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On k-Circulant Matrices with Arithmetic Sequence

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Abstract. Let k be a nonzero complex number. In this paper we consider k-circulant matrices with arithmetic sequence and investigate the eigenvalues, determinants and Euclidean norms of such matrices. Also, for k = 1, the inverses of such (invertible) matrices are obtained (in a way different from the way presented in [1]), and the Moore-Penrose inverses of such (singular) matrices are derived.

1. Introduction

By $\mathbb{C}^{m\times n}$ and $\mathbb{C}_r^{m\times n}$ we denote the set of all $m \times n$ complex matrices and the set of all $m \times n$ complex matrices of rank r, respectively. Similarly, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices and $\mathbb{R}_r^{m\times n}$ denotes the set of all $m \times n$ real matrices of rank r. Let $C \in \mathbb{C}^{n\times n}$. The eigenvalues, rank, determinant and the Euclidean (or Frobenius) norm of C are denoted by λ_j, $j=\overline{0,\, n-1}$, $r(C)$, $|C|$ and $\|C\|_E$, respectively. Symbols C^*, $C_{i\rightarrow}$ and $C_{\downarrow j}$ stand for the conjugate transpose of C, the i^{th} row of C and the j^{th} column of C, respectively. O denotes the zero matrix of appropriate dimensions.

Let C be a complex matrix of order n such that $(c_0, c_1, c_2, \ldots, c_{n-1})$ is its first row and $k\in\mathbb{C}\setminus\{0\}$. Then C is called a *k-circulant matrix* if C satisfies the following conditions:

$$c_{i,j} = \begin{cases} c_{j-i}, & i \le j \\ k\, c_{n+j-i}, & \text{otherwise} \end{cases}, \quad (i = \overline{2,n},\ j = \overline{1,n})$$

i.e. C has the following form:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \ldots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \ldots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \ldots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \ldots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \ldots & kc_{n-1} & c_0 \end{bmatrix}. \tag{1}$$

Let C be a k-circulant matrix and $(c_0, c_1, c_2, \ldots, c_{n-1})$ is its first row, then we shall write $C = circ_n\{_k(c_0, c_1, c_2, \ldots, c_{n-1})\}$. If the order of a matrix is known, then the designation for the order of a matrix can be omitted. Instead of

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"C is a 1*-circulant matrix"* we say *"C is a circulant matrix"*. Instead of *"C is a* −1*-circulant matrix"* we say *"C is an anti-circulant matrix"* or *"C is a negacyclic matrix"*.

In [9] R. E. Cline, R. J. Plemmons and G. Worm considered k-circulant matrices. Namely, they proved the following lemmas which yield necessary and sufficient conditions for a complex square matrix to be a k-circulant matrix.

Lemma 1.1. (Lemma 2. [9]) Let $k\in\mathbb{C}\setminus\{0\}$ and $W=circ_n\{_k(0,1,0,\ldots,0)\}$. Then a complex matrix C of order n is a k-circulant matrix if and only if it commutes with W. In this case C can be expressed as

$$C=\sum_{i=0}^{n-1} c_i W^i, \tag{2}$$

where $(c_0,c_1,c_2,\ldots,c_{n-1})$ is the first row of C.

Lemma 1.2. (Lemma 3. [9]) Let ψ be any n^{th} root of k and

$$\Psi=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \psi & 0 & \cdots & 0 & 0 \\ 0 & 0 & \psi^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \psi^{n-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \psi^{n-1} \end{bmatrix}.$$

Then a matrix C is a k-circulant matrix if and only if

$$C=\Psi Q \Psi^{-1} \tag{3}$$

for some circulant matrix Q.

In this paper we shall investigate the eigenvalues of k-circulant matrices with arithmetic sequence, among other things, and we need the following lemma.

