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Semi-Fredholmness of the Discrete Gauss-Bonnet Operator

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Abstract. In the context of an infinite locally finite weighted graph, we give a necessary and sufficient condition for semi-Fredholmness of the Gauss-Bonnet operator. This result is a discrete version of the theorem of Gilles Carron in the continuous case [5]. In addition, using a criterion of Anghel [2], we give a sufficient condition to have an operator of Gauss-Bonnet with closed range. Finally, this work can be considered as an extension of the work of Colette Anné and Nabila Torki-Hamza [3].

1. Introduction

Dirac type operators have become of central importance in many branches of mathematics such as PDE's, differential geometry and topology (see [4], [7], [12]..), since the introduction in 1928 by the physicist Paul Dirac of a first-order linear differential operator whose square is the Laplacian operator. In particular, this paper focuses on the conditions to have semi-Fredholmness of the discrete Gauss-Bonnet operator needed to approach the Hodge decomposition theorem [3]. In fact, we present a discrete version of the work of G. Carron [5], which defines a new concept "non-parabolicity at infinity" to have the Gauss-Bonnet operator with closed range. Indeed, G. Carron's condition is quite weaker than the one given by Anghel [2]. Moreover, we provide a new sufficient condition to obtain a Gauss-Bonnet operator semi-Fredholm. Finally, we give two explicit examples one example verifying the property of non-parabolicity at infinity, and the other not.

2. Preliminaries

2.1. *Definitions and notations*

- A graph G is a couple $(\mathcal{V},\mathcal{E})$ where $\mathcal{V}$ is a set at most countable whose elements are called vertices and $\mathcal{E}$ is a set of oriented edges, considered as a subset of $\mathcal{V}\times\mathcal{V}$.
- If the graph G has a finite set of vertices, it's called a finite graph. Otherwise, G is called infinite graph.
- We assume that $\mathcal{E}$ is symmetric without loops:

$$v \in \mathcal{V} \Rightarrow (v,v) \notin \mathcal{E}, \quad (v_1,v_2) \in \mathcal{E} \Rightarrow (v_2,v_1) \in \mathcal{E}.$$

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- Choosing an orientation of G consists of defining a partition of $\mathcal{E}$: $\mathcal{E}^+ \sqcup \mathcal{E}^- = \mathcal{E}$
$$(v_1, v_2) \in \mathcal{E}^+ \Leftrightarrow (v_2, v_1) \in \mathcal{E}^-.$$
- For $e = (v_1, v_2)$, we denote
$$e^- = v_1,\ e^+ = v_2 \ and \ -e = (v_2, v_1).$$
- The graph G is connected if, any two vertices x, y in $\mathcal{V}$ can be joined by a path of edges γ_{xy}, that means,
$$\gamma_{xy} = \{e_k\}_{k=1,\ldots,n} \ with \ e_1^- = x,\ e_n^+ = y \ and \ if \ n \geq 2\ ,\ \forall j\ ; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$
- The degree (or valence) of a vertex x is the number of edges emanating from x. We denote
$$deg(x) := \sharp\{e \in \mathcal{E};\ e^- = x\}.$$
- If $deg(x) < \infty,\ \forall x \in \mathcal{V}$, we say that G is a locally finite graph.

2.2. The weighted graph

The weighted graph (G, c, r) is given by the graph $G = (\mathcal{V}, \mathcal{E})$, a weight on the vertices $c : \mathcal{V} \to]0, \infty[$ and a weight on the edges $r : \mathcal{E} \to]0, \infty[$such that $r(-e) = r(e)$.

Examples: - An infinite electrical network is a weighted graph (G, c, r) where the weights of the edges called resistances r; their reciprocals are called conductances. And the weights of the vertices given by $c(x) = \sum_{y \in \mathcal{V}} \frac{1}{r(x,y)} < \infty,\ \forall x \in \mathcal{V}$.

-The graph G called a simple graph where the weights of the edges and the vertices equals 1.

All the graphs we shall consider on the sequel will be weighted, connected and locally finite.

2.3. The notion of subgraph

A subgraph of a graph G is a graph $G_K := (K, \mathcal{E}_K)$ such that $K \subset \mathcal{V}$ and $\mathcal{E}_K := \{e \in \mathcal{E};\ e^-,\ e^+ \in K\}$.
For such a subgraph we define:

- the vertex boundary :
$$\partial K := \{x \in \mathcal{V} \setminus K; \exists y \in K,\ (x, y) \in \mathcal{E}\},$$
- the edge boundary:
$$\partial \mathcal{E}_K := \{e \in \mathcal{E};\ e^- \in K \ and \ e^+ \notin K \ or \ e^+ \in K \ and \ e^- \notin K\}.$$

2.4. Functional spaces

We denote the set of real functions on $\mathcal{V}$ by:
$$C(\mathcal{V}) = \{f : \mathcal{V} \to \mathbb{R}\}$$
and the set of functions of finite support by $C_0(\mathcal{V})$.

Moreover, we denote the set of real skewsymmetric functions on $\mathcal{E}$ by:
$$C^a(\mathcal{E}) = \{\varphi : \mathcal{E} \to \mathbb{R}\ ; \varphi(-e) = -\varphi(e)\}$$
and the set of functions of finite support by $C_0^a(\mathcal{E})$.

We define on the weighted graph (G, c, r) the following function spaces endowed of the scalar products.

a)

$$l^2(\mathcal{V}) := \left\{ f \in C(\mathcal{V});\ \sum_{x\in\mathcal{V}} c(x) f^2(x) < \infty \right\},$$

with the inner product

$$\langle f, g\rangle_{\mathcal{V}} = \sum_{x\in\mathcal{V}} c(x) f(x) g(x)$$

and the norm

$$\|f\|_{l^2(\mathcal{V})} = \sqrt{\langle f, f\rangle_{\mathcal{V}}}.$$

b)

$$l^2(\mathcal{E}) := \left\{ \varphi \in C^a(\mathcal{E});\ \frac{1}{2}\sum_{e\in\mathcal{E}} r(e)\varphi^2(e) < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{E}} = \frac{1}{2}\sum_{e\in\mathcal{E}} r(e)\varphi(e)\psi(e)$$

and the norm

$$\|\varphi\|_{l^2(\mathcal{E})} = \sqrt{\langle \varphi, \varphi\rangle_{\mathcal{E}}}.$$

As a consequence, we define the direct sums of $l^2(\mathcal{V})$ and $l^2(\mathcal{E})$ by:

$$l^2(G) := l^2(\mathcal{V}) \oplus l^2(\mathcal{E}) = \left\{ (f,\varphi),\ f \in l^2(\mathcal{V}) \text{ and } \varphi \in l^2(\mathcal{E}) \right\},$$

with the norm

$$\|(f,\varphi)\|^2_{l^2(G)} := \|f\|^2_{l^2(\mathcal{V})} + \|\varphi\|^2_{l^2(\mathcal{E})}.$$

2.5. Operators and properties

The difference operator: it is the operator

$$d : C_0(\mathcal{V}) \longrightarrow C_0^a(\mathcal{E}),$$

given by

$$d(f)(e) = f(e^+) - f(e^-).$$

The coboundary operator: it is δ the formal adjoint of d. Thus it satisfies

$$\langle df, \varphi\rangle_{\mathcal{E}} = \langle f, \delta\varphi\rangle_{\mathcal{V}} \tag{2.1}$$

for all $f \in C_0(\mathcal{V})$ and for all $\varphi \in C_0^a(\mathcal{E})$.

