$$q(t) \sim \frac{\alpha}{3^{\alpha}}\, t^{\alpha-1}\, \frac{r(t)\, (\log t)^{\frac{\beta}{3}-2\alpha}}{\log(\log t)^{-\frac{1}{3}}}, \quad t \to \infty, \tag{65}$$

where $r(t)$ is continuous function on $[t_0, \infty)$ such that $\lim_{t\to\infty} r(t) = 1$. Then, $q(t) \in \mathrm{RV}(\alpha - 1)$, so that $\sigma = \eta - \alpha - 1$ and we see that

$$\begin{aligned} \int_t^\infty \varphi^{-1}\left(p(s)^{-1} \int_{t_0}^s q(r)dr\right) ds \quad &\sim \quad \frac{1}{3}\int_t^\infty (\log s)^{\frac{\beta}{3\alpha}-\frac{4}{3}} \left(\log(\log s)^{-\frac{1}{3}}\right)^{-\frac{1}{\alpha}} \frac{ds}{s} \\ &\sim \quad \frac{\alpha}{\alpha-\beta} u^{\frac{\beta-\alpha}{3\alpha}} (\log u^{-\frac{1}{3}})^{-\frac{1}{\alpha}}\Big|_{u=\log t} \longrightarrow 0, \quad t \to \infty, \end{aligned}$$

implying that (20) holds. Therefore, by Theorem 3.3 there exist nontrivial slowly varying solutions of (62), and any such solution $x(t)$ has asymptotic behavior

$$\Psi(x(t)) \sim \frac{\alpha}{\alpha-\beta} (\log t)^{\frac{\beta-\alpha}{3\alpha}} (\log(\log t)^{-\frac{1}{3}})^{-\frac{1}{\alpha}}, \quad t \to \infty.$$

In view of (14) we have

$$x(t)^{\frac{\alpha-\beta}{\alpha}}(\log x(t))^{-\frac{1}{\alpha}} \sim (\log t)^{-\frac{\alpha-\beta}{3\alpha}}(\log(\log t)^{-\frac{1}{3}})^{-\frac{1}{\alpha}}, \quad t \to \infty$$

implying that $x(t) \sim (\log t)^{-\frac{1}{3}}$, $t \to \infty$. If in (65) instead of " $\sim$ " one has " $=$ " and in particular $r(t) = 1 - \frac{2}{\log t}$, then (62) possesses an exact solution $x(t) = (\log t)^{-\frac{1}{3}}$.

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