Lemma 1.3. (Lemma 4. [9]) Let C be a k-circulant matrix. Then the eigenvalues of C are:

$$\lambda_j=\sum_{i=0}^{n-1} c_i(\psi\omega^{-j})^i,\ j=\overline{0,\,n-1}, \tag{4}$$

where $(c_0,c_1,c_2,\ldots,c_{n-1})$ is the first row of C, ψ is any n^{th} root of k and ω is any primitive n^{th} root of unity. Moreover, in this case

$$c_i=\frac{1}{n}\sum_{j=0}^{n-1} \lambda_j(\psi\omega^{-j})^{-i},\ i=\overline{0,\,n-1}\ . \tag{5}$$

In [9] the authors also investigated generalized inverses (see [2] and [11]) of k-circulant matrices. The inverse C^{-1} of (an invertible) k-circulant matrix C is always k-circulant, but the Moore-Penrose inverse $C^{\dagger}$ (i.e. the unique matrix which satisfies $CC^{\dagger}C=C$, $C^{\dagger}CC^{\dagger}=C^{\dagger}$, $(CC^{\dagger})^*=CC^{\dagger}$ and $(C^{\dagger}C)^*=C^{\dagger}C$) of (a singular) k-circulant matrix C need not be k-circulant. Namely, they proved the following theorem.

Theorem 1.1. (Theorem 3. [9]) Let C be a singular k-circulant matrix. Then $C^{\dagger}$ is k-circulant if and only if k lies on the unit circle.

They did not solve the problem of characterizing $C^{\dagger}$ for an arbitrary k-circulant matrix C. That problem was solved by E. Boman in [3].

In [7] the author investigated the eigenvalues, determinants, Euclidean norms and spectral norms of circulant matrices with geometric sequence and their inverses. The eigenvalues and determinants of circulant matrices with binomial coefficients were determined in [13], and the spectral norms and Euclidean norms of such matrices were derived in [12]. In [15] ([4]) the authors investigated the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers (with Jacobsthal and Jacobsthal-Lucas numbers) while the eigenvalues and Euclidean norms of circulant matrices with Fibonacci sequence were obtained by Bueno in [8] and the Euclidean norms of circulant and anti-circulant matrices with Jacobsthal and Jacobsthal-Lucas sequence were obtained in [10]. In [10] the author also investigated the eigenvalues, determinants, Euclidean norms and spectral norms of circulant and anti-circulant matrices with modified Pell numbers, while the determinants and inverses of circulant matrices with Pell and Pell-Lucas numbers were derived in [6]. The paper [5] is devoted to obtaining the determinants and inverses of k-circulant matrices associated with a number sequence. Circulant matrices with arithmetic sequence were considered in [1]. In this paper we consider k-circulant matrices with arithmetic sequence. The main aim of this paper is to obtain the formulae for the eigenvalues, determinants and Euclidean norms of k-circulant matrices with arithmetic sequence. Also, we shall obtain the inverse of (an invertible) circulant matrix with arithmetic sequence (the result of Theorem 2.6 [1]) using the new method (for obtaining the inverse of an invertible k-circulant matrix) which was illustrated in [14].

Lemma 1.4. (Lemma 2.2. [14]) Let $C = circ\{_k(c_0, c_1, c_2, \dots, c_{n-1})\}$ be an invertible matrix with complex entries. Then $C^{-1} = circ\{_k(c'_0, c'_1, c'_2, \dots, c'_{n-1})\}$, where $(c'_0, c'_1, c'_2, \dots, c'_{n-1})$ is the unique solution of the following system of linear equations:

$$C\begin{bmatrix} x_0 \\ kx_{n-1} \\ \vdots \\ kx_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6}$$

Using the full-rank factorization of matrices, the Moore-Penrose inverse $C^\dagger$ (i.e. the unique matrix which satisfies $CC^\dagger C = C$, $C^\dagger CC^\dagger = C^\dagger$, $(CC^\dagger)^* = CC^\dagger$ and $(C^\dagger C)^* = C^\dagger C$) of (a singular) circulant matrix C with arithmetic sequence will be derived. Namely,

Lemma 1.5. (Lemma 5. [2], p. 22) Let $C \in \mathbb{C}_r^{m\times n}$, $r > 0$. Then there exist matrices $M \in \mathbb{C}_r^{m\times r}$ and $N \in \mathbb{C}_r^{r\times n}$ such that

$$C = MN. \tag{7}$$

A factorization (7) with the properties stated in Lemma 1.5 is called a *full-rank factorization of* C.