As consequence, we have the following formula characterizing δ :

Lemma 2.1. *The coboundary operator δ is characterized by the formula*

$$\delta\varphi(x) = \frac{1}{c(x)} \sum_{e, e^+ = x} r(e)\varphi(e),$$

for all $\varphi \in C_0^a(\mathcal{E})$.

Proof: For $f \in C_0(\mathcal{V})$ and $\varphi \in C_0^a(\mathcal{E})$, using (2.1), we get

$$\begin{aligned}
\langle df, \varphi \rangle_{\mathcal{E}} &= \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) df(e) \varphi(e) \\
&= \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \left(f(e^+) - f(e^-) \right) \varphi(e) \\
&= \frac{1}{2} \sum_{x \in \mathcal{V}} f(x) \left(\sum_{e, e^+ = x} r(e) \varphi(e) - \sum_{e, e^- = x} r(e) \varphi(e) \right).
\end{aligned}$$

But, $r(-e) = r(e)$ and $\sum_{e, e^+ = x} r(e) \varphi(e) = - \sum_{e, e^- = x} r(e) \varphi(e)$.

So we have,

$$\begin{aligned}
\langle df, \varphi \rangle_{\mathcal{E}} &= \sum_{x \in \mathcal{V}} c(x) f(x) \left(\frac{1}{c(x)} \sum_{e, e^+ = x} r(e) \varphi(e) \right) \\
&= \langle f, \delta \varphi \rangle_{\mathcal{V}}.
\end{aligned}$$

□

We introduce now a very important result inspired by [11].

Lemma 2.2. *Let x and x_0 in $\mathcal{V}$, then there exists a positive constant C_{xx_0} such that*

$$\left| f(x) \right| \leq C_{xx_0} \left(\left| f(x_0) \right| + \left\| df \right\|_{l^2(\mathcal{E})} \right), \tag{2.2}$$

for all $f \in C_0(\mathcal{V})$.

Proof: As G is connected, then we can find a path γ_{xx_0} joining x to x_0, i.e,

$$\gamma_{xx_0} = \{e_k\}_{k=1,\dots,n} \; with \; e_1^- = x, \; e_n^+ = x_0 \; and \; if \; n \geq 2 \;, \; \forall j \,; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

Then, using the triangle inequality, we have

$$\begin{aligned}
\left| f(x) - f(x_0) \right| &= \left| f(x) - f(e_1^+) + f(e_1^+) - f(e_2^+) + \dots + f(e_{n-1}^+) - f(x_0) \right| \\
&\leq \left| df(e_1) \right| + \left| df(e_2) \right| + \dots + \left| df(e_n) \right| \\
&\leq \sum_{e \in \gamma_{xx_0}} \frac{1}{\sqrt{r(e)}} \sqrt{r(e)} \left| df(e) \right|.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left| f(x) - f(x_0) \right| &\leq \left(\sum_{e \in \gamma_{xx_0}} \frac{1}{r(e)} \right)^{\frac{1}{2}} \left(\sum_{e \in \gamma_{xx_0}} r(e) (df(e))^2 \right)^{\frac{1}{2}} \\
&\leq S_{xx_0} \left(\sum_{e \in \mathcal{E}} r(e) (df(e))^2 \right)^{\frac{1}{2}} \\
&\leq S_{xx_0} \left\| df \right\|_{l^2(\mathcal{E})},
\end{aligned}$$

with $S_{xx_0} = \left(\sum_{e \in \gamma_{xx_0}} \frac{1}{r(e)} \right)^{\frac{1}{2}}$.

Thus, we deduce that

$$\begin{aligned} \left|f(x)\right| &\leq \left|f(x)-f(x_0)\right| + \left|f(x_0)\right| \\ &\leq S_{xx_0}\left\|df\right\|_{l^2(\mathcal{E})} + \left|f(x_0)\right| \\ &\leq C_{xx_0}\left(\left\|df\right\|_{l^2(\mathcal{E})} + \left|f(x_0)\right|\right), \end{aligned}$$

with $C_{xx_0} = max(S_{xx_0}, 1)$. □

Before giving another important result, for $f \in C_0(\mathcal{V})$, we define the mean value $\overline{f}$ of f by

$$\overline{f}(e) = \frac{f(e^+) + f(e^-)}{2}$$

for all $e \in \mathcal{E}$.

And we have from [10] the following derivation property:

Lemma 2.3. *For $f, g \in C_0(\mathcal{V})$ and $\varphi \in C_0^a(\mathcal{E})$, it follows*

$$d(fg)(e) = f(e^+)dg(e) + g(e^-)d(f)(e). \tag{2.3}$$

$$\delta(\overline{f}\varphi)(x) = f(x)\delta\varphi(x) - \frac{1}{2c(x)}\sum_{e,e^+=x} r(e)d(f)(e)\varphi(e). \tag{2.4}$$

Proof: For $f, g \in C_0(\mathcal{V})$ and $e \in \mathcal{E}$,

$$\begin{aligned} d(fg)(e) &= (fg)(e^+) - (fg)(e^-) \\ &= f(e^+)\left(g(e^+) - g(e^-)\right) + g(e^-)\left(f(e^+) - f(e^-)\right) \\ &= f(e^+)d(g)(e) + g(e^-)d(f)(e). \end{aligned}$$

On the other hand, for $\varphi \in C_0^a(\mathcal{E})$ applying the characterization of δ from Lemma (2.1) to the function $\overline{f}\varphi \in C_0^a(\mathcal{E})$, we have

$$\begin{aligned} \delta(\overline{f}\varphi)(x) &= \frac{1}{c(x)}\sum_{e,e^+=x} r(e)(\overline{f}\varphi)(e) \\ &= \frac{1}{c(x)}\sum_{e,e^+=x} r(e)\left(\frac{f(e^+)+f(e^-)}{2}\right)\varphi(e) \\ &= \frac{1}{c(x)}\sum_{e,e^+=x} r(e)f(e^+)\varphi(e) + \frac{1}{c(x)}\sum_{e,e^+=x} r(e)\left(\frac{f(e^-)-f(e^+)}{2}\right)\varphi(e) \\ &= f(x)\frac{1}{c(x)}\sum_{e,e^+=x} r(e)\varphi(e) + \frac{1}{2c(x)}\sum_{e,e^+=x} r(e)d(f)(-e)\varphi(e) \\ &= f(x)\delta(\varphi)(x) - \frac{1}{2c(x)}\sum_{e,e^+=x} r(e)d(f)(e)\varphi(e). \end{aligned}$$

□

The Gauss-Bonnet operator: it is the endomorphism

$$D = d + \delta : C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E}) \longrightarrow C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$$

with,

$$D(f,\varphi) = \delta\varphi + df, \quad \forall (f,\varphi) \in C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E}).$$

And it is a symmetric operator.

3. Non-Parabolicity at Infinity

Now we introduce the discrete result of Carron [5]:

Definition 3.1. *We say that D is non-parabolic at infinity if there is a finite subgraph G_K of G such that for all finite subset U of $G \setminus G_K$, there exists a positive constant $C = C(U)$ such that holds the following inequality*

$$C \left\|(f, \varphi)\right\|_{l^2(U)} \leq \left\|D(f, \varphi)\right\|_{l^2(G \setminus G_K)}, \ \forall (f, \varphi) \in C_0(\mathcal{V} \setminus K) \times C_0^a(\mathcal{E} \setminus \mathcal{E}_K).$$

Remark 3.2. *We call a finite subset U of G a couple $U := (\mathcal{V}_U, \mathcal{E}_U)$ such that $\mathcal{V}_U$ is a finite subset of $\mathcal{V}$ and $\mathcal{E}_U$ is a finite subset of $\mathcal{E}$. And, we denote*

$$\left\|(f, \varphi)\right\|^2_{l^2(U)} = \left\|f\right\|^2_{l^2(\mathcal{V}_U)} + \left\|\varphi\right\|^2_{l^2(\mathcal{E}_U)}.$$

Definition 3.3. *$G_{\widetilde{K}}$ is **a neighborhood** of G_K if $G_{\widetilde{K}} := (\widetilde{K}, \mathcal{E}_{\widetilde{K}})$ is a finite subgraph of G such that*

$$\begin{cases} i)\ K \subset \widetilde{K} \ finite, \\ ii)\ \mathcal{E}_K \sqcup \partial\mathcal{E}_K \subset \mathcal{E}_{\widetilde{K}}, \\ iii)\ e = (x, y) \in \mathcal{E}_{\widetilde{K}} \Rightarrow x,\ y \in \widetilde{K}. \end{cases}$$

*Since we can define **the smallest neighborhood** of G_K by $G_{\widetilde{K}_0}$, where $G_{\widetilde{K}_0}$ is a finite subgraph of G contains G_K and its boundary.*

Remark 3.4. *In [9], $G_{\widetilde{K}_0}$ is called a combinatorial neighborhood of G_K.*

Lemma 3.5. *If D is non-parabolic at infinity then, for every finite subset U of G there exists a positive constant $C' = C'(U)$ such that*

$$C' \left\|(f, \varphi)\right\|_{l^2(U)} \leq \left\|D(f, \varphi)\right\|_{l^2(G)} + \left\|(f, \varphi)\right\|_{l^2(G_{\widetilde{K}})}, \ \forall (f, \varphi) \in C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E}), \tag{3.5}$$

where $G_{\widetilde{K}}$ is a neighborhood of G_K.

Proof: Since U is a finite subset of G it can be reduced to a point or an edge.

Let x any vertex of G, we start by proving

$$C' \left|f(x)\right| \leq \left\|df\right\|_{l^2(\mathcal{E})} + \left\|f\right\|_{l^2(\widetilde{K})}, \ \forall f \in C_0(\mathcal{V}).$$

$G_{\widetilde{K}}$ is a finite subgraph of G, so according to Lemma 2.2, we obtain

$$f^2(x) \leq C_1 \left(\left\|f\right\|^2_{l^2(\widetilde{K})} + \left\|df\right\|^2_{l^2(\mathcal{E})}\right), \tag{3.6}$$

where C_1 is a positive constant which depends on x and $\widetilde{K}$. Indeed:
let $x \in \mathcal{V}$ and $x_0 \in \widetilde{K}$, using Lemma 2.2, we obtain

$$f^2(x) \leq C_{xx_0} \left(f^2(x_0) + \left\|df\right\|^2_{l^2(\mathcal{E})}\right). \tag{3.7}$$

Multiplying (3.7) by $c(x_0) > 0$, we get

$$\begin{aligned} c(x_0) f^2(x) &\leq C_{xx_0} \left(c(x_0) f^2(x_0) + c(x_0) \left\|df\right\|^2_{l^2(\mathcal{E})}\right) \\ &\leq C_{xx_0} \left(\left\|f\right\|^2_{l^2(\widetilde{K})} + c(x_0) \left\|df\right\|^2_{l^2(\mathcal{E})}\right) \\ &\leq C'_{xx_0} \left(\left\|f\right\|^2_{l^2(\widetilde{K})} + \left\|df\right\|^2_{l^2(\mathcal{E})}\right), \end{aligned}$$

where $C'_{xx_0} = max(C_{xx_0}, c(x_0)C_{xx_0})$.
Then, we have

$$f^2(x) \leq \frac{C'_{xx_0}}{c(x_0)} \left(\|f\|^2_{l^2(\widetilde{K})} + \|df\|^2_{l^2(\mathcal{E})} \right).$$

Finally, we obtain

$$f^2(x) \leq C_1 \left(\|f\|^2_{l^2(\widetilde{K})} + \|df\|^2_{l^2(\mathcal{E})} \right)$$

where $C_1 = \frac{C'_{xx_0}}{c(x_0)}$.

On the other hand, we want to show the following inequality, for any edge $e \in \mathcal{E}$

$$C'' |\varphi(e)| \leq \|\delta\varphi\|_{l^2(\mathcal{V})} + \|\varphi\|_{l^2(\mathcal{E}_{\widetilde{K}})}, \ \forall \varphi \in C_0(\mathcal{E}).$$

For $e \in \mathcal{E}_K \subset \mathcal{E}_{\widetilde{K}}$ finite, we have

$$\varphi^2(e) \leq \|\varphi\|^2_{l^2(\mathcal{E}_{\widetilde{K}})} \leq \|\varphi\|^2_{l^2(\mathcal{E}_{\widetilde{K}})} + \|\delta\varphi\|^2_{l^2(\mathcal{V})}.$$

And if $e \in \mathcal{E} \setminus \mathcal{E}_K$, we consider the indicator function of K^c, denoted by χ

$$\chi(x) = \begin{cases} 0 \text{ if } x \in K \\ 1 \text{ otherwise.} \end{cases} \tag{3.8}$$

which gives

$$d\chi(e) = \begin{cases} 0 \text{ if } e \in \mathcal{E}_K, \\ \pm 1 \text{ if } e \in \partial\mathcal{E}_K, \\ 0 \text{ otherwise.} \end{cases} \quad \& \quad \overline{\chi}(e) = \begin{cases} 0 \text{ if } e \in \mathcal{E}_K, \\ \frac{1}{2} \text{ if } e \in \partial\mathcal{E}_K, \\ 1 \text{ otherwise.} \end{cases}$$

Let $\varphi \in C_0^a(\mathcal{E})$, we have then $\overline{\chi}\varphi$ with finite support in $\mathcal{E} \setminus \mathcal{E}_K$. Thus, applying the definition of the non-parabolicity at infinity of D to the function $(0, \overline{\chi}\varphi)$, we obtain

$$\|\overline{\chi}\varphi\|^2_{l^2(U)} \leq C \|\delta(\overline{\chi}\varphi)\|^2_{l^2(\mathcal{V})},$$

where $C = \frac{1}{C(U)}$.

Since we have $e \in \mathcal{E} \setminus \mathcal{E}_K$, this implies that

$$\varphi^2(e) \leq C \|\delta(\overline{\chi}\varphi)\|^2_{l^2(\mathcal{V})}. \tag{3.9}$$

The derivation property of Lemma (2.3), gives

$$\delta(\overline{\chi}\varphi)(x) = \chi(x)\delta\varphi(x) - \frac{1}{2c(x)} \sum_{e, e^+ = x} r(e) d(\chi)(e)\varphi(e).$$

And by the inequality $(a-b)^2 \leq 2(a^2 + b^2)$, we obtain

$$\begin{aligned} \|\delta(\overline{\chi}\varphi)\|^2_{l^2(\mathcal{V})} &= \sum_{x \in \mathcal{V}} c(x)(\delta(\overline{\chi}\varphi))^2 \\ &\leq 2 \left[\underbrace{\sum_{x \in \mathcal{V}} c(x) \left(\chi(x)\delta\varphi(x)\right)^2}_{I} + \underbrace{\sum_{x \in \mathcal{V}} c(x) \left(\frac{1}{2c(x)} \sum_{e, e^+ = x} r(e) d(\chi)(e)\varphi(e) \right)^2}_{J} \right]. \end{aligned}$$