Let us mention two ways to obtain a full-rank factorization of C.

i) Choose the columns of M as any maximal linearly independent set of columns of C, and then N is uniquely determined by (7);

ii) Choose the rows of N as any maximal linearly independent set of rows of C, and then M is uniquely determined by (7).

Theorem 1.2. (Theorem 5. [2], p. 23) (*MacDuffee*) If $C \in \mathbb{C}_r^{m\times n}$, $r > 0$, has the full-rank factorization (7), then

$$C^\dagger = N^*(M^*MNN^*)^{-1}M^*. \tag{8}$$

Before we present our main results, let us recall that *an arithmetic sequence* is a sequence having the following form:

$$a_0 = a,\ a_1 = a + d,\ a_2 = a + 2d,\ a_3 = a + 3d,\ \dots \tag{9}$$

where $a \in \mathbb{R}$ and $d \in \mathbb{R}\setminus\{0\}$. As we can see the difference between the consecutive terms of an arithmetic sequence is constant. For example, the sequence

$$3,\ \frac{7}{2},\ 4,\ \frac{9}{2}, 5,\ \dots$$

is an arithmetic sequence ($a = 3$, $d = \frac{1}{2}$). The sum of the first n terms of an arithmetic sequence is given by the following formula:

$$\sum_{i=0}^{n-1} a_i = \frac{n}{2}(a_0 + a_{n-1}), \tag{10}$$

i.e.

$$\sum_{i=0}^{n-1} a_i = \frac{n}{2}\left[2a_0 + (n-1)d\right]. \tag{11}$$

Our main results will be presented in the next section.

2. Main Results

Throughout this section, ψ is any n^{th} root of $k \in \mathbb{C}\setminus\{0\}$ and ω is any primitive n^{th} root of unity i.e. $\omega = e^{\frac{2\pi i}{n}}$. First, we investigate the eigenvalues of

$$circ\{_k(a, a+d, \ldots, a+(n-1)d)\}, \tag{12}$$

where $a \in \mathbb{R}$ and $d \in \mathbb{R}\setminus\{0\}$.

Theorem 2.1. Let A be a matrix of the form (12). Then the eigenvalues of A are given by the following formulae:

1) If $\psi\omega^{-j} = 1$, then

$$\lambda_j = \frac{n}{2}\left[2a + (n-1)d\right], \tag{13}$$

2) If $\psi\omega^{-j} \neq 1$, then

$$\lambda_j = a\,\frac{k-1}{\psi\omega^{-j}-1} + d\,\frac{\psi\omega^{-j}(1+nk-k) - nk}{(1-\psi\omega^{-j})^2}. \tag{14}$$

Proof.

1) Suppose that $\psi\omega^{-j} = 1$. Then, using Lemma 1.3, we obtain

$$\begin{aligned} \lambda_j &= \sum_{i=0}^{n-1} a_i(\psi\omega^{-j})^i \\ &= \sum_{i=0}^{n-1} a_i \\ &= \frac{n}{2}\left[2a + (n-1)d\right], \end{aligned}$$

2) Suppose that $\psi\omega^{-j} \neq 1$ and let $x := \psi\omega^{-j}$. Then, using Lemma 1.3, we obtain

$$\begin{aligned} \lambda_j &= \sum_{i=0}^{n-1} a_i(\psi\omega^{-j})^i \\ &= \sum_{i=0}^{n-1} (a+id)x^i \\ &= a\sum_{i=0}^{n-1} x^i + d\sum_{i=0}^{n-1} ix^i \end{aligned}$$

$$
\begin{aligned}
&= a\frac{x^n-1}{x-1} + dx(1+2x+3x^2+\ldots+(n-1)x^{n-2})\\
&= a\frac{k-1}{x-1} + dx\left(\frac{x-x^n}{1-x}\right)'\\
&= a\,\frac{k-1}{x-1} + dx\,\frac{(1-nx^{n-1})(1-x)+x-x^n}{(1-x)^2}\\
&= a\,\frac{k-1}{x-1} + dx\,\frac{1-x-nx^{n-1}+nk+x-k}{(1-x)^2}\\
&= a\,\frac{k-1}{x-1} + d\,\frac{x-nk+xnk-xk}{(1-x)^2}\\
&= a\,\frac{k-1}{x-1} + d\,\frac{x(1+nk-k)-nk}{(1-x)^2}\\
&= a\,\frac{k-1}{\psi\omega^{-j}-1} + d\,\frac{\psi\omega^{-j}(1+nk-k)-nk}{(1-\psi\omega^{-j})^2}\ . \quad \square
\end{aligned}
$$