So, for the first term we have

$$I = \sum_{x\in\mathcal{V}\setminus K} c(x)\left(\delta\varphi(x)\right)^2 \le \left\|\delta\varphi\right\|^2_{l^2(\mathcal{V})} \tag{3.10}$$

and for the second one, we get

$$J = \underbrace{\sum_{x\in K}\frac{1}{2c(x)}\left(\sum_{e,e^+=x} r(e)d(\chi)(e)\varphi(e)\right)^2}_{J_1} + \underbrace{\sum_{x\in\mathcal{V}\setminus K}\frac{1}{2c(x)}\left(\sum_{e,e^+=x} r(e)d(\chi)(e)\varphi(e)\right)^2}_{J_2}. \tag{3.11}$$

Using that $supp(d\chi) = \partial\mathcal{E}_K \subset \mathcal{E}_{\widetilde{K}}$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
J_1 &= \sum_{x\in K}\frac{1}{2c(x)}\left(\sum_{\substack{e,e^+=x\\ e\in supp d\chi}} r(e)\varphi(e)\right)^2\\
&= C_K\left(\sum_{e\in supp d\chi} r(e)\varphi(e)\right)^2\\
&\le C_K\left(\sum_{e\in supp d\chi} r(e)\right)\left(\sum_{e\in supp d\chi} r(e)\varphi^2(e)\right)\\
&\le C_K C'_K\sum_{e\in\mathcal{E}_{\widetilde{K}}} r(e)\varphi^2(e)\\
&= C_2\left\|\varphi\right\|^2_{l^2(\mathcal{E}_{\widetilde{K}})},
\end{aligned}$$

where $C_K = \max\limits_{x\in K}\frac{1}{2c(x)}$, $C'_K = \sharp\mathcal{E}_{\widetilde{K}}\max\limits_{e\in\mathcal{E}_{\widetilde{K}}} r(e)$ and $C_2 = C_K C'_K$.

And for J_2, we have $e = (e^-, e^+) \in supp d\chi = \partial\mathcal{E}_K$, so if $e^- \in K$, $e^+ \in \partial K$.

$$\begin{aligned}
J_2 &= \sum_{x\in\partial K}\frac{1}{2c(x)}\left(\sum_{\substack{e,e^+=x\\ e\in supp d\chi}} r(e)\varphi(e)\right)^2\\
&= C''_K\left(\sum_{e\in supp d\chi} r(e)\varphi(e)\right)^2\\
&\le C''_K\left(\sum_{e\in supp d\chi} r(e)\right)\left(\sum_{e\in supp d\chi} r(e)\varphi^2(e)\right)\\
&\le C''_K C'_K\sum_{e\in\mathcal{E}_{\widetilde{K}}} r(e)\varphi^2(e)\\
&= C'_2\left\|\varphi\right\|^2_{l^2(\mathcal{E}_{\widetilde{K}})},
\end{aligned}$$

where $C''_K = \max\limits_{x\in\partial K}\frac{1}{2c(x)}$ and $C'_2 = C''_K C'_K$.

Thus, (3.11) becomes

$$J \leq C_2'' \|\varphi\|^2_{l^2(\mathcal{E}_{\widetilde{K}})}, \tag{3.12}$$

where $C_2'' = max(C_2, C_2')$.

So by (3.10) and (3.12), we get

$$\|\delta(\overline{\chi}\varphi)\|^2_{l^2(\mathcal{V})} \leq max(2, 2C_2'')\left(\|\delta\varphi\|^2_{l^2(\mathcal{V})} + \|\varphi\|^2_{l^2(\mathcal{E}_{\widetilde{K}})}\right). \tag{3.13}$$

Finally, (3.9) and (3.13) give

$$\varphi^2(e) \leq \widetilde{C}\left(\|\delta\varphi\|^2_{l^2(\mathcal{V})} + \|\varphi\|^2_{l^2(\mathcal{E}_{\widetilde{K}})}\right)$$

where $\widetilde{C} = \frac{2max(1,C_2'')}{C}$. □

Proposition 3.6. *If D is non-parabolic at infinity, then we can construct a Hilbert space W such that :*

1. *$C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ is dense in W.*
2. *The injection of $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ to $C(\mathcal{V}) \oplus C^a(\mathcal{E})$ extends by continuity to W.*
3. *$D : W \longrightarrow l^2(G)$ is a bounded operator.*

Remark 3.7. *In 1) and 2) we use the topology of ponctual convergence on $C(\mathcal{V}) \oplus C^a(\mathcal{E})$, it means, the sequence (f_n, φ_n) converges ponctually to (f, φ) on $C(\mathcal{V}) \oplus C^a(\mathcal{E})$ if $f_n(x)$ converges to $f(x)$, $\forall x \in \mathcal{V}$ and $\varphi_n(e)$ converges to $\varphi(e)$, $\forall e \in \mathcal{E}$.*

Remark 3.8. *In Carron's paper [5], the injection of the space of functions with compact support to l^2_{loc} extends by continuity to W. But, in our case we didn't need to introduce the space l^2_{loc} because in discrete case this notion is trivial.*

Proof: Let us denote by W the closure of $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ for the norm

$$N_{\widetilde{K}}(f, \varphi) = \left(\|(f, \varphi)\|^2_{l^2(G_{\widetilde{K}})} + \|D(f, \varphi)\|^2_{l^2(G)}\right)^{\frac{1}{2}},$$

where $G_{\widetilde{K}}$ is a neighborhood of G_K (see Definition (3.3)).

Aim i): $N_{\widetilde{K}}$ is a norm on W, we just look at the nullity, we have

$$\begin{aligned} N_{\widetilde{K}}(f, \varphi) = 0 \quad &\Leftrightarrow \quad \|(f, \varphi)\|_{l^2(G_{\widetilde{K}})} = 0 \text{ and } \|D(f, \varphi)\|_{l^2(G)} = 0 \\ &\Leftrightarrow \quad \|f\|_{l^2(\widetilde{K})} = 0, \ \|\varphi\|_{l^2(\mathcal{E}_{\widetilde{K}})} = 0, \ \|df\|_{l^2(\mathcal{E})} = 0 \text{ and } \|\delta\varphi\|_{l^2(\mathcal{V})} = 0. \end{aligned}$$

For any $x \in \mathcal{V}$ and as $\sharp\widetilde{K} < \infty$, from Lemma (3.5), we get

$$f^2(x) \leq C_1\left(\|f\|^2_{l^2(\widetilde{K})} + \|df\|^2_{l^2(\mathcal{E})}\right). \tag{3.14}$$

But, $\|f\|_{l^2(\widetilde{K})} = 0$ and $\|df\|_{l^2(\mathcal{E})} = 0$. So it follows immediately that $f = 0$ on $\mathcal{V}$.

It remains to show that if $\|\varphi\|_{l^2(\mathcal{E}_{\widetilde{K}})} = 0$ and $\|\delta\varphi\|_{l^2(\mathcal{V})} = 0$ then $\varphi = 0$. We suppose that $\varphi \neq 0$.

φ is a finite support function in $\mathcal{E} \setminus \mathcal{E}_{\widetilde{K}}$ and therefore, by Lemma (3.5) where U equals to the support of φ, there exists a positive constant C such that

$$C\|\varphi\|_{l^2(\mathcal{E}_U)} \leq \|\varphi\|_{l^2(\mathcal{E}_{\widetilde{K}})} + \|\delta\varphi\|_{l^2(\mathcal{V})}.$$

But, $\left\|\varphi\right\|_{l^2(\mathcal{E}_{\widetilde{K}})} = \left\|\delta\varphi\right\|_{l^2(\mathcal{V})} = 0$, since we get $\varphi = 0$ on $\mathcal{E}_U$, which is impossible.

Aim ii) Show that the space W is independent of the choice of $G_{\widetilde{K}}$.

Let $G_{\widetilde{K}_1}$ be another neighborhood of G_K such that $K \subset \widetilde{K}_0 \subset \widetilde{K}_1$.

So, we have

$$N_{\widetilde{K}_0}(f,\varphi) \le N_{\widetilde{K}_1}(f,\varphi).$$

Moreover, to show the existence of a constant $C > 0$ such that $N_{\widetilde{K}_1}(f,\varphi) \le C N_{\widetilde{K}_0}(f,\varphi)$, it suffices to show the existence of a constant $C > 0$ such that $\left\|(f,\varphi)\right\|^2_{l^2(\widetilde{K}_1 \setminus \widetilde{K}_0)} \le C N^2_{\widetilde{K}_0}(f,\varphi)$. Indeed, we have

$$\begin{aligned}
N^2_{\widetilde{K}_1}(f,\varphi) &= \left\|(f,\varphi)\right\|^2_{l^2(\widetilde{K}_1)} + \left\|D(f,\varphi)\right\|^2_{l^2(G)} \\
&= \left\|(f,\varphi)\right\|^2_{l^2(\widetilde{K}_1 \setminus \widetilde{K}_0)} + \left\|(f,\varphi)\right\|^2_{l^2(\widetilde{K}_0)} + \left\|D(f,\varphi)\right\|^2_{l^2(G)} \\
&= \left\|(f,\varphi)\right\|^2_{l^2(\widetilde{K}_1 \setminus \widetilde{K}_0)} + N^2_{\widetilde{K}_0}(f,\varphi).
\end{aligned}$$

Using lemma (3.5) and as we have $\sharp(\widetilde{K}_1 \setminus \widetilde{K}_0) < \infty$, we get

$$\left\|f\right\|^2_{l^2(\widetilde{K}_1 \setminus \widetilde{K}_0)} \le C\left(\left\|f\right\|^2_{l^2(\widetilde{K}_0)} + \left\|df\right\|^2_{l^2(\mathcal{E})}\right),$$

where $C = C(\widetilde{K}_1 \setminus \widetilde{K}_0, \widetilde{K}_0)$.

And

$$\left\|\varphi\right\|^2_{l^2(\mathcal{E}_{\widetilde{K}_1} \setminus \mathcal{E}_{\widetilde{K}_0})} \le C\left(\left\|\varphi\right\|^2_{l^2(\mathcal{E}_{\widetilde{K}_0})} + \left\|\delta\varphi\right\|^2_{l^2(\mathcal{V})}\right).$$

where $C = C(\widetilde{K}_1 \setminus \widetilde{K}_0, \widetilde{K}_0)$.

So, we obtain

$$\left\|(f,\varphi)\right\|^2_{l^2(G_{\widetilde{K}_1} \setminus G_{\widetilde{K}_0})} \le C N^2_{\widetilde{K}_0}(f,\varphi).$$

Thus, we have shown that the construction of a norm on W is independent of the choice of the neighborhood associated to the subgraph G_K. We set:

$$\left\|(f,\varphi)\right\|_W := \left(\left\|(f,\varphi)\right\|^2_{l^2(G_{\widetilde{K}_0})} + \left\|D(f,\varphi)\right\|^2_{l^2(G)}\right)^{\frac{1}{2}},$$

for $(f,\varphi) \in C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$.

Aim iii): By Lemma (3.5), we have the injection of $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ to $C(\mathcal{V}) \oplus C^a(\mathcal{E})$ extends by continuity to W.

Aim iv): we have

$$\left\|D(f,\varphi)\right\|^2_{l^2(G)} \le \left\|(f,\varphi)\right\|^2_{l^2(G_{\widetilde{K}})} + \left\|D(f,\varphi)\right\|^2_{l^2(G)} = \left\|(f,\varphi)\right\|^2_W.$$

Consequently, $D : W \longrightarrow l^2(G)$ is a bounded operator. □

4. Semi-Fredholmness of the Discrete Gauss-Bonnet Operator

Definition 4.1. *An operator is semi-Fredholm if its range is closed and its kernel is finite dimensional .*

Now we come to our main result:

Theorem. Let W be a Hilbert space satisfying:

1. $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ is dense in W.
2. The injection of $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ to $C(\mathcal{V}) \oplus C^a(\mathcal{E})$ extends by continuity to W.
3. $D : W \longrightarrow l^2(G)$ is a bounded operator.

Then, the following conditions are equivalent:
i) $D : W \longrightarrow l^2(G)$ is semi-Fredholm.
ii) There exists a finite subgraph G_K of G and a positive constant $C = C_K$ such that

$$C \left\|(f,\varphi)\right\|_W \leq \left\|D(f,\varphi)\right\|_{l^2(G)}, \ \forall (f,\varphi) \in C_0(\mathcal{V} \setminus K) \times C_0^a(\mathcal{E} \setminus \mathcal{E}_K). \tag{4.15}$$

Proof: We take the same arguments used by Carron [5]. We start by showing the direct implication, we assume that the conclusion is false. Then, we can find an increasing sequence of finite subgraph $\{G_{K_n}\}_n$ such that $G = \bigcup_n G_{K_n}$ and a sequence $\{\sigma_n\}_n$ with finite support in $\mathcal{V} \setminus K_n$ satisfying the following conditions, for all $n \geq 1$

$$\begin{cases} \sigma_n = (f_n, \varphi_n) \in C_0(\mathcal{V} \setminus K_n) \times C_0^a(\mathcal{E} \setminus \mathcal{E}_{K_n}), \\ \|\sigma_n\|_W = 1, \\ \|D\sigma_n\|_{l^2(G)} \leq \frac{1}{n}. \end{cases}$$

On the other hand, it was assumed that $D : W \longrightarrow l^2(G)$ is semi-Fredholm. Therefore, by [13] there exists a bounded operator $P : l^2(G) \longrightarrow W$ such that

$$P \circ D = Id_W - H, \tag{4.16}$$

where H is the orthogonal projection onto the kernel of D, it is an operator with finite rank.
Then, we obtain

$$\begin{aligned} \|\sigma_n\|_W &\leq \|(P \circ D)\sigma_n\|_W + \|H\sigma_n\|_W \\ &\leq \|P\| \, \|D\sigma_n\|_{l^2(G)} + \|H\sigma_n\|_W \\ &\leq \left(\frac{\|P\|}{n} + \|H\sigma_n\|_W\right). \end{aligned}$$

If

$$\lim_{n\to\infty} \|H\sigma_n\|_W = 0 \Longrightarrow \lim_{n\to\infty} \|\sigma_n\|_W = 0,$$

which contradicts the assumption $\|\sigma_n\|_W = 1$.

So, our aim is to prove that $\{H\sigma_n\}_n$ converges to 0 in W. Indeed, we set

$$\sigma_n = \sigma_n^1 + \sigma_n^2 \tag{4.17}$$

with $\sigma_n^1 (= H\sigma_n) \in KerD$ and $\sigma_n^2 \in (KerD)^\perp$.

Such as

$$\begin{cases} (P \circ D)\sigma_n = \sigma_n^2, \\ \|P \circ D\sigma_n\|_W \leq \|P\| \, \|D\sigma_n\|_{l^2(G)} \longrightarrow_{n\to\infty} 0. \end{cases}$$

Then, for the norm of W

$$\lim_{n\to\infty} \sigma_n^2 = 0. \tag{4.18}$$

Moreover, $\{\sigma_n^1\}_n$ is a bounded sequence of $KerD$ which is of finite dimension. So we can extract a subsequence converging to σ in W, which we denote $\{\sigma^1_{\varphi(n)}\}_n$.