Remark 2.1. If $k=1$, then we obtain the result of M. Bahsi and S. Solak (see Theorem 2.1 [1]).

Now, we obtain the determinant of the matrix in (12).

Theorem 2.2. Let A be a matrix of the form (12). Then the determinant of A is:

$$
|A| = aq^{n-1} + (-1)^{n-1}kd^2\left[\sum_{i=1}^{n-1}(-1)^{i-1}ir^{n-(i+1)}q^{i-1}\right], \tag{15}
$$

where $q := (1-k)a - nkd$ and $r := (k-1)a + [1+(n-1)k]\,d$.

Proof. Applying the properties of the determinant to the determinant of A we obtain the following equalities:

$$
|A| = \begin{vmatrix}
a & a+d & a+2d & \cdots & a+(n-2)d & a+(n-1)d\\
k\,[a+(n-1)d] & a & a+d & \cdots & a+(n-3)d & a+(n-2)d\\
k\,[a+(n-2)d] & k\,[a+(n-1)d] & a & \cdots & a+(n-4)d & a+(n-3)d\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
k\,[a+2d] & k\,[a+3d] & k\,[a+4d] & \cdots & a & a+d\\
k\,[a+d] & k\,[a+2d] & k\,[a+3d] & \cdots & k\,[a+(n-1)d] & a
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a & d & 2d & \cdots & (n-2)d & (n-1)d\\
0 & q & r & \cdots & 0 & 0\\
0 & 0 & q & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & q & r\\
kd & 0 & 0 & \cdots & 0 & q
\end{vmatrix},
$$

where $q := (1-k)a - nkd$ and $r := (k-1)a + [1+(n-1)k]\,d$.

Therefore,

$$
\begin{aligned}
|A| &= aq^{n-1} + (-1)^{n-1}kd^2\left[r^{n-2} - 2r^{n-3}q + 3r^{n-4}q^2 + \ldots + (-1)^{n-2}(n-1)q^{n-2}\right]\\
&= aq^{n-1} + (-1)^{n-1}kd^2\left[\sum_{i=1}^{n-1}(-1)^{i-1}ir^{n-(i+1)}q^{i-1}\right]. \quad \square
\end{aligned}
$$

Remark 2.2. If $k=1$, then we obtain the result of M. Bahsi and S. Solak (see Theorem 2.4 [1]).

In order to obtain the Euclidean norm of the matrix in (12) we shall use the following formulae:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{16}$$

and

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 . \tag{17}$$

Recall that the Euclidean norm of $C=\left[c_{i,j}\right]\in\mathbb{C}^{n\times n}$ is $\|C\|_E=\left(\sum_{i,j=1}^{n} | c_{i,j}|^2\right)^{\frac{1}{2}}$.

Theorem 2.3. Let A be a matrix of the form (12). Then the Euclidean norm of A is:

$$\|A\|_E=\sqrt{\frac{n}{2}a^2[n+1+|k|^2(n-1)]+(n-1)nad\left[\frac{n+1}{3}+|k|^2\frac{2n-1}{3}\right]+\frac{(n-1)n^2}{4}d^2\left[\frac{n+1}{3}+|k|^2(n-1)\right]} . \tag{18}$$

Proof.