Using (4.17) and (4.18), $\{\sigma_{\varphi(n)}\}_n$ converges in W to σ (as a sum of two converging sequences) and as a consequence $\|\sigma\|_W = 1$.

Let us prove that $\sigma = 0$ where $\sigma = \lim \sigma_{\varphi(n)} = \lim \sigma^1_{\varphi(n)}$.

We suppose that $\sigma \neq 0$. As W is injected continuously in $C(\mathcal{V}) \oplus C^a(\mathcal{E})$, there exists $x \in V$ such that $\{\sigma_{\varphi(n)}(x)\}_n$ converges to $\sigma(x) \neq 0$. But, by construction the sequence $\{\sigma_{\varphi(n)}\}_n$ converges ponctually to 0 (the sequence $\{\sigma_{\varphi(n)}\}_n$ has a finite support outside of G_{K_n}). Hence, we conclude that $\sigma(x) = 0$ which is absurd.

It remains to prove *ii*) $\Rightarrow$ *i*).

First step: We construct a bounded operator $Q : l^2(G) \longrightarrow W$ such that $Q \circ D - Id_W$ is a compact operator, this will show that $D : W \longrightarrow l^2(G)$ has a finite kernel and a closed range [13].

Let D_1 be the restriction of D on $G \setminus G_K$, so $D_1 : W(G \setminus G_K) \longrightarrow l^2(G)$ is bounded, where $W(G \setminus G_K) = \{\sigma = (f, \varphi) \in W;\ \sigma = 0 \ on \ G_K\}$. Moreover, by assumption we have

$$C \left\|(f, \varphi)\right\|_W \leq \left\|D(f, \varphi)\right\|_{l^2(G)},\ \forall (f, \varphi) \in C_0(\mathcal{V} \setminus K) \times C_0^a(\mathcal{E} \setminus \mathcal{E}_K).$$

Then, D_1 is injective with closed range, which allows the existence of a left inverse P_1 such that

$$P_1 \circ D_1 = Id.$$

On the other hand, we denote

$$D_2 : l^2(\widetilde{K}_1) \longrightarrow l^2(G)$$

where $\widetilde{K}_1$ is a neighborhood (see Definition (3.3)) of $\widetilde{K}_0$, such that $\widetilde{K}_0$ is the smallest neighborhood of K.

Since $l^2(\widetilde{K}_1)$ is a vector space of finite dimension, then D_2 is continuous with closed range. We denote P_2 "the parametrix" which is a continuous operator satisfying

$$P_2 \circ D_2 = Id - H_2,$$

where H_2 is the orthogonal projection onto the kernel of D_2.

We consider now the indicator function χ as in (3.8) by replacing K by $\widetilde{K}_0$, which gives $d\chi$, $\overline{\chi}$, $1 - \chi$ and $1 - \overline{\chi}$ where

$$(1-\chi)(x) = \begin{cases} 1 \text{ if } x \in \widetilde{K}_0 \\ 0 \text{ otherwise.} \end{cases} \quad and \ (1-\overline{\chi})(e) = \begin{cases} 1 \text{ if } e \in \mathcal{E}_{\widetilde{K}_0}, \\ \frac{1}{2} \text{ if } e \in \partial\mathcal{E}_{\widetilde{K}_0}, \\ 0 \text{ otherwise.} \end{cases}$$

Furthermore, we define the operator $\chi.$ depending on the domain by:

If $\chi. : C_0(\mathcal{V}) \longrightarrow C_0(\mathcal{V})$ so we have $\chi.f = \chi f$, for all $f \in C_0(\mathcal{V})$.

If $\chi. : C_0^a(\mathcal{E}) \longrightarrow C_0^a(\mathcal{E})$ we get $\chi.\varphi = \overline{\chi}\varphi$, for all $\varphi \in C_0^a(\mathcal{E})$.

If $\chi. : C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E}) \longrightarrow C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ hence we obtain $\chi.(f,\varphi) = (\chi f, \overline{\chi}\varphi)$,

for all $(f,\varphi) \in C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$.

We set

$$Q\sigma := P_2(1-\chi)\sigma + P_1\chi\sigma,$$

where $\sigma = (f,\varphi)$.

Second step: Let us check that the operator $Q \circ D - Id$ is compact.
We denote the following bracket for any two operators A and B:

$$[A,B] = AB - BA.$$

Then, we obain

$$\begin{aligned}
Q \circ D &= P_2(1-\chi)D + P_1\chi D \\
&= P_2D(1-\chi) + P_2[1-\chi, D] + P_1D\chi + P_1[\chi, D] \\
&= P_2D_2(1-\chi) + P_2[1-\chi, D] + P_1D_1\chi + P_1[\chi, D] \\
&= (Id - H_2)(1-\chi) + P_2[1-\chi, D] + Id(\chi) + P_1[\chi, D] \\
&= Id - H_2(1-\chi) + P_2[1-\chi, D] + P_1[\chi, D].
\end{aligned}$$

We just calculate $P_2[\chi, D]$. We have

$$[\chi, D] = [\chi, d] + [\chi, \delta].$$

For the first bracket, we obtain

$$\begin{aligned}
[\chi, d]f(e) &= \overline{\chi}(e)d(f)(e) - d(\chi f)(e) \\
&= \frac{1}{2}\left[\chi(e^+) + \chi(e^-)\right] d(f)(e) - \chi(e^+)d(f)(e) - f(e^-)d\chi(e) \\
&= -\frac{1}{2}d\chi(e)d(f)(e) - f(e^-)d\chi(e).
\end{aligned}$$

And for the second one, we get

$$\begin{aligned}
[\chi, \delta]\varphi(x) &= \chi(x)\delta(\varphi)(x) - \delta(\overline{\chi}\varphi)(x) \\
&= \chi(x)\delta(\varphi)(x) - \chi(x)\delta(\varphi)(x) + \frac{1}{2}\sum_{e,e^+=x} d(\chi)(e)\varphi(e) \\
&= \frac{1}{2}\sum_{e,e^+=x} d(\chi)(e)\varphi(e).
\end{aligned}$$

But, the support of $d(\chi)$ is included in $\partial\mathcal{E}_{\widetilde{K}_0} \subset \widetilde{K}_1$ which is finite. Then, $[\chi, D]$ has a finite range so it is a compact operator.
Finally, $Q \circ D = Id + H$ where H is a compact operator . □

Remark 4.2. *In the Theorem , we obtain D Fredholm if it is an essential-selfadjoint operator [5].*

Remark 4.3. *There is a second method inspired from [2] to show ii) ⇒ i) of the Theorem . This can be demonstrated with the aid of the following* ***claim****: "If* $\sigma_n = (f_n, \varphi_n) \in C_0(\mathcal{V}) \times C_0^a(\mathcal{E})$ *is W-bounded and* $(D\sigma_n)_n$ *is convergent in* $l^2(G)$*, then* $(\sigma_n)_n$ *has a W-convergent subsequence".*

We have the following result:

Proposition 4.4. *Let W be a Hilbert space satisfying:*

1. $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ *is dense in W.*
2. *The injection of* $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ *to* $C(\mathcal{V}) \oplus C^a(\mathcal{E})$ *extends by continuity to W.*
3. $D : W \longrightarrow l^2(G)$ *is a bounded operator.*

Then if there exists a finite subgraph G_K *of G and a positive constant* $C = C_K$ *such that*

$$C \left\|(f, \varphi)\right\|_W \leq \left\|D(f, \varphi)\right\|_{l^2(G)}, \ \forall (f, \varphi) \in C_0(\mathcal{V} \setminus K) \times C_0^a(\mathcal{E} \setminus \mathcal{E}_K), \tag{4.19}$$

so necessarily, the operator $D : W \longrightarrow l^2(G)$ *is semi-Fredholm.*

Proof: We start by proving the following **claim**: if $\sigma_n = (f_n, \varphi_n) \in C_0(\mathcal{V}) \times C_0^a(\mathcal{E})$ is W-bounded and $(D\sigma_n)_n$ is convergent in $l^2(G)$, then $(\sigma_n)_n$ has a W-convergent subsequence.