$$\begin{aligned}
(\|A\|_E)^2 &= \sum_{i,j=1}^{n} |a_{i,j}|^2 \\
&= na^2+\left[(n-1)+|k|^2\right][a+d]^2+\ldots+\left[1+(n-1)|k|^2\right][a+(n-1)d]^2 \\
&= a^2\left[\sum_{i=1}^{n} i+|k|^2\sum_{i=1}^{n-1} i\right]+2ad\left[\sum_{i=1}^{n-1} i(n-i)+|k|^2\sum_{i=1}^{n-1} i^2\right]+d^2\left[\sum_{i=1}^{n-1} i^2(n-i)+|k|^2\sum_{i=1}^{n-1} i^3\right] \\
&= a^2\left[\sum_{i=1}^{n} i+|k|^2\sum_{i=1}^{n-1} i\right]+2ad\left[n\sum_{i=1}^{n-1} i-\sum_{i=1}^{n-1} i^2+|k|^2\sum_{i=1}^{n-1} i^2\right]+d^2\left[n\sum_{i=1}^{n-1} i^2-\sum_{i=1}^{n-1} i^3+|k|^2\sum_{i=1}^{n-1} i^3\right] \\
&= a^2\left[\frac{n(n+1)}{2}+|k|^2\frac{(n-1)\,n}{2}\right]+2ad\left[\frac{(n-1)\,n(n+1)}{6}+|k|^2\frac{(n-1)\,n(2n-1)}{6}\right]+ \\
&\quad d^2\left[\frac{(n-1)\,n^2(n+1)}{12}+|k|^2\frac{(n-1)^2n^2}{4}\right] \\
&= \frac{n}{2}a^2\left[n+1+|k|^2(n-1)\right]+(n-1)nad\left[\frac{n+1}{3}+|k|^2\frac{2n-1}{3}\right]+\frac{(n-1)n^2}{4}d^2\left[\frac{n+1}{3}+|k|^2(n-1)\right] .
\end{aligned}$$

Therefore,

$$\|A\|_E=\sqrt{\frac{n}{2}a^2[n+1+|k|^2(n-1)]+(n-1)nad\left[\frac{n+1}{3}+|k|^2\frac{2n-1}{3}\right]+\frac{(n-1)n^2}{4}d^2\left[\frac{n+1}{3}+|k|^2(n-1)\right]}. \qquad \square$$

Remark 2.3. If $k=1$, then we obtain the result of M. Bahsi and S. Solak (see Theorem 2.3 [1]).

Remark 2.4. It follows from Theorem 2.4 [1] (or Theorem 2.2, for $k = 1$) that

$$circ\{(a, a+d, a+2d, \ldots, a+(n-1)d)\}, \tag{19}$$

where $a\in\mathbb{R}$ and $d\in\mathbb{R}\setminus\{0\}$, is an invertible matrix if and only if

$$a+\frac{n-1}{2}d\neq 0 . \tag{20}$$

The inverse of the matrix in (19) was obtained in [1] (see Theorem 2.6 [1]), but in this paper we obtain the inverse of the matrix in (19) using Lemma 1.4.

Theorem 2.4. Let A be an invertible matrix of the form (19). Then the inverse of A is:

$$A^{-1}=\frac{1}{n^2(a+\frac{n-1}{2}d)}\,circ_n\{(-\frac{na+\frac{n^2-n-2}{2}d}{d},\frac{na+\frac{n^2-n+2}{2}d}{d},1,\dots,1)\}\,. \tag{21}$$

Proof. Let $A^{-1}=circ\{(a'_0,a'_1,a'_2,\dots,a'_{n-1})\}$. Based on Lemma 1.4 $(a'_0,a'_1,a'_2,\dots,a'_{n-1})$ is the unique solution of the following system of linear equations:

$$A\begin{bmatrix}x_0\\x_{n-1}\\\vdots\\x_1\end{bmatrix}=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}. \tag{22}$$

Applying elementary row operations to the augmented matrix, we get:

$$\overline{A}=\begin{bmatrix}a & a+d & a+2d & \dots & a+(n-2)d & a+(n-1)d & 1\\ a+(n-1)d & a & a+d & \dots & a+(n-3)d & a+(n-2)d & 0\\ a+(n-2)d & a+(n-1)d & a & \dots & a+(n-4)d & a+(n-3)d & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ a+2d & a+3d & a+4d & \dots & a & a+d & 0\\ a+d & a+2d & a+3d & \dots & a+(n-1)d & a & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}a & a+d & a+2d & \dots & a+(n-3)d & a+(n-2)d & a+(n-1)d & 1\\ -(n-1)d & d & d & \dots & d & d & d & 1\\ -nd & nd & 0 & \dots & 0 & 0 & 0 & 1\\ 0 & -nd & nd & \dots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -nd & nd & 0 & 0\end{bmatrix}.$$

Therefore, the linear system (22) is equivalent to the following system:

$$\begin{cases}\sum_{i=1}^{n-1}[a+id]x_{n-i}=1-ax_0,\\ \sum_{i=1}^{n-1}dx_i=1+(n-1)dx_0,\\ x_{n-1}-x_0=\frac{1}{nd},\\ x_i=x_{i+1},\ i=\overline{2,n-2}\,.\end{cases} \tag{23}$$

The solution of the system (23) is:

$$\begin{cases}x_0=-\frac{na+\frac{n^2-n-2}{2}d}{dn^2(a+\frac{n-1}{2}d)},\\ x_1=\frac{na+\frac{n^2-n+2}{2}d}{dn^2(a+\frac{n-1}{2}d)},\\ x_i=\frac{1}{n^2(a+\frac{n-1}{2}d)},\ i=\overline{2,n-1}\,.\end{cases} \tag{24}$$

Since the system (22) is equivalent to the system (23), it follows that (24) is also the solution of the system (22). □

If $a+\frac{n-1}{2}\,d=0$ (i.e. $d=\frac{2a}{1-n}$), then the matrix in (19) is a singular matrix (i.e. its inverse does not exist). But, the Moore-Penrose inverse of such matrix exists and will be given by the following theorem. Since $a_i=(\frac{1-n+2i}{1-n})\,a$, $i=\overline{0,n-1}$, we shall assume that $a\in\mathbb{R}\backslash\{0\}$.

Theorem 2.5. Let n be an arbitrary natural number greater than 1, $a\in\mathbb{R}\backslash\{0\}$ and

$$A = circ\{(a_0,a_1,\dots,a_{n-1})\}, \tag{25}$$

where $a_i=(\frac{1-n+2i}{1-n})\,a$, $i=\overline{0,n-1}$. Then the Moore-Penrose inverse of A is:

$$A^{\dagger} = \frac{n-1}{2na}\,circ_n\{(1,-1,0,\dots,0)\}\,. \tag{26}$$

Proof. Notice that $r(A)=n-1$ and $A_{\downarrow j}\neq O$, $j=\overline{1,n}$. Let $A=[A_1|A_2]$, where $A_1\in\mathbb{R}^{n\times(n-1)}$ and $A_2\in\mathbb{R}^{n\times 1}$. Since $A_{\downarrow j}$, $j=\overline{1,n-1}$, are linearly independent columns of A, it follows that $r(A_1)=n-1$. Therefore, A has the full-rank factorization (Lemma 1.5) as

$$A = A_1 N,$$

where

$$N=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1\\ 0 & 1 & 0 & \cdots & 0 & 0 & -1\\ 0 & 0 & 1 & \cdots & 0 & 0 & -1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0 & -1\\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}\in\mathbb{R}^{(n-1)\times n}_{n-1}.$$

Based on Theorem 1.2 it follows that

$$\begin{aligned} A^{\dagger} &= N^*(A_1^*A_1NN^*)^{-1}A_1^*\\ &= circ_n\{(\frac{n-1}{2na},-\frac{n-1}{2na},0,\dots,0)\}\\ &= \frac{n-1}{2na}\;circ_n\{(1,-1,0,\dots,0)\}\,. \;\square \end{aligned}$$

3. Conclusion

In this paper we determined the eigenvalues, determinants and Euclidean norms of k-circulant matrices with a non-constant arithmetic sequence, where $k\in\mathbb{C}\backslash\{0\}$, and we extended some results presented in [1]. Also, for $k=1$, we obtained the inverses of such (invertible) matrices using the method for obtaining the inverse of an invertible k-circulant matrix which was described in [14] and the explicit expression for the Moore-Penrose inverses of such (singular) matrices using the full-rank factorization of matrices.

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