Let $G_{\widetilde{K}}$ be a neighborhood of the subgraph G_K (see Definition 3.3), then $(\sigma_n \upharpoonright_{\widetilde{K}})_n$ is a bounded sequence in a vector space with finite dimension. Hence, it admits a convergent subsequence.

In $G \setminus G_{\widetilde{K}}$, we consider the indicator function χ as in (3.8) by replacing K by $\widetilde{K}$. Then, we obtain a function $\chi\sigma_n$ with finite support in $G \setminus G_K$ and we can apply the inequality (4.19) to $\chi\sigma_n$, in particular to $(\chi f_n, 0)$ and $(0, \overline{\chi}\varphi_n)$. First, we obtain

$$\left\|\chi f_n\right\|_W \leq C \left\|d(\chi f_n)\right\|_{l^2(\mathcal{E})}.$$

But, from the equality (2.3) of Lemma (2.3), we get

$$d(\chi f_n)(e) = \chi(e^+)d(f_n)(e) + f_n(e^-)d(\chi)(e).$$

We have $(d(f_n))_n$ is a convergent sequence and $supp(d\chi) \subset \mathcal{E}_{\widetilde{K}}$ is finite, thus, $f_n(x) \upharpoonright_{\widetilde{K}}$ admits a convergent subsequence.
Then we may conclude that χf_n admits a W-convergent subsequence, i.e, $(f_n \upharpoonright_{V \setminus \widetilde{K}})_n$ admits a W-convergent subsequence.

Second, we have

$$\left\|\overline{\chi}\varphi_n\right\|_W \leq C \left\|\delta(\overline{\chi}\varphi_n)\right\|_{l^2(\mathcal{V})}.$$

Since the equality (2.4) of Lemma (2.3) gives

$$\delta(\overline{\chi}\varphi_n)(x) = \chi(x)\delta(\varphi_n)(x) - \frac{1}{2c(x)} \sum_{e, e^+ = x} r(e)d(\chi)(e)\varphi_n(e), \ \forall x \in \mathcal{V}.$$

Furthermore by assumptions the sequence $(\delta(\varphi_n))_n$ is convergent and $supp(d\chi) \subset \mathcal{E}_{\widetilde{K}}$ is finite, hence, $(\varphi_n \upharpoonright_{\mathcal{E}_{\widetilde{K}}})$ admits a convergent subsequence. As a result, we deduce that the sequence $(\overline{\chi}\varphi_n)_n$ admits a W-convergent subsequence. So, the sequence $(\varphi_n \upharpoonright_{\mathcal{E} \setminus \mathcal{E}_{\widetilde{K}}})_n$ admits a W-convergent subsequence.

Now we can show that our operator D is semi-Fredholm.

1. We start by proving that $KerD$ is finite dimensional, which is equivalent to show that $\{\sigma \in KerD;\ \|\sigma\|_W = 1\}$ is compact.

   Let $(\sigma_n)_n \subset KerD$ be such that $\|\sigma_n\|_W = 1$ and $D\sigma_n = 0$. Then, by the claim, $(\sigma_n)_n$ admits a convergent subsequence. So the result occurs.
2. Let us show that ImD is closed.

   Let $(y_n)_n$ be a sequence of ImD such that $(y_n)_n$ converges to y in $l^2(G)$. Is that y in ImD?

   Since $(y_n)_n \subset ImD$, then there exist $(\sigma_n)_n \subset KerD^\perp$ and $\sigma_n \neq 0\ \forall n$, such that $y_n = D\sigma_n$. $(\sigma_n)_n$ must be bounded. If not, by extraction we can construct $s_n = \frac{\sigma_n}{\|\sigma_n\|_W}$, such that

$$\begin{cases} (s_n)_n \subset KerD^\perp \\ \|s_n\|_W = 1 \\ Ds_n \to 0. \end{cases}$$

   Using the claim, we can conclude that $(s_n)_n$ admits a convergent subsequence with limit denoted s such that

$$\begin{cases} s \in KerD^\perp \\ \|s\|_W = 1 \\ Ds = 0. \end{cases}$$

   Then, $s \in KerD \cap KerD^\perp = \{0\}$. So $s = 0$, which is absurd.

   Hence the sequence $(\sigma_n)_n$ is bounded and since $(D\sigma_n)_n$ converges to y, using the claim, the sequence $(\sigma_n)_n$ admits a convergent subsequence and let σ be this limit. But, the operator D is bounded. Then, $D\sigma_n$ converges to $D\sigma$ and by uniqueness of the limit $y = D\sigma$.

□

Corollary 4.5. *D is non-parabolic at infinity if and only if there exists a finite subgraph G_K of G such that if we complete $C_0(\mathcal{V}) \times C_0^a(\mathcal{E})$ by the norm*

$$\left\|(f,\varphi)\right\|_W = \left(\left\|(f,\varphi)\right\|^2_{l^2(\widetilde{K})} + \left\|D(f,\varphi)\right\|^2_{l^2(G)}\right)^{\frac{1}{2}},$$

in order to obtain W satisfying

1. *$C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ is dense in W.*
2. *The injection of $C_0(\mathcal{V}) \oplus C_0^a(\mathcal{E})$ to $C(\mathcal{V}) \oplus C^a(\mathcal{E})$ extends by continuity to W.*
3. *$D : W \longrightarrow l^2(G)$ is semi-Fredholm.*

5. Examples

5.1. A star-like graph

Definition 5.1. *The disjoint union of two graphs $G_\alpha = (\mathcal{V}_\alpha, \mathcal{E}_\alpha)$ and $G_\beta = (\mathcal{V}_\beta, \mathcal{E}_\beta)$ is the disjoint union of their vertex and edge with no edge joining $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$.*

According to [6], we have the following definition:

Definition 5.2. *An infinite graph $G = (\mathcal{V}, \mathcal{E})$ is called **star-like**, if there exists a finite subgraph G_K of G so that $G \setminus G_K$ is the union of a finite number of disjoint copies G_α of the graph $\mathbb{N}$.*

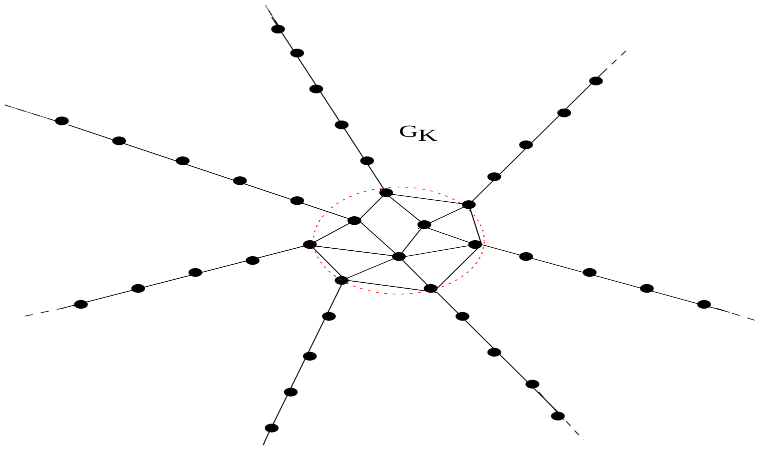

Figure 1: Star-like graph

Proposition 5.3. *In the case where $c = r = 1$, D is non-parabolic at infinity in the star-like graph.*

Proof: By the definition of the star-like graph, there exists a finite subgraph G_K of G so that $G \setminus G_K = \bigsqcup_{\alpha\in J} G_\alpha$. Let U be a finite subset of $G \setminus G_K$ then, there exists $\alpha \in J$ such that $U \subset G_\alpha$. We look for a positive constant $C = C(U)$ such that

$$C \left\|(f,\varphi)\right\|_{l^2(U)} \leq \left\|D(f,\varphi)\right\|_{l^2(G)},\ \forall (f,\varphi) \in C_0(\mathcal{V} \setminus K) \oplus C_0^a(\mathcal{E} \setminus \mathcal{E}_K). \tag{5.20}$$

Let $f \in C_0(\mathcal{V} \setminus K)$ such that U is included in the support of f.

For $U = \{a\}$, we have

$$\left\|f\right\|^2_{l^2(U)} = f^2(a).$$

For $o \in K$ and as G is connected we can find a path γ_{oa} joining o to a. Suppose that this path is of length n such that $x_0 = a$ and $x_n = o$, using the Jensen's inequality and $f(x_n) = 0$, we obtain

$$\begin{aligned} f^2(a) &= \left(f(x_0) - f(x_1) + f(x_1) - f(x_2) + f(x_2) - \ldots - f(x_{n-1}) + f(x_{n-1}) - f(x_n) + f(x_n)\right)^2 \\ &\leq n\left(\left(f(a) - f(x_1)\right)^2 + \left(f(x_1) - f(x_2)\right)^2 + \ldots + \left(f(x_{n-1}) - f(x_n)\right)^2\right), \end{aligned}$$

which implies

$$f^2(a) \leq n \left\|df\right\|^2_{l^2(V)}. \tag{5.21}$$

Remark 5.4. *n depends only on U and K.*

Similarly, for $\varphi \in C_0^a(\mathcal{E} \setminus \mathcal{E}_K)$, we obtain

$$\left\|\varphi\right\|^2_{l^2(\mathcal{E}_U)} \leq C_U \left\|\delta\varphi\right\|^2_{l^2(\mathcal{V})}.$$

Moreover, for $U = \{a_1, \ldots, a_n\}$, we prove the inequality (5.20).

By the inequality (5.21), for all $i \in \{1, \ldots, n\}$, we get

$$f^2(a_i) \leq n_i \left\|df\right\|^2_{l^2(\mathcal{V})}$$

where n_i is the number of edge of the shortest path between a_i and any vertex of K.

For thus, we have

$$\sum_{i=1}^{n} f^2(a_i) \leq \sum_{i=1}^{n} n_i \left\|df\right\|^2_{l^2(\mathcal{V})}.$$

Hence

$$\left\|f\right\|^2_{l^2(U)} \leq C_U \left\|df\right\|^2_{l^2(\mathcal{V})}.$$

And similarly, we show that

$$\left\|\varphi\right\|^2_{l^2(\mathcal{E}_U)} \leq C_U \left\|\delta\varphi\right\|^2_{l^2(\mathcal{V})}.$$

□

5.2. The triadic tree

Definition 5.5. *A tree is a connected graph containing no cycles. The **triadic tree** is the tree such that all the vertices have degree 3.*

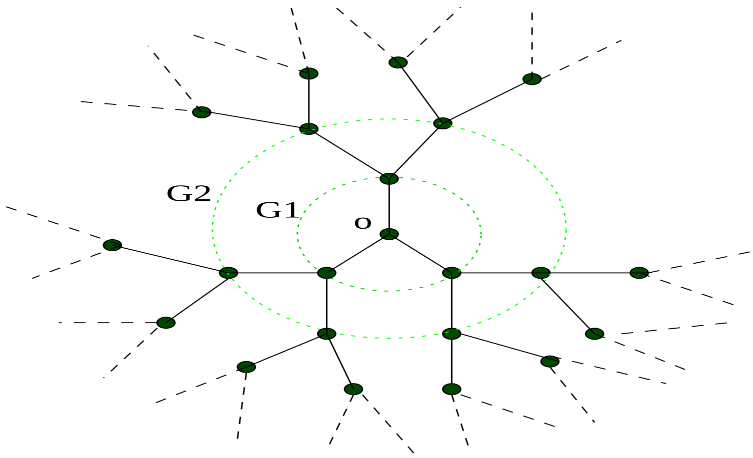

Figure 2: Triadic tree

Proposition 5.6. *In the triadic graph the condition of "non-parabolicity at infinity" is not verified.*

Proof: We fix a vertex o, see the figure 2, we can find an increasing sequence of finite subgraph $\{G_n\}_n$ such that $G_n = \{x \in \mathcal{V};\ d(o,x) \leq n\}$ and $G = \bigcup_n G_n$. The contradiction of non-parabolicity at infinity property could be: for all n there exists U outside of G_n and a 1-form φ_n with finite support outside of G_n such that $\delta\varphi_n = 0$ and $\left\|\varphi_n\right\|_{l^2(U)} \neq 0$. Such φ_n exist. Indeed one can construct a skewsymmetric function φ_n supported on the outward tree of every vertex $x_n \in G_n$ with $\delta\varphi_n = 0$ in the following way: let e_0 and b_0 be the two outward edges of x_n (the third one rely x_n to x_{n-1}) and denote e^k_m, $m \geqslant 1$, $1 \leqslant k \leqslant 2^m$, resp. b^k_m, $m \geqslant 1$, $1 \leqslant k \leqslant 2^m$, the outward edges emanating from e_0, resp. b_0, of generation m. We define φ_n to be 0 excepted on these edges where $\varphi_n(e^k_m) = \frac{1}{2^m}$ and $\varphi_n(b^k_m) = -\frac{1}{2^m}$ (the edge are oriented outward). So, we deduce that δ does not satisfy the property of non-parabolicity at infinity. □

Remark 5.7. *We can generalize this example for the tree with degree $d \geq 3$, we can use the same argument with $\varphi_n = \pm(\frac{1}{d-1})^m$.*

Remark 5.8. *a) The importance of non-parabolicity at infinity appears with the operator δ. In fact, this property for the operator d is always true on any connected graph.*
b) In probability [8] and potential theory [15] there exists an interesting notion of non-parabolic for the graph which is equivalent ([1] Theorem 2.1) to the following statement: there exists $x \in \mathcal{V}$ and $C > 0$ such that

$$f^2(x) \leq C \left\| df \right\|^2_{\ell^2(\mathcal{E})} , \forall f \in C_0(G).$$

This notion is different from the non-parabolicity at infinity. Indeed, the graph $\mathbb{Z}$ and $\mathbb{Z}^2$ are parabolic, but $\mathbb{Z}^n$, $n \geq 3$ is non-parabolic. On the other side, we have δ is non-parabolic at infinity in $\mathbb{Z}$ but in $\mathbb{Z}^n$, $n \geq 2$, δ does not verify this property (since it has cycles supported outside any finite subgraph).